* 1. **Difference Equations and Impulse Responses**
		1. **Format of Difference Equation**

An LTI (linear-Time-Invariant), causal system can be represented by a difference equation having the following general form:

|  |  |
| --- | --- |
| $$y\left(n\right)+a\_{1}y\left(n-1\right)+a\_{2}y\left(n-2\right)+…+a\_{N}y\left(n-N\right)$$$$ =b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+b\_{1}x\left(n-2\right)+…+b\_{M}x(n-M)$$ | (3.12) |

where$ a\_{1}, a\_{2}, …, a\_{N} $and$ b\_{0}, b\_{1}, b\_{2}, …, b\_{M}$ are coefficients of the difference equation. Equation (3.12) can further be written as:

|  |  |
| --- | --- |
| $$y\left(n\right)=-a\_{1}y\left(n-1\right)-a\_{2}y\left(n-2\right)-…-a\_{N}y\left(n-N\right)$$$$ +b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+b\_{1}x\left(n-2\right)+…+b\_{M}x(n-M)$$ | (3.13) |

or

|  |  |
| --- | --- |
| $$y\left(n\right)=-\sum\_{i=1}^{N}a\_{i}y\left(n-i\right)+\sum\_{j=0}^{M}b\_{j}x\left(n-j\right)$$ | (3.14) |

It should be noted that$ y\left(n\right) $is the current output, which depends on the past output samples$ y\left(n-1\right), y\left(n-2\right), …, y(n-N)$, the current input sample$ x\left(n\right)$, and the past input samples$ x\left(n-1\right), x\left(n-2\right), …, x(n-M)$.

**Example 3.5**

Given the following difference equation

$$y\left(n\right)=0.25 y\left(n-1\right)+ x(n)$$

Identify the nonzero system coefficients.

**Solution**

The general format of the difference equation for a causal LTI system is given by Equation (3.13), i.e.,

|  |  |
| --- | --- |
| $$y\left(n\right)=-a\_{1}y\left(n-1\right)-a\_{2}y\left(n-2\right)-…-a\_{N}y\left(n-N\right)$$$$ +b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+b\_{1}x\left(n-2\right)+…+b\_{M}x(n-M)$$ | (3.13) |

Therefore, in this case (by comparing coefficients), we can write

$$a\_{1}=-0.25, b\_{0}=1$$

**Example 3.6**

Given a linear system described by the following difference equation

$$y\left(n\right)=x\left(n\right)+0.5 x\left(n-1\right)$$

Determine the nonzero system coefficients.

**Solution**

The general format of the difference equation for a causal LTI system is given by Equation (3.13), i.e.,

|  |  |
| --- | --- |
| $$y\left(n\right)=-a\_{1}y\left(n-1\right)-a\_{2}y\left(n-2\right)-…-a\_{N}y\left(n-N\right)$$$$ +b\_{0}x\left(n\right)+b\_{1}x\left(n-1\right)+b\_{1}x\left(n-2\right)+…+b\_{M}x(n-M)$$ | (3.13) |

Therefore, in this case (by comparing coefficients), we can write

$$b\_{0}=1$$

$$b\_{1}=0.5$$

* + 1. **System Representation Using Its Impulse Response**

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input$ δ\left(n\right)$ with zero initial condition, depicted in Figure 3.12.

Linear Time-Invariant System

$$δ\left(n\right)$$

$$h\left(n\right)$$

|  |  |
| --- | --- |
| **Figure 3.13** | **Unit-impulse response of the linear time-invariant system** |

With the obtained unit impulse response , we can represent the linear time-invariant system as in Figure 3.14.

$$h\left(n\right)$$

$$x\left(n\right)$$

$$y\left(n\right)$$

|  |  |
| --- | --- |
| **Figure 3.14** | **Representation of a linear time-invariant system using the impulse response** |

**Example 3.7**

Given the linear time-invariant system

$$y\left(n\right)=0.5x\left(n\right)+0.25 x\left(n-1\right) with an initial condition x\left(-1\right)=0,$$

1. Determine the unit-impulse response$ h\left(n\right)$.
2. Draw the system block diagram.
3. Write the output using the obtained impulse response.

**Solution**

1. According to Figure 3.13, let$ x\left(n\right)=δ\left(n\right)$ , then

$$h\left(n\right)=0.5 δ\left(n\right)+0.25 δ\left(n-1\right)$$

Thus, for this particular linear system, we have

$$h\left(n\right)=\left\{\begin{matrix}0.5&n=0\\0.25&n=1\\0.0&otherwise\end{matrix}\right.$$

1. The block diagram of the linear-time-invariant system is shown as

$$h\left(n\right)=0.5 δ\left(n\right)+0.25 δ\left(n-1\right)$$

$$x\left(n\right)$$

$$y\left(n\right)$$

1. The system output can be written as

$$y\left(n\right)=h(0)x\left(n\right)+h(1) x\left(n-1\right)$$

From this result, it could be noted that if the difference equation without the past output terms,$ y\left(n-1\right), y\left(n-2\right), …, y\left(N\right)$, that is the corresponding coefficients$ a\_{1}, a\_{2}, …, a\_{N}$, are zeros, the impulse response$ h(n)$ has a finite number of terms. We call this a ***finite impulse response (FIR) system***.

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as:

|  |  |
| --- | --- |
| $$y\left(n\right)=…+h(-1)x\left(n+1\right)+h(0)x\left(n\right)+h(1)x\left(n-1\right)+h(2)x\left(n-2\right)+…$$ | (3.15) |

Equation (3.15) is called the ***digital convolution sum***. We can verify Equation (3.15) by substituting the impulse sequence$ x\left(n\right)=δ\left(n\right)$ to get the impulse response

$$h\left(n\right)=…+h(-1)δ\left(n+1\right)+h(0)δ\left(n\right)+h(1)δ\left(n-1\right)+h(2)δ\left(n-2\right)+…$$

where$ …,h\left(-1\right), h\left(0\right), h\left(1\right),h\left(2\right),…$ are the amplitudes of the impulse response at the corresponding time indices.

**Example 3.8**

Given the difference equation

$$y\left(n\right)=0.25 y\left(n-1\right)+x\left(n\right) for n\geq 0 and y\left(-1\right)=0,$$

1. Determine the unit-impulse response$ h\left(n\right)$.
2. Draw the system block diagram.
3. Write the output using the obtained impulse response.
4. For a step input$ x\left(n\right)=u(n)$, verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3.15).

**Solution**

1. Let$ x\left(n\right)=δ\left(n\right)$ , then

$$h\left(n\right)=0.25 h\left(n-1\right)+δ\left(n\right)$$

To solve for $h\left(n\right)$, we evaluate

$$h\left(0\right)=0.25 h\left(0-1\right)+δ\left(0\right)=0.25×0+1=1$$

$$h\left(1\right)=0.25 h\left(1-1\right)+δ\left(1\right)=0.25×1+0=0.25$$

$$h\left(2\right)=0.25 h\left(2-1\right)+δ\left(2\right)=0.25×0.25+0=0.0625$$

With the calculated results, the impulse response can be predicted as

$$h\left(n\right)=δ\left(n\right)+0.25δ\left(n-1\right)+0.0625δ\left(n-2\right)+…=\left(0.25\right)^{n}u(n)$$

1. The system block diagram is shown below

$$h\left(n\right)= δ\left(n\right)+0.25 δ\left(n-1\right)+…$$

$$x\left(n\right)$$

$$y\left(n\right)$$

1. The output sequence is a sum of the infinite terms expressed as

$$y\left(n\right)=h\left(0\right)x\left(n\right)+h\left(1\right)x\left(n-1\right)+h\left(2\right)x\left(n-2\right)+…=x\left(n\right)+0.25x\left(n-1\right)+0.0625x\left(n-2\right)+…$$

1. From the difference equation and using the zero-initial condition, we have

$$y\left(n\right)=0.25 y\left(n-1\right)+x\left(n\right) for n\geq 0 and y\left(-1\right)=0,$$

Therefore, for $x\left(n\right)=u(n)$, we have

$$for n=0, y\left(0\right)=0.25y\left(-1\right)+x\left(0\right)=0.25×0+u\left(0\right)=1$$

$$for n=1, y\left(1\right)=0.25y\left(0\right)+x\left(1\right)=0.25×1+u\left(1\right)=1.25$$

$$for n=2, y\left(2\right)=0.25y\left(1\right)+x\left(2\right)=0.25×1.25+u\left(1\right)=1.3125$$

and so on.

We can obtain the same results using the convolution Equation (3.15), as

$$y\left(n\right)= x\left(n\right)+0.25x\left(n-1\right)+0.0625x\left(n-2\right)+…$$

Therefore, we have

$$for n=0, y\left(0\right)= x\left(0\right)+0.25x\left(0-1\right)+0.0625x\left(0-2\right)+…== u\left(0\right)+0.25u\left(-1\right)+0.0625u\left(-2\right)+…=1+0+0+…=1$$

$$for n=1, y\left(1\right)= x\left(1\right)+0.25x\left(1-1\right)+0.0625x\left(1-2\right)+…== u\left(1\right)+0.25u\left(0\right)+0.0625u\left(-1\right)+…=1+0.25+0+0+…=1.25$$

$$for n=2, y\left(2\right)= x\left(2\right)+0.25x\left(2-1\right)+0.0625x\left(2-2\right)+…== u\left(2\right)+0.25u\left(1\right)+0.0625u\left(0\right)+…=1+0.25+0.0625+0+…=1.3125$$

Therefore, we verify that a linear time-invariant system can be represented by the convolution sum using its impulse response and input sequence.

We also note that we have verified this only for the causal systems for the sake of simplicity. However, this principle works for both causal and non-causal LTI systems.

It should also be noted that, in Example 3.8, the impulse response$ h\left(n\right)$ contains an infinite number of terms in its duration due to the past output term $y\left(n-1\right)$. Such a system is called an ***infinite impulse response (IIR) system***.

* 1. **Bound-in-and-Bound-out (BIBO) Stability**

We are interested in designing and implementing stable linear systems. ***A stable system is one for which every bounded input produces a bounded output (BIBO)***.

To find the stability criterion, consider a linear time-invariant representation with all the inputs reaching the maximum value$ M$ for the worst case. In that case, Equation (3.15) becomes

|  |  |
| --- | --- |
| $$y\left(n\right)=M×(…+h\left(-1\right)+h\left(0\right)+h\left(1\right)+h\left(2\right)+…)$$ | (3.16) |

Using the absolute values of the impulse response lead to

|  |  |
| --- | --- |
| $$y\left(n\right)<M×(…+\left|h\left(-1\right)\right|+\left|h\left(0\right)\right|+\left|h\left(1\right)\right|+\left|h\left(2\right)\right|+…)$$ | (3.17) |

If the absolute sum in Equation (3.17) is a finite number, the product of the absolute sum and the maximum input value is therefore a finite number. Hence, we have a bounded input and a bounded output.

In terms of the impulse response, a linear system is stable if the sum of its absolute impulse response coefficients is a finite number. We can apply Equation (3.18) to determine whether a linear time-invariant system is stable or not stable; that is,

|  |  |
| --- | --- |
| $$S=\sum\_{k=-\infty }^{+\infty }\left|h\left(k\right)\right|=…+\left|h\left(-1\right)\right|+\left|h\left(0\right)\right|+\left|h\left(1\right)\right|+\left|h\left(2\right)\right|+…<\infty $$ | (3.18) |

Figure 3.17 illustrates a linear stable system, where the impulse response decreases to zero in finite amount of time, so that the summation of its absolute impulse response coefficients is guaranteed to be finite.

**Linear stable system**

$$n$$

$$δ\left(n\right)$$

$$n$$

$$h\left(n\right)$$

|  |  |
| --- | --- |
| **Figure 3.17** | **Illustration of stability of the digital linear system** |

**Example 3.9**

Given the linear system of Example 3.8,

$$y\left(n\right)=0.25 y\left(n-1\right)+x\left(n\right) for n\geq 0 and y\left(-1\right)=0,$$

which is described by the unit-impulse response,

$$h\left(n\right)=\left(0.25\right)^{n}u(n)$$

1. Determine whether this system is stable or not.

**Solution**

1. Using Equation (3.18), we have

$$S=\sum\_{k=-\infty }^{+\infty }\left|h\left(k\right)\right|=\sum\_{k=-\infty }^{+\infty }\left|\left(0.25\right)^{k}u(k)\right|$$

Applying the definition of the unit-step function$ u\left(k\right)=1 for k\geq 0$, we have

$$S=\sum\_{k=0}^{+\infty }\left(0.25\right)^{k}=1+0.25+0.25^{2}+…$$

Using the formula for a sum of the geometric series, i.e.,

$$\sum\_{k=0}^{+\infty }a^{k}=\frac{1}{1-a}$$

where$ a=0.25<1$, we compute

$$S=1+0.25+0.25^{2}+…=\frac{1}{1-0.25}=\frac{1}{0.75}=\frac{4}{3}<\infty $$

As the summation is a finite number, the linear system is stable.

* 1. **Digital Convolution**

Digital convolution plays an important role in digital filtering. Given a linear time-invariant system, we can find out the output sequence$ y(n)$ in terms of any input sequence$ s(n)$ using the convolution sum, i.e.,

|  |  |
| --- | --- |
| $$y\left(n\right)=\sum\_{k=-\infty }^{+\infty }h\left(k\right)x(n-k)=…+h\left(-1\right)x\left(n+1\right)+h\left(0\right)x\left(n\right)+h\left(1\right)x\left(n-1\right)+h\left(2\right)x\left(n-2\right)+…$$ | (3.19) |

The sequences and in Equation (3.19) are interchangeable. Hence, we can also write

|  |  |
| --- | --- |
| $$y\left(n\right)=\sum\_{k=-\infty }^{+\infty }x\left(k\right)h(n-k)=…+x\left(-1\right)h\left(n+1\right)+x\left(0\right)h\left(n\right)+x\left(1\right)h\left(n-1\right)+x\left(2\right)h\left(n-2\right)+…$$ | (3.20) |

Using conventional notations, it can be written as

|  |  |
| --- | --- |
| $$y\left(n\right)=h\left(n\right)\*x(n)$$ | (3.21) |

It could be noted that for a causal system, which implies its impulse response

$$h\left(n\right)=0 for n<0$$

The lower limit of the convolution sum begins at$ 0$ instead of$ \infty $, that is

|  |  |
| --- | --- |
| $$y\left(n\right)=\sum\_{k=0}^{+\infty }h\left(k\right)x(n-k)=\sum\_{k=0}^{+\infty }x\left(k\right)h(n-k)$$ | (3.22) |

We shall focus on calculating the convolution sum based on Equation (3.20). The first few outputs are give as follows

$$y\left(0\right)=\sum\_{k=-\infty }^{+\infty }x\left(k\right)h(-k)=…+x\left(-1\right)h\left(1\right)+x\left(0\right)h\left(0\right)+x\left(1\right)h\left(-1\right)+x\left(2\right)h\left(-2\right)+…$$

$$y\left(1\right)=\sum\_{k=-\infty }^{+\infty }x\left(k\right)h(1-k)=…+x\left(-1\right)h\left(2\right)+x\left(0\right)h\left(1\right)+x\left(1\right)h\left(0\right)+x\left(2\right)h\left(-1\right)+…$$

$$y\left(1\right)=\sum\_{k=-\infty }^{+\infty }x\left(k\right)h(2-k)=…+x\left(-1\right)h\left(3\right)+x\left(0\right)h\left(2\right)+x\left(1\right)h\left(1\right)+x\left(2\right)h\left(0\right)+…$$

We note that the convolution sum requires the sequence$ h\left(n\right)$ to be reversed and shifted.

The digital convolution can be evaluated through

1. The graphical method
2. The formula method
3. The table method

We will need to use the reversed and shifted sequence. The reversed sequence is defined as follows: If$ h\left(n\right)$ is the given sequence, then$ h\left(-n\right)$ is the reversed sequence. The reversed sequence is a mirror image of the original sequence, assuming the vertical axis as the mirror.

**Example 3.10**

Given a sequence

$$h\left(k\right)=\left\{\begin{matrix}3,&k=0,1\\1,&k=2,3\\0,&elsewhere\end{matrix}\right.$$

where$ k$ is the time index or sample number,

1. Sketch the sequence $h(k)$ and reversed sequence$ h(-k)$.
2. Sketch the shifted sequence$ h(k+3)$ and$ h(-k-2)$.

**Solution**

$$k$$

$$h\left(k\right)$$

1

2

3

**-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6**

$$k$$

$$h\left(-k\right)$$

1

2

3

**-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6**

1. Shifted sequences are shown in the following

**-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6**

$$k$$

$$h\left(k+3\right)$$

1

2

3

**-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6**

$$k$$

$$h\left(-k-2\right)$$

1

2

3