King Saud University: Mathematics Department Math-254
Second Semester 1444 H Solution of Final Examination
Maximum Marks = 40 Time: 180 mins.

Note: Check the total number of pages are Six (6).

(**15 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d															

Question No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(Math)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	b	a	c	b	a	c	b	c	a	c	b	a	c	a	c

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MAth)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	c	b	a	c	b	a	c	b	b	a	c	c	a	b	b

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.(MATh)

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	a	c	b	a	c	b	a	a	c	b	a	b	b	c	a

Question 16: Let $x_0 = 1, x_1 = 1, x_2 = 1, x_3 = 2,\ \ f(x) = \dfrac{2}{x}$, and the third divided difference is $f[1,1,1,2] = -1$. Compute the absolute error and an error bound for the approximation of $f(1.5)$ using cubic Newton's polynomial.

Solution. Since $f(x) = \dfrac{2}{x}$ and $x_0 = 1, x_1 = 1, x_2 = 1, x_3 = 2$, so we have the first four derivatives of the function are

$$f^{(1)}(x) = \frac{-2}{x^2}, \qquad f^{(2)}(x) = \frac{4}{x^3}, \qquad f^{(3)}(x) = \frac{-12}{x^4}, \qquad f^{(4)}(x) = \frac{48}{x^5}.$$

Since the cubic Newton's interpolating polynomial has the following form

$$p_3(x) = f[x_0] + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_0)f[x_0,x_1,x_2] + (x-x_0)(x-x_0)(x-x_1)f[x_0,x_1,x_2,x_3],$$

then using the given x points and interpolating point $x = 1.5$, we have

$$p_3(1.5) = f(1) + (1.5-1)f[1,1] + (1.5-1)(1.5-1)f[1,1,1] + (1.5-1)(1.5-1)(1.5-1)f[1,1,1,2],$$

Now the values of first, second and third-order divided differences are as follows:

$$f[1,1] = f^{(1)}(1) = -2,$$

$$f[1,1,1] = \frac{f^{(2)}(1)}{2!} = \frac{4}{2} = 2,$$

$$f[1,1,1,2] = -1.$$

Thus

$$f(1.5) \approx p_3(1.5) = 2 + (0.5)(-2) + (0.25)(2) + (0.1250)(-1) = 1.3750,$$

the required approximation of $f(1.5)$ and

$$|f(1.5) - p_3(1.5)| = |f(1.5) - p_3(1.5)| = |1.3333 - 1.3750| = 0.0417,$$

the possible absolute error in the approximation.
Taking the fourth derivative of the given function, we obtain

$$f^{(4)}(x) = \frac{48}{x^5} \quad \text{and} \quad |f^{(4)}(\eta(x))| = \left|\frac{48}{(\eta(x))^5}\right|, \quad \text{for} \quad \eta(x) \in (1,2).$$

Since

$$|f^{(4)}(1)| = 48 \quad \text{and} \quad |f^{(4)}(2)| = 1.5,$$

so $|f^{(4)}(\eta(x))| \leq \max\limits_{1\leq x\leq 2}\left|\dfrac{48}{x^5}\right| = 48$ and it gives

$$|f(1.5) - p_3(1.5)| \leq \frac{48}{24}|(1.5-1)(1.5-1)(1.5-1)(1.5-2)| = 0.1250,$$

which is the required error bound for the approximation $p_3(1.5)$. •

Question 17: Find the approximation of $f''(0.8)$ by using the following set of data points using three-point central difference rule:

x	0.0	0.11	0.24	0.3	0.4	0.5	0.6	0.72	0.8	0.9	1.05	1.11	1.2
$f(x)$	1.00	1.10	1.2	1.26	1.32	1.38	1.43	1.47	1.50	1.52	1.55	1.55	1.56

The function tabulated is $f(x) = x + \cos x$ (x in radian), how many subintervals approximate the given derivative to within accuracy of 10^{-6} using the differentiation rule of $f''(x)$?

Solution. Given $x_1 = 0.8, h = 0.4$, then the iterative formula for f'' becomes

$$f''(0.8) \approx \frac{f(0.8+0.4) - 2f(0.8) + f(0.8-0.4)}{(0.4)^2} = D_h^2 f(1),$$

or

$$f''(0.8) \approx \frac{f(1.2) - 2f(0.8) + f(0.4)}{0.16} = \frac{1.56 - 2(1.50) + (1.32)}{0.16} = -0.75 = D_h^2 f(1),$$

is the required approximation of $f''(0.8)$.
The fourth derivative of the given function at $\eta(x_1)$ is

$$f^{(4)}(\eta(x_1)) = \cos \eta(x_1),$$

and it cannot be computed exactly because $\eta(x_1)$ is not known. But one can bound the error by computing the largest possible value for $|f^{(4)}(\eta(x_1))|$. So bound $|f^{(4)}|$ on the interval $(0.4, 1.2)$ is

$$M = \max_{0.4 \leq x \leq 1.2} |\cos x)| = 0.9211,$$

at $x = 0.4$.

Since the given accuracy required is 10^{-6}, so

$$|E_C(f,h)| = \Big| -\frac{h^2}{12} f^{(4)}(\eta(x_1)) \Big| \leq 10^{-6},$$

for $\eta(x_1) \in (0.4, 1.2)$. Then for $|f^{(4)}(\eta(x_1))| \leq M$, we have

$$\frac{h^2}{12} M \leq 10^{-6},$$

or using $h = \dfrac{b-a}{n} = \dfrac{1.2 - 0.4}{n}$, gives

$$\frac{(1.2-0.4)^2}{n^2} M \leq 10^{-6},$$

solving for n, we get

$$n \geq \sqrt{(0.64 \times 0.9211 \times 10^6)/12} \geq 221.6424, \quad \text{gives,} \quad n = 222,$$

the required subintervals for approximating the given derivative to the given accuracy . •