King Saud University: Mathematics Department Math-254
First Semester 1442-43 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————————— **I.D. No.** ————————————

Name of the Teacher:———————————— **Section No.** ————————————

Note: Check the total number of pages are Seven (7).

(**16 Multiple choice questions and Three (3) Full questions**)

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d																

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 16		24
Q. 17		6
Q. 18		5
Q. 19		5
Total		40

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(Math)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	c	b	a	c	b	a	c	b	b	a	c	c	a	b	b	c

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MAth)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	b	a	b	b	a	c	b	c	a	c	b	a	c	a	c	a

The Answer Tables for Q.1 to Q.16 : Marks: 1.5 for each one ($1.5 \times 16 = 24$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATh)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a,b,c,d	a	c	c	a	c	b	a	a	c	b	a	b	b	c	a	b

Question 17: Consider the following linear system of equations

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & & & = & 3 \\ x_1 & + & 8x_2 & + & x_3 & = & 10 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$$

Use Jacobi iterative method and the initial solution $\mathbf{x^{(0)}} = [0.5, 0.5, 0.5]^T$ to compute second approximation $\mathbf{x^{(2)}}$. Use the computed second approximation to find the error bound for the relative error.

Solution. The Jacobi method for the given system is

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big] \\ x_2^{(k+1)} &= \frac{1}{8}\Big[10 - x_1^{(k)} - x_3^{(k)}\Big] \\ x_3^{(k+1)} &= \frac{1}{2}\Big[3 - x_2^{(k)}\Big] \end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 0.5, x_2^{(0)} = 0.5, x_3^{(0)} = 0.5$, and for $k = 0, 1$, we obtain the first and the second approximations as

$$\mathbf{x^{(1)}} = [\mathbf{1.25}, \mathbf{1.125}, \mathbf{1.25}]^{\mathbf{T}} \quad \text{and} \quad \mathbf{x^{(2)}} = [\mathbf{0.9375}, \mathbf{0.9375}, \mathbf{0.9375}]^{\mathbf{T}}.$$

Since the inverse of the matrix is

$$A^{-1} = \begin{bmatrix} 15/28 & -1/14 & 1/28 \\ -1/14 & 1/7 & -1/14 \\ 1/28 & -1/14 & 15/28 \end{bmatrix} = \begin{bmatrix} 0.5357 & -0.0714 & 0.0357 \\ -0.0714 & 0.r1429 & -0.0714 \\ 0.0357 & -0.0714 & 0.5357 \end{bmatrix},$$

so the condition number of the matrix is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = 10 \times 0.6429 = 6.4286.$$

The residual vector can be calculated as

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix} 3 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0.9375 \\ 0.9375 \\ 0.9375 \end{pmatrix} = \begin{pmatrix} 0.1875 \\ 0.6250 \\ 0.1875 \end{pmatrix},$$

and it gives

$$\|\mathbf{r}\|_\infty = 0.6250.$$

From relative error formula, we have

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq K(A) \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

By using $K(A) = 6.4286$, $\|\mathbf{r}\|_\infty = 0.6250$, and $\|\mathbf{b}\|_\infty = 10$, we obtain

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|}{\|\mathbf{x}\|} \leq (6.4286) \frac{(0.6250)}{10} = 0.4018.$$

Question 18: If $f(x) = x^2 + \cos 2x$ and x-values are $\{-0.5, 0.0, 0.3, 0.5, 0.6, 1.0\}$. Use the quadratic Lagrange interpolating polynomial for equally spaced data points to find the best approximation of $0.16 + \cos 0.8$. Compute an error bound and the absolute error.

Solution. Since the given function is $x^2 + \cos 2x$, so by taking $2x = 0.8$, we have $x = 0.4$, therefore, the best points for the equally spaced data points Lagrange quadratic polynomial are, $x_0 = 0.0, x_1 = 0.3$, and $x_2 = 0.6$ with $h = 0.3$. Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \tag{1}$$

$$f(0.4) \approx p_2(0.4) = L_0(0.4)(1.0000) + L_1(0.4)(0.9153) + L_2(0.4)(0.7224). \tag{2}$$

The Lagrange coefficients can be calculate as follows:

$$L_0(0.4) = \frac{(0.4-0.3)(0.4-0.6)}{(0.0-0.3)(0.0-0.6)} = -0.1111,$$

$$L_1(0.4) = \frac{(0.4-0.0)(0.4-0.6)}{(0.3-0.0)(0.3-0.6)} = 0.8889,$$

$$L_2(0.4) = \frac{(0.4-0.0)(0.4-0.3)}{(0.6-0.0)(0.6-0.3)} = 0.2222.$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-0.1111)(1.0000) + (0.8889)(0.9153) + (0.7224)(0.2222) = 0.8630,$$

which is the required approximation of the given exact solution $0.16 + \cos 0.8 \approx 0.8567$.
To compute an error bound for the approximation of the given function in the interval $[0.0, 0.6]$, we use the following quadratic error formula

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}}.$$

As

$$M = \max_{0.0 \leq x \leq 0.6} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x - 2\sin 2x, \qquad f''(x) = 2 - \cos 2x, \qquad f^{(3)}(x) = 8\sin 2x,$$

$$M = \max_{0.0 \leq x \leq 0.6} \left| 8\sin 2x \right| = 7.4563.$$

Hence

$$|f(0.4) - p_2(0.4)| \leq \frac{(7.4563)(0.3^3)}{9\sqrt{(3)}} = 0.0129,$$

which is desired error bound. Also, we have

$$|f(0.4) - p_2(0.4)| = |(0.16 + \cos 0.8) - 0.8630| = |0.8567 - 0.8630| = 0.0063,$$

the desired absolute error.

Question 19: The following table is for the data point $(x, x + \cos x)$:

x	0.0	0.1	0.21	0.3	0.4	0.5	0.6	0.75	0.8	0.92	1.0	1.1	1.2
$f(x)$	1.00	1.10	1.19	1.26	1.32	1.38	1.43	1.48	1.50	1.53	1.54	1.55	1.56

Find the approximation of $\int_0^{1.2} f(x)\, dx$ by using the best integration rule and then compute absolute error and error bound.

Solution. Using equally spaced data, so we select the following set of data points as

x	0.0	0.6	1.2
$f(x)$	1.00	1.43	1.56

which gives $h = 0.6$ and $n = 2$(even), so the best rule is simple Simpson's rule for three points which can be used as

$$\int_0^{1.2} f(x)\, dx \approx S_2(f) = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big],$$

$$\int_0^{1.2} f(x)\, dx \approx 0.2\Big[1.00 + 4(1.43) + 1.56\Big] = 1.6560.$$

We can easily computed the exact value of the given integral as

$$\int_0^{1.2} (x + \cos x)\, dx = (x^2/2 + \sin x)\Big|_0^{1.2} = 1.6520.$$

Thus the absolute error $|E|$ in our approximation is given as

$$|E| = |0.3298 - S_2(f)| = |1.6520 - 1.6560| = 0.0040.$$

The fourth derivative of the function $f(x) = x + \cos x$ can be obtain as

$$f'(x) = 1 - \sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x.$$

Since $\eta(x)$ is unknown point in $(0, 1.2)$, therefore, the bound $|f^{(4)}|$ on $[0, 1.2]$ is

$$M = \max_{0 \le x \le 1.2} |f^{(4)}| = \max_{0 \le x \le 1.2} |\cos x| = 1.0,$$

at $x = 0$. Thus the error bound formula takes form

$$|E_{S_2}(f)| \le \frac{(0.6)^5}{90}(1.0) = 0.000864.$$