FINAL EXAMINATION, SEMESTER I, 2022
DEPT. MATH., COLLEGE OF SCIENCE, KSU
MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [Marks: 3+3+3=9]

(a) Find the values of λ for which the following system of equations has a unique solution:

$$\begin{aligned} 2x_1 + 3x_2 + \quad x_3 &= -1 \\ x_1 + 2x_2 + \quad x_3 &= 0 \\ 3x_1 + x_2 + (\lambda^2 - 6)x_3 &= \lambda - 3. \end{aligned}$$

(b) Let A be a square matrix with $\det(A) = 1$ and

$$adj(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute A^{-1} and deduce A.

(c) By using Cramer's Rule solve the system of equations:

$$\begin{aligned} -p + q - 2r &= 1 \\ p - q + 9r &= -2 \\ 5q + r &= 4. \end{aligned}$$

Q2. [Marks: 3+3+3=9]

(a) Let $\mathbf{u} = \langle 1, 2, -1 \rangle$, $\mathbf{v} = \langle 0, 1, -1 \rangle$ and $\mathbf{w} = \langle 2, 3, 1 \rangle$. (i) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

(ii) Find the component of $\mathbf{u} + 2\mathbf{v}$ along $\mathbf{w}$.

(b) Find an equation of the plane that contains the point $P(4, -3, 0)$ and the line: $x = t + 5$, $y = 2t - 1$, $z = -t + 7$.

(c) Identify the surface $x - y^2 - z^2 = 0$, give traces, and sketch.

Q3. [Mark: 6+3+3=12]

(a) The position vector of a moving point at time t is $\mathbf{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$. Find the tangential and normal components of acceleration, and curvature at time t.

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{x^2+y^2}$ does not exist.

(c) The temperature T at (x, y) is given by $T = 5(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Use differential to approximate the temperature difference between points $(1, 1)$ and $(1.01, 0.98)$.

Q4. [Marks: 3+3+4=10]

(a) Use partial derivative to find $\frac{\partial z}{\partial x}$ if $(x^2 + 1)z^3 - y^2z^2 + xyz = 3$.

(b) Let $f(x, y, z) = xyz$. Find the directional derivative of f at the point $A(1, 1, 1)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(c) Let $g(x, y) = x^2 + y^2 + xy^2 + 9$. Find the local extrema and saddle point if any.

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1

Model Answer of
Final Exam S1/1444

M 107

Q1 [3+3+3=9]

(a) The augmented Mx is

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & \lambda^2-6 & \lambda-3 \end{array}\right]$$

$$\xRightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & -1 \\ 3 & 1 & \lambda^2-6 & \lambda-3 \end{array}\right]$$

$$\xRightarrow{-2R_1+R_2,\ -3R_1+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -5 & \lambda^2-9 & \lambda-3 \end{array}\right]$$

$$\xRightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & \lambda^2-9 & \lambda-3 \end{array}\right]$$

$$\xRightarrow{5R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda+2 \end{array}\right]$$

For $\lambda^2-4=0 \Rightarrow (\lambda-2)(\lambda+2)=0$
i.e $\lambda=2,\ \lambda=-2$

$\therefore$ the system of Eqns has a unique solution as
$\lambda \in \mathbb{R} \sim \{-2,2\}$ i.e $\lambda \neq 2,\ \lambda \neq -2$

(b) $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$= \operatorname{adj}(A)$, $\det(A)=1$

$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = (A^{-1})^{-1}$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ #

(c)

$\det(A) = \begin{vmatrix} -1 & 1 & -2 \\ 1 & -1 & 9 \\ 0 & 5 & 1 \end{vmatrix} = 35$

$\det(A_1) = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -1 & 9 \\ 4 & 5 & 1 \end{vmatrix} = 4$, $\det(A_2) = \begin{vmatrix} -1 & 1 & -2 \\ 1 & -2 & 9 \\ 0 & 4 & 1 \end{vmatrix} = 29$

$\det(A_3) = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 5 & 4 \end{vmatrix} = 5$

The solution is $p = \frac{4}{35}$, $q = \frac{29}{35}$, $r = \frac{-1}{7}$

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Q_2 [3+3+3=9]

(a)

(i) $\|\vec{u}\| = \sqrt{6}$, $\|\vec{v}\| = \sqrt{2}$

and $\vec{u}\cdot\vec{v} = \langle 1,2,-1\rangle \cdot \langle 0,1,-1\rangle$

$\therefore \vec{u}\cdot\vec{v} = 0+2+1 = 3$

$\because \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\therefore \cos\theta = \dfrac{3}{\sqrt{6}\sqrt{2}} = \dfrac{3}{\sqrt{12}} = \dfrac{3}{2\sqrt{3}}$

$\Rightarrow \theta = 30^\circ$

(ii) $\vec{u}+2\vec{v} = \langle 1,2,-1\rangle + 2\langle 0,1,-1\rangle$
$= \langle 1,4,-3\rangle$

$$\mathrm{Comp}_{\vec{w}}(\vec{u}+2\vec{v}) = \frac{\langle 1,4,-3\rangle \cdot \langle 2,3,1\rangle}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

where $\vec{w} = \langle 2,3,1\rangle$, $\|\vec{w}\| = \sqrt{14}$

(b) we have the two points $P(4,-3,0)$ and $Q(5,-1,7)$ in the plane, So $\vec{PQ} = \langle 1,2,7\rangle$, the direction vector for the line in the plane is $\vec{a} = \langle 1,2,-1\rangle$ So the normal vector to the plane is

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 7 \\ 1 & 2 & -1 \end{vmatrix} = \langle -16,8,0\rangle$$

The eq$\underline{n}$ of the plane is

$-16(x-4)+8(y+3)+0(z-0)=0$

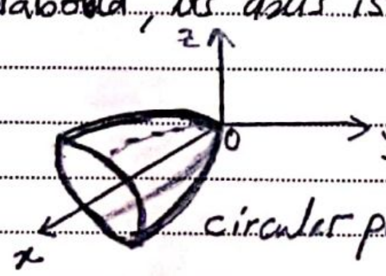
$\therefore 16x-8y-88=0$

or $2x-y=11$ is the eq$\underline{n}$ of the required plane.

(c) $x-y^2-z^2=0 \Rightarrow x=y^2+z^2$

Trace	Eq. of Trace	Description
xy-plane $z=0$	$x=y^2$	parabola
xz-plane $y=0$	$x=z^2$	parabola
yz-plane $x=0$, and $x=c$, $c>0$	$y^2+z^2=0$, $y^2+z^2=c$	origin, circle

$\therefore$ The given surface is a circular paraboloid, its axis is x-axis.



circular paraboloid

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$\frac{3}{(a)}$ Q3 [6+3+3 = 12]

(i) $\vec{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$

$\vec{r}'(t) = \langle -4\sin t, 9\cos t, 1 \rangle$

$\vec{r}''(t) = \langle -4\cos t, -9\sin t, 0 \rangle$

$\vec{r}'(t) \cdot \vec{r}''(t) = 16\sin t\cos t - 81\sin t\cos t$

$= -65\sin t\cos t$

$\|\vec{r}'(t)\| = \sqrt{16\sin^2 t + 81\cos^2 t + 1}$

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|} = \frac{-65\sin t\cos t}{(16\sin^2 t + 81\cos^2 t + 1)^{1/2}} \quad (1)$$

$$r'(t) \times r''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4\sin t & 9\cos t & 1 \\ -4\cos t & -9\sin t & 0 \end{vmatrix} = \langle 9\sin t, -4\cos t, 36 \rangle$$

$$a_N = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

$$a_N = \frac{(81\sin^2 t + 16\cos^2 t + 1296)^{1/2}}{(16\sin^2 t + 81\cos^2 t + 1)^{1/2}} \quad (2)$$

$$K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{(81\sin^2 t + 16\cos^2 t + 1296)^{1/2}}{(16\sin^2 t + 81\cos^2 t + 1)^{3/2}} \quad (3)$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{x^2+y^2} \to \frac{0}{0}$$ (Indeterminate form)

$\therefore$ At $x=0$, $$\lim_{(0,y)\to(0,0)} \frac{0^2+3y^2}{0^2+y^2} = \frac{3y^2}{y^2} = 3$$

Also, at $y=0$,

$$\lim_{(x,0)\to(0,0)} \frac{x^2+3(0)^2}{x^2+0^2} = \frac{x^2}{x^2} = 1$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{x^2+y^2}$ D.N.E

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(c)

$T = 5(x^2+y^2)^2$

$dT = T_x dx + T_y dy$

$dx \simeq 0.01$, $dy \simeq -0.02$

$\therefore dT = 2(5)(x^2+y^2)2x\,dx$
$+ 2(5)(x^2+y^2)2y\,dy$

$\therefore dT = 20(1)(1+1)(0.01)$
$+ 20(1)(1+1)(-0.02)$
$= 0.4 - 0.8$

$\therefore dT = -0.4$, where $(x,y) = (1,1)$

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Q4 [3+3+4=10]

(a)

let $F(x,y,z) = (x^2+1)z^3 - y^2z^2 + xyz - 3 = 0$

then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{(2xz^3+yz)}{3(x^2+1)z^2 - 2y^2z + xy}$

(b)

$f(x,y,z) = xyz$

$\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{<1,1,1>}{\sqrt{3}}$

$\nabla f = <yz, xz, xy>$

The direction derivative at the point A(1,1,1) is

$D_u f(x,y,z) = \nabla f \cdot \vec{u}$

$= <1,1,1> \cdot <\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>$

$\therefore D_u f(x,y,z) = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

(c)

$g(x,y,z) = x^2 + y^2 + xy^2 + 9$

To get the critical points, solve the following two Eqs

$g_x = 2x + y^2 = 0$ ① $g_y = 2y + 2xy = 0$ ②

Eq. ① $\Rightarrow x = -\frac{1}{2}y^2$, substitute in ② we get

$y^3 - 2y = 0 \Rightarrow y(y^2-2) = 0 \therefore y = 0, y = \pm\sqrt{2}$

and $x = 0, x = -1$

$\therefore$ The critical points are $(0,0), (-1,\sqrt{2}), (-1,-\sqrt{2})$

$D(x,y) = \begin{vmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2y \\ 2y & 2+2x \end{vmatrix}$

$\therefore D(0,0) = 4 > 0$, $g_{xx} = 2 > 0 \Rightarrow g(0,0) = 9$ is a local minimum

$D(-1,\sqrt{2}) = \begin{vmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{vmatrix} = -8 < 0 \Rightarrow (-1,\sqrt{2})$ is a saddle point

and $D(-1,-\sqrt{2}) = \begin{vmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{vmatrix} = -8 < 0 \Rightarrow (-1,-\sqrt{2})$ is a saddle point.

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