1. **The Difference Equation and the Digital Filtering**

**Consider a DSP system in the form of an LTI system as shown in the figure below**

**An LTI System**

**Digital Input**

$$x(n)$$

**Digital Output**

$$y(n)$$

**The relationship between its input and output can be expressed in the form of a *difference equation* as,**

|  |  |
| --- | --- |
| $$y\left(n\right)=b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+…+b\_{M}x\left(n-M\right)$$$$ -a\_{1}y\left(n-1\right)-a\_{2}y\left(n-2\right)-…-a\_{N}y\left(n-N\right)$$ | **(1)** |

**where**$ b\_{i}, 0\leq i\leq M$ **and**$ a\_{j}, 1\leq j\leq N$ **represent the coefficients of the system and**$ n$ **is the time index. This equation can also be written as**

|  |  |
| --- | --- |
| $$y\left(n\right)=\sum\_{i=0}^{M}b\_{i}x\left(n-i\right)-\sum\_{j=1}^{N}a\_{j}y\left(n-j\right)$$ | **(2)** |

**The equation shows that the current value of the output**$ y\left(n\right)$ **depends on the current**$ x\left(n\right)$ **and past values**$ x\left(n-1\right), x\left(n-2\right), …, x(n-M)$ **of the input as well as the past values**$ y\left(n-1\right), y\left(n-2\right), …, y(n-N)$ **of the output.**

**We have already seen that a system expressed in this form of difference equation fulfills the conditions of linearity, time-invariance and causality.**

**If the initial conditions are given, the system output (i.e. time response),** $y\left(n\right)$**, can be obtained recursively (illustrated below by the examples). This process is called *digital filtering*.**

**Example 1**

**Compute the system output**

$$y\left(n\right)=0.5 y\left(n-2\right)+x(n-1)$$

**for the first four samples using the following initial conditions**

1. **Initial conditions:**$ y\left(-2\right)=1, y\left(-1\right)=0, x\left(-1\right)=-1$**, and input** $x\left(n\right)=\left(0.5\right)^{n} u(n)$**.**
2. **Zero initial conditions:**$ y\left(-2\right)=0, y\left(-1\right)=0, x\left(-1\right)=0$**, and input** $x\left(n\right)=\left(0.5\right)^{n} u(n)$**.**

**Solution**

1. **Setting**$ n=0$**, and using the initial conditions, we obtain the input and output as**

$$x\left(0\right)=\left(0.5\right)^{0}u\left(0\right)=1$$

$$y\left(0\right)=0.5 y\left(-2\right)+x\left(-1\right)=0.5×1-1=-0.5$$

**Setting**$ n=1$**, and using the initial conditions, we obtain the input and output as**

$$x\left(1\right)=\left(0.5\right)^{1}u\left(1\right)=0.5$$

$$y\left(1\right)=0.5 y\left(-1\right)+x\left(0\right)=0.5×0+1=1.0$$

**Setting**$ n=2$**, and using the past values of the input and output,**

$$x\left(2\right)=\left(0.5\right)^{2}u\left(2\right)=0.25$$

$$y\left(2\right)=0.5 y\left(0\right)+x\left(1\right)=0.5×\left(-0.5\right)+0.5=0.25$$

**Setting**$ n=3$**, and using the past values of the input and output,**

$$x\left(3\right)=\left(0.5\right)^{3}u\left(1\right)=0.125$$

$$y\left(3\right)=0.5 y\left(1\right)+x\left(2\right)=0.5×1+0.25=0.75$$

**Clearly, it can be seen that the further value of the output can be obtained recursively.**

1. **Setting**$ n=0$**, and using the initial conditions, we obtain the input and output as**

$$x\left(0\right)=\left(0.5\right)^{0}u\left(0\right)=1$$

$$y\left(0\right)=0.5 y\left(-2\right)+x\left(-1\right)=0.5×0+0=0$$

**Setting**$ n=1$**, and using the initial conditions, we obtain the input and output as**

$$x\left(1\right)=\left(0.5\right)^{1}u\left(1\right)=0.5$$

$$y\left(1\right)=0.5 y\left(-1\right)+x\left(0\right)=0.5×0+1=1.0$$

**Setting**$ n=2$**, and using the past values of the input and output,**

$$x\left(2\right)=\left(0.5\right)^{2}u\left(2\right)=0.25$$

$$y\left(2\right)=0.5 y\left(0\right)+x\left(1\right)=0.5×0+0.5=0.5$$

**Setting**$ n=3$**, and using the past values of the input and output,**

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**Clearly, it can be seen that the further value of the output can be obtained recursively.**

**Example 2**

**Compute the DSP system output**

$$y\left(n\right)=2x\left(n\right)-4x\left(n-1\right)-0.5 y\left(n-1\right)-y(n-2)$$

**with the initial conditions**$ y\left(-2\right)=1, y\left(-1\right)=0, x\left(-1\right)=-1$**, and input** $x\left(n\right)=\left(0.8\right)^{n} u(n)$**.**

1. **Compute the system response**$ y\left(n\right)$ **for 20 samples using MATLAB.**

**Solution**

**A MATLAB program to compute the system response for 20 samples is given below along with the corresponding output shown in graphical form.**

**% Example 2**

**%**

**% Compute the response y(n) of a DSP system expressed by**

**% y(n)=2x(n)-4x(n-1)-0.5y(n-1)-y(n-2)**

**% for the first 20 samples. Initial conditions are**

**% y(-2)=1, y(-1)=0, x(-1)=-1 and the system input is**

**% x(n)=(0.8)^n\*u(n).**

**%**

**% Initialize the input and output vectors**

**xi = [0 -1]; % for n=-2 and n=-1**

**yi = [1 0]; % for n=-2 and n=-1**

**% Compute time indices**

**n = 0:1:19;**

**% Compute the input samples x(n) for these time instants n**

**x = (0.8).^n;**

**% Include the initial values of input into this vector**

**x = [xi x];**

**% Now compute the system response**

**y = []; % an empty vector**

**y = [yi y]; % after including the initial conditions**

**% compute y(n) recursively**

**for k = 3:1:22**

 **r = 2\*x(k-2)-4\*x(k-1)-0.5\*y(k-1)-0.5\*y(k-2);**

 **y = [y r];**

**end**

**subplot(2,1,1), stem(n,x(3:22),'filled','LineWidth',2), grid on**

**xlabel('Sample number'); ylabel('Input x(n)');**

**subplot(2,1,2), stem(n,y(3:22),'filled','LineWidth',2), grid on**

**xlabel('Sample number'); ylabel('Output x(n)');**



**Figure 2: Plots of the input and system output for Example 2.**

**There are two MATLAB functions (syntax given below), that can used to perform this filtering process:**

**Zi = filtic(B, A, Yi, Xi)**

**y = filter(B, A, x, Zi)**

**where B and A are vectors for the coefficients given as**

$A=[1 a\_{1} a\_{2}… a\_{N}]$ **and** $B=[ b\_{0} b\_{1} b\_{2}… b\_{M}]$

**Xi and Yi are the vectors containing the initial conditions. Also x, y are the input and system output vectors.**

**The function filtic is used to obtain the initial states required by the second function filter. The function filter is based on the *direct-form II realization* to implement a digital filter from its difference equation form. This will be studied in a coming lecture.**

**The following MATLAB code, illustrates how to solve Example 1, using filtic and filter MATLAB functions.**

**>> B = [0 1];**

**>> A = [1 0 -0.5];**

**>> Xi = [-1 0];**

**>> Yi = [0 1];**

**>> Zi = filtic(B, A, Yi, Xi);**

**>> n = 0:3;**

**>> x = (0.5).^n;**

**>> y = filter(B, A, x, Zi)**

**y =**

 **-0.5000 1.0000 0.2500 0.7500**

**These are the same results as obtained in Example 1.**

1. **The Difference Equation and the Transfer Function**

**From Equation 1, we have**

|  |  |
| --- | --- |
| $$y\left(n\right)=b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+…+b\_{M}x\left(n-M\right)$$$$ -a\_{1}y\left(n-1\right)-a\_{2}y\left(n-2\right)-…-a\_{N}y\left(n-N\right)$$ |  |

**Assuming that all initial conditions for this system are zero, we take the z-transform of both sides to get**

|  |  |
| --- | --- |
| $$Y\left(z\right)=b\_{0}X\left(z\right)+b\_{1}z^{-1}X\left(z\right)+…+b\_{M}z^{-M}X\left(z\right)$$$$ -a\_{1}z^{-1}Y\left(z\right)-a\_{2}z^{-2}Y\left(z\right)-…-a\_{N}z^{-N}Y\left(z\right)$$ | **(3)** |

**We have made use of the shift-theorem in the above equation. Rearranging, we obtain**

|  |  |
| --- | --- |
| $$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}=\frac{b\_{0}+b\_{1}z^{-1}+…+b\_{M}z^{-M}}{1+a\_{1}z^{-1}+…+a\_{N}z^{-N}}=\frac{B\left(z\right)}{A\left(z\right)}$$ | **(4)** |

**where**$ H\left(z\right)$ **is defined as the z-transfer function with its numerator and denominator polynomials given by**

|  |  |
| --- | --- |
| $$B\left(z\right)=b\_{0}+b\_{1}z^{-1}+…+b\_{M}z^{-M}$$ | **(5)** |
| $$A\left(z\right)=1+a\_{1}z^{-1}+…+a\_{N}z^{-N}$$ | **(6)** |

**It can clearly be notices that the z-tranfer function is the ratio of the z-transform of the output with the z-transform of the input. This can diagrammatically be shown as**

**The z-transfer function can be used to determine the stability and frequency response of the digital filter.**

**Example 3**

**A DSP system is described by the following difference equation**

$$y\left(n\right)=x\left(n\right)-x\left(n-2\right)-1.3y\left(n-1\right)-0.36y(n-2)$$

**Find the z-transfer function**$ H\left(z\right)$**, the denominator polynomial**$ A\left(z\right)$**, and the numerator polynomial**$ B\left(z\right)$**.**

**Solution**

**Taking the z-transform of both sides of the given difference equation, and using the shift-theorem, we get**

$$Y\left(z\right)=X\left(z\right)-z^{-2}X\left(z\right)-1.3z^{-1}Y\left(z\right)-0.36z^{-2}Y(z)$$

**It can also be written as**

$$Y\left(z\right)+1.3z^{-1}Y\left(z\right)+0.36z^{-2}Y\left(z\right)=X\left(z\right)-z^{-2}X\left(z\right)$$

$$Y\left(z\right)\left(1+1.3z^{-1}+0.36z^{-2}\right)=X\left(z\right)\left(1-z^{-2}\right)$$

$$\frac{Y\left(z\right)}{X\left(z\right)}=\frac{\left(1-z^{-2}\right)}{\left(1+1.3z^{-1}+0.36z^{-2}\right)}$$

**The transfer function, is therefore, given by**

$$H\left(z\right)= \frac{Y\left(z\right)}{X\left(z\right)}=\frac{\left(1-z^{-2}\right)}{\left(1+1.3z^{-1}+0.36z^{-2}\right)}$$

**The denominator and numerator polynomials are**

$$A\left(z\right)=1+1.3z^{-1}+0.36z^{-2}$$

$$B\left(z\right)= 1-z^{-2}$$

**Example 4**

**A digital system is described by the following difference equation**

$$y\left(n\right)=x\left(n\right)-0.5x\left(n-1\right)+0.36x(n-2)$$

**Find the z-transfer function**$ H\left(z\right)$**, the denominator polynomial**$ A\left(z\right)$**, and the numerator polynomial**$ B\left(z\right)$**.**

**Solution**

**Taking the z-transform of both sides of the given difference equation, and using the shift-theorem, we get**

$$Y\left(z\right)=X\left(z\right)-0.5z^{-1}X\left(z\right)+0.36z^{-2}X(z)$$

**It can also be written as**

$$Y\left(z\right)=X\left(z\right)\left(1-0.5z^{-1}+0.36z^{-2}\right)$$

$$\frac{Y\left(z\right)}{X\left(z\right)}=\frac{1}{\left(1-0.5z^{-1}+0.36z^{-2}\right)}$$

**The transfer function, is therefore, given by**

$$H\left(z\right)= \frac{Y\left(z\right)}{X\left(z\right)}=\frac{1}{\left(1-0.5z^{-1}+0.36z^{-2}\right)}$$

**The denominator and numerator polynomials are**

$$A\left(z\right)=1-0.5z^{-1}+0.36z^{-2}$$

$$B\left(z\right)=1$$

**In some DSP applications, the given transfer function of a digital system can be converted into a difference equation for DSP implementation. The following example illustrates this procedure.**

**Example 5**

**Convert each of the following transfer functions into its difference equation**

1. $H\left(z\right)=\frac{z^{2} - 1}{z^{2} + 1.3 z + 0.36}$
2. $H\left(z\right)=\frac{z^{2}- 0.5 z + 0.36}{z^{2}}$

**Solution**

**Part (a): We first divide the numerator and denominator by**$ z^{2}$ **to obtain the transfer function whose numerator and the denominator polynomials have the negative powers of**$ z$**, it follows that**

$$H\left(z\right)=\frac{\left(z^{2} - 1\right)/z^{2}}{\left(z^{2} + 1.3 z + 0.36\right)/z^{2}}=\frac{\left(1 - z^{-2}\right)}{\left(1 + 1.3 z^{-1} + 0.36 z^{-2}\right)}$$

**According to the definition of the transfer function**

$$H\left(z\right)= \frac{Y\left(z\right)}{X\left(z\right)}$$

**Therefore, in this case,**

$$\frac{Y\left(z\right)}{X\left(z\right)}=\frac{\left(1 - z^{-2}\right)}{\left(1 + 1.3 z^{-1} + 0.36 z^{-2}\right)}$$

**Cross multiplication gives**

$$\left(1 + 1.3 z^{-1} + 0.36 z^{-2}\right)Y\left(z\right)=\left(1 - z^{-2}\right)X\left(z\right)$$

$$Y\left(z\right) + 1.3 z^{-1}Y\left(z\right) + 0.36 z^{-2}Y\left(z\right)=X\left(z\right) - z^{-2}X\left(z\right)$$

**Applying the inverse z-transform and applying the shift-theorem**

$$y\left(n\right) + 1.3 y\left(n-1\right) + 0.36 y\left(n-2\right)=x\left(n\right) - x\left(n-2\right)$$

**This equation can be re-arranged to give the required difference equation for the DSP system, as**

$$y\left(n\right)= x\left(n\right) - x\left(n-2\right)- 1.3 y\left(n-1\right)- 0.36 y\left(n-2\right)$$

**Part (b): In this case also, we first divide the numerator and denominator by**$ z^{2}$ **to obtain the transfer function whose numerator and the denominator polynomials have the negative powers of**$ z$**, it follows that**

$$H\left(z\right)=\frac{\left(z^{2}- 0.5 z + 0.36\right)/z^{2}}{\left(z^{2} \right)/z^{2}}=\frac{\left(1- 0.5 z^{-1} + 0.36 z^{-2}\right)}{1}$$

**According to the definition of the transfer function**

$$H\left(z\right)= \frac{Y\left(z\right)}{X\left(z\right)}$$

**Therefore, in this case,**

$$\frac{Y\left(z\right)}{X\left(z\right)}=\left(1- 0.5 z^{-1} + 0.36 z^{-2}\right)$$

**It can be written as**

$$Y\left(z\right)=\left(1- 0.5 z^{-1} + 0.36 z^{-2}\right)X\left(z\right)$$

$$Y\left(z\right) =X\left(z\right)- 0.5 z^{-1}X\left(z\right) + 0.36 z^{-2} X\left(z\right)$$

**Applying the inverse z-transform and applying the shift-theorem**

$$y\left(n\right) =x\left(n\right)- 0.5 x\left(n-1\right) + 0.36 x\left(n-2\right)$$

**This is the required difference equation for the DSP system.**

**Transfer Function in Pole-Zero Form**

**From Equation 4, we know that the transfer function for a digital filter can be written as**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}=\frac{b\_{0}+b\_{1}z^{-1}+…+b\_{M}z^{-M}}{1+a\_{1}z^{-1}+…+a\_{N}z^{-N}}=\frac{B\left(z\right)}{A\left(z\right)}$$

**The numerator**$ B\left(z\right)$ **and the denominator**$ A\left(z\right)$ **polynomials of the transfer function**$ H\left(z\right)$ **can be factorized. The transfer function**$ H\left(z\right)$ **can therefore, be written in its pole-zero form as**

|  |  |
| --- | --- |
| $$H\left(z\right)=\frac{b\_{0}\left(z-z\_{1}\right)\left(z-z\_{2}\right)… \left(z-z\_{M}\right)}{\left(z-p\_{1}\right)\left(z-p\_{2}\right)… \left(z-p\_{N}\right)}$$ | **(7)** |

**where the zeros**$z\_{i}$ **and poles**$ p\_{j}$ **can be found by solving (finding the roots of) the polynomial equations**

$$z^{M}+\left(\frac{b\_{1}}{b\_{0}}\right)z^{M-1}+…+\left(\frac{b\_{M}}{b\_{0}}\right)=0$$

$$a\_{1}z^{N}+a\_{2}z^{N-1}+…+b\_{N}=0$$

**This is explained with the following example.**

**Example 6**

**Given the following transfer function**

$$H\left(z\right)=\frac{\left(1-z^{-2}\right)}{\left(1+1.3z^{-1}+0.36z^{-2}\right)}$$

**Convert it into its pole-zero form.**

**Solution**

**We first multiply the numerator and denominator by**$ z^{2}$ **to obtain the transfer function whose numerator and the denominator polynomials have the positive powers of**$ z$**, as follows**

$$H\left(z\right)=\frac{\left(1-z^{-2}\right) z^{2}}{\left(1+1.3z^{-1}+0.36z^{-2}\right) z^{2}}=\frac{z^{2}-1}{z^{2}+1.3z+0.36}$$

**Putting the numerator polynomial equal to zero and then finding the roots, gives us the zeros of the transfer function,**

$$z^{2}-1=0$$

$$\left(z-1\right)(z+1)=0$$

**Therefore, we get**$ z\_{1}=1$ **and**$ z\_{2}=-1$ **as the roots.**

**Now, setting the denominator polynomial equal to zero and find the roots, gives us the poles of the transfer function,**

$$z^{2}+1.3z+0.36=0$$

$$z=\frac{-1.3\pm \sqrt{\left(1.3\right)^{2}-4\left(1\right)(0.36)}}{2(1)}=\frac{-1.3\pm \sqrt{1.69-1.44}}{2}=\frac{-1.3\pm \sqrt{0.25}}{2}=\frac{-1.3\pm 0.5}{2}=-0.4, -0.9$$

**Therefore, the poles are**$ p\_{1}=-0.4$ **and**$ p\_{21}=-0.9$**. The transfer function can now be written in the pole-zero form as**

$$H\left(z\right)=\frac{\left(z-1\right)\left(z+1\right)}{\left(z+0.4\right)\left(z+0.9\right)}$$

**Impusle Response, Step Response and System Response**

**Example 6.7**

**Given a transfer function depicting a DSP system**

$$H\left(z\right)=\frac{z+1}{z-0.5}$$

**Determine**

1. **The impulse response**$ h\left(n\right)$
2. **The step response**$ y\left(n\right)$**, and**
3. **The system response**$ y\left(n\right)$**, if the input is given as**$ x\left(n\right)=\left(0.25\right)^{n} u(n)$**.**

**Solution**

**Part (a): In this case**$ x\left(n\right)=δ\left(n\right)$**, thus**$ X\left(z\right)=1$**. As**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}$$

**Therefore, in this case, the z-transform of the output is equal to the transfer function:**

$$H\left(z\right)=Y\left(z\right)$$

**By taking the inverse z-transform of the transfer function we can find out the unit impulse response**$ h\left(n\right)$ **of the system. The transfer function can be written as**

$$\frac{H\left(z\right)}{z}=\frac{z+1}{z\left(z-0.5\right)}$$

**This can further be written in the form of partial fractions as**

$$\frac{H\left(z\right)}{z}=\frac{A}{z}+\frac{B}{\left(z-0.5\right)}$$

**where**

$$A=\left.\frac{z+1}{\left(z-0.5\right)}\right|\_{z=0}=\frac{0+1}{\left(0-0.5\right)}=-2$$

$$B=\left.\frac{z+1}{z}\right|\_{z=0.5}=\frac{0.5+1}{0.5}=3$$

**Thus we have**

$$\frac{H\left(z\right)}{z}=\frac{-2}{z}+\frac{3}{\left(z-0.5\right)}$$

**Or**

$$H\left(z\right)=-2+\frac{3z}{\left(z-0.5\right)}$$

**Taking inverse z-transform of both sides (and using Table 5.1), we get**

$$h\left(n\right)=-2 δ\left(n\right)+3\left(0.5\right)^{n} u\left(n\right)$$

**which is the required impulse response of the system.**

**Part (b): In this case**$ x\left(n\right)=u\left(n\right)$**, thus**$ X\left(z\right)=\frac{z}{z-1}$**. As**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}$$

**Therefore, in this case,**

$$Y\left(z\right)=H\left(z\right)X\left(z\right)=\frac{\left(z+1\right)}{\left(z-0.5\right)}\frac{z}{\left(z-1\right)}$$

**It can be written as**

$$\frac{Y\left(z\right)}{z}=\frac{z+1}{\left(z-0.5\right)\left(z-1\right)}$$

**This can further be written in the form of partial fractions as**

$$\frac{Y\left(z\right)}{z}=\frac{A}{\left(z-0.5\right)}+\frac{B}{\left(z-1\right)}$$

**where**

$$A=\left.\frac{z+1}{\left(z-1\right)}\right|\_{z=0.5}=\frac{0.5+1}{\left(0.5-1\right)}=-3$$

$$B=\left.\frac{z+1}{\left(z-0.5\right)}\right|\_{z=1}=\frac{1+1}{1-0.5}=4$$

**Thus we have**

$$\frac{Y\left(z\right)}{z}=\frac{-3}{\left(z-0.5\right)}+\frac{4}{\left(z-1\right)}$$

**Or**

$$Y\left(z\right)=\frac{-3z}{\left(z-0.5\right)}+\frac{4z}{\left(z-1\right)}$$

**Taking inverse z-transform of both sides (and using Table 5.1), we get**

$$y\left(n\right)=-3\left(0.5\right)^{n} u\left(n\right)+4 u\left(n\right)$$

**which is the required step response of the system.**

**Part (c): In this case**$ x\left(n\right)=\left(0.25\right)^{n} u\left(n\right)$**, thus from Table 5.1,**$ X\left(z\right)=\frac{z}{z-0.25}$**. As**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}$$

**Therefore, in this case,**

$$Y\left(z\right)=H\left(z\right)X\left(z\right)=\frac{\left(z+1\right)}{\left(z-0.5\right)}\frac{z}{\left(z-0.25\right)}$$

**It can be written as**

$$\frac{Y\left(z\right)}{z}=\frac{z+1}{\left(z-0.5\right)\left(z-0.25\right)}$$

**This can further be written in the form of partial fractions as**

$$\frac{Y\left(z\right)}{z}=\frac{A}{\left(z-0.5\right)}+\frac{B}{\left(z-0.25\right)}$$

**where**

$$A=\left.\frac{z+1}{\left(z-0.25\right)}\right|\_{z=0.5}=\frac{0.5+1}{\left(0.5-0.25\right)}=\frac{1.5}{0.25}=6$$

$$B=\left.\frac{z+1}{\left(z-0.5\right)}\right|\_{z=0.25}=\frac{0.25+1}{0.25-0.5}=\frac{1.25}{-0.25}=-5$$

**Thus we have**

$$\frac{Y\left(z\right)}{z}=\frac{6}{\left(z-0.5\right)}+\frac{-5}{\left(z-0.25\right)}$$

**Or**

$$Y\left(z\right)=\frac{6z}{\left(z-0.5\right)}-\frac{5z}{\left(z-0.25\right)}$$

**Taking inverse z-transform of both sides (and using Table 5.1), we get**

$$y\left(n\right)=6\left(0.5\right)^{n} u\left(n\right)+5 \left(0.25\right)^{n} u\left(n\right)$$

**which is the required system response.**

**Table 5.1 Table of z-transform pairs (for causal sequences)**

|  |  |  |  |
| --- | --- | --- | --- |
| Line No. | Signal$$x\left(n\right), n\geq 0$$ | z-Transform$$Z\left(x\left(n\right)\right)=X(z)$$ | Region of Convergence |
| 1 | $$x(n)$$ | $$\sum\_{n=0}^{\infty }x\left(n\right) z^{-n}$$ |  |
| 2 | $$δ(n)$$ | **1** | **Entire z-plane** |
| 3 | $$a u(n)$$ | $$\frac{az}{z-1}$$ | $$\left|z\right|>1$$ |
| 4 | $$n u(n)$$ | $$\frac{z}{\left(z-1\right)^{2}}$$ | $$\left|z\right|>1$$ |
| 5 | $$n^{2} u(n)$$ | $$\frac{z(z+1)}{\left(z-1\right)^{3}}$$ | $$\left|z\right|>1$$ |
| 6 | $$a^{n} u(n)$$ | $$\frac{z}{z-a}$$ | $$\left|z\right|>\left|a\right|$$ |
| 7 | $$e^{-na} u(n)$$ | $$\frac{z}{z-e^{-a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 8 | $$n a^{n} u(n)$$ | $$\frac{az}{\left(z-a\right)^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 9 | $$sin\left(an\right) u(n)$$ | $$\frac{z sin\left(a\right)}{z^{2}-2z cos\left(a\right)+1}$$ | $$\left|z\right|>\left|1\right|$$ |
| 10 | $$cos\left(an\right) u(n)$$ | $$\frac{z \left(z-cos\left(a\right)\right)}{z^{2}-2z cos\left(a\right)+1}$$ | $$\left|z\right|>\left|1\right|$$ |
| 11 | $$a^{n} sin\left(bn\right) u(n)$$ | $$\frac{ \left[a sin\left(b\right)\right] z}{z^{2}-\left[2a cos\left(b\right)\right]z +a^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 12 | $$a^{n} cos\left(bn\right) u(n)$$ | $$\frac{z \left[z-a cos\left(b\right)\right] }{z^{2}-\left[2a cos\left(b\right)\right]z +a^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 13 | $$e^{-an} sin\left(bn\right) u(n)$$ | $$\frac{ \left[e^{-a} sin\left(b\right)\right] z}{z^{2}-\left[2e^{-a} cos\left(b\right)\right]z +e^{-2a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 14 | $$e^{-an} cos\left(bn\right) u(n)$$ | $$\frac{z \left[z-e^{-a} cos\left(b\right)\right] }{z^{2}-\left[2e^{-a} cos\left(b\right)\right]z +e^{-2a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 15 | $2\left|A\right|\left|P\right|^{n} cos\left(nθ+ϕ\right) u\left(n\right)$ **where**$ P$ **and**$ A$ **are complex constants defined by**$P=\left|P\right|∠θ$**,** $A=\left|A\right|∠ϕ$ | $$\frac{Az}{z-P}+\frac{A^{\*}z}{z-P^{\*}}$$ |  |

**Shift Theorem:** $ Z\left(x\left(n-m\right)\right)=z^{-m} Z\left(x\left(n\right)\right)$