CEN352
Digital Signal Processing

By

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Lecture No. 5

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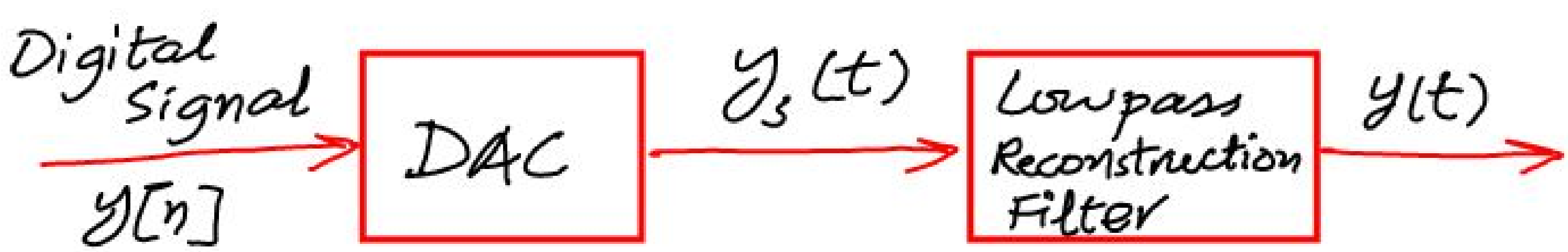
September, 2012 - January, 2013

Signal Reconstruction

This is a two step process:

(1) Digitally processed data $y[n]$ are converted to the ideal impulse train $y_s(t)$.

(2) The analog reconstruction filter is applied to the ideally recovered sampled signal $y_s(t)$.



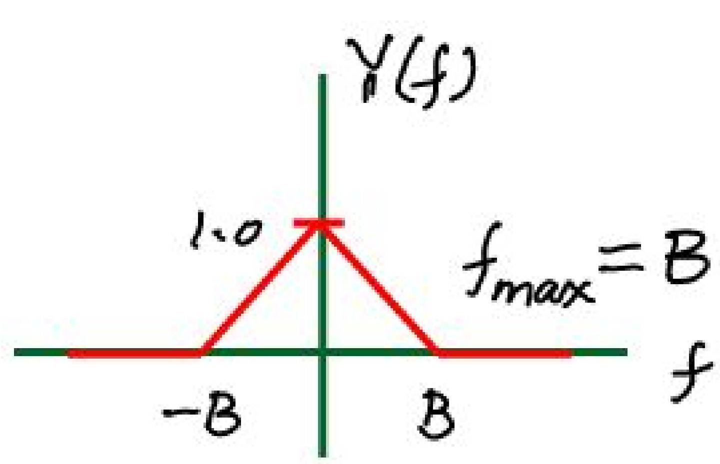
We consider a situation in which $y[n] = x[n]$, that is there is no DSP, and that the reconstructed sampled signal and the input sampled signals are the same:

$$y_s(t) = x_s(t).$$

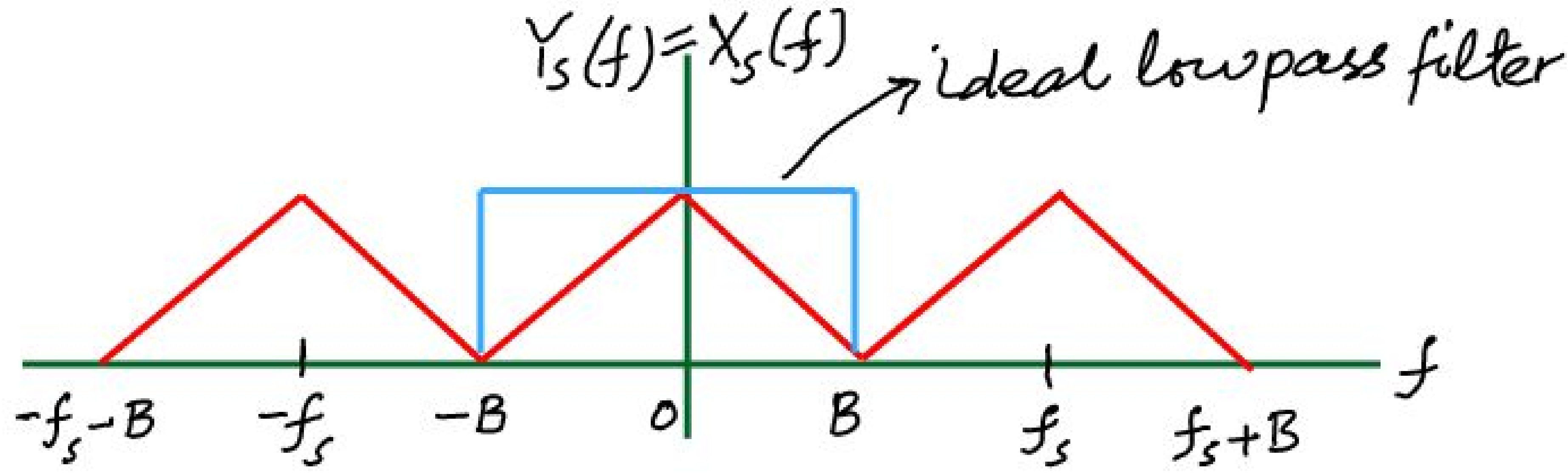
Hence

$$Y(f) = X(f)$$

Same frequency content

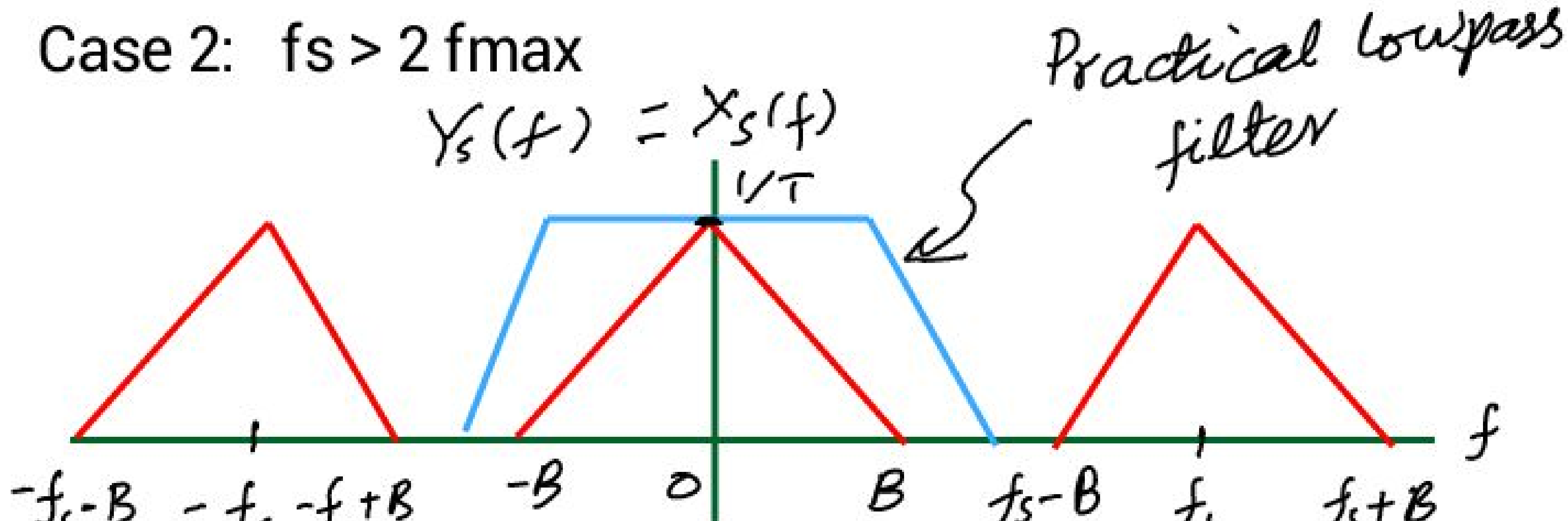


Case 1: fs = 2 fmax



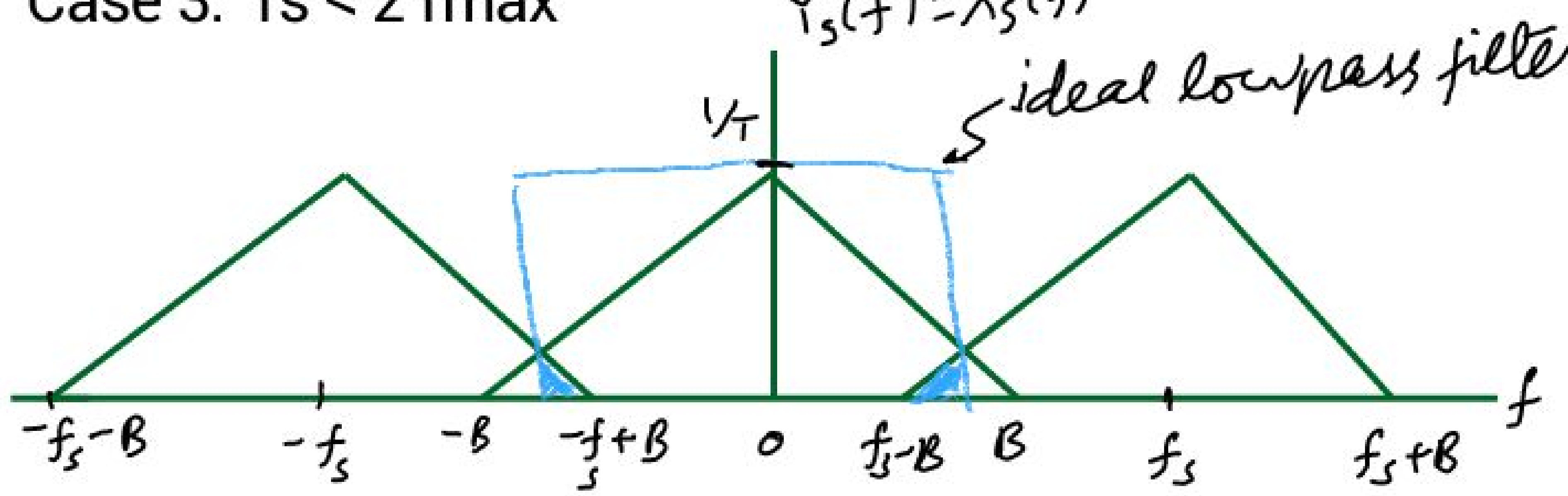
In this case the Nyquist frequency is equal to the maximum frequency of the analog signal. An ideal lowpass reconstruction filter is required in this case to recover the analog spectrum. This is impractical in realistic situations.

Case 2: fs > 2 fmax



In this case, there is a separation between the highest-frequency edge of the baseband spectrum and the lower edge of the first replica (image). Therefore, a lowpass filter can be designed to reject all images & achieve original spectrum.

Case 3: $f_s < 2 f_{max}$



This case violates the condition of the Shannon sampling theorem. There are spectral overlap-ings between the original baseband spectrum and the adjoining replicas. Even when we apply a lowpass filter, there is still some foldover frequency components from the adjoining replicas (images). The recovered analog signal may contain the aliasing noise spectrum. In time domain, the recovered signal may contain distortions due to the frequencies originating from the adjoining aliases.

It should be noted that if an analog signal with a frequency 'f' is undersampled, the aliasing frequency component 'f_{alias}' in the baseband is given by

$$f_{alias} = f_s - f$$

Example 1.4.2 (Proakis's Book)

Consider the analog signal

$$x_a(t) = 3\cos 100\pi t$$

(a) Determine the minimum sampling rate required to avoid aliasing.

(b) Suppose that the signal is sampled at the rate $f_s = 100 Hz$. What is the discrete-time signal obtained after sampling?

(c) Suppose that the signal is sampled at the rate $f_s = 75 Hz$. What is the discrete-time signal obtained after sampling?

(d) What is the required frequency $0 < F < f_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

Solution

(a) Given analog signal can be written as

$$x_a = 3\cos 2\pi .\, 50t \qquad = 3\cos 2\pi ft$$

The frequency of this signal is $f = 50 Hz$. The minimum sampling rate required to avoid aliasing (according to the samping theorem) is

$$f_s = 2 \times 50 = 100\ Hz$$

Answer

(b) $$x_s(t) = x_a(nT) = x\left(\frac{n}{f_s}\right)$$

Here $f_s = 200\text{Hz}$

$$\Rightarrow x[n] = x_s(t) = 3\cos\left(100\pi \cdot \frac{n}{200}\right) = 3\cos\left(\frac{\pi}{2}n\right)$$

Answer

(c) In this case, the signal is sampled at $f_s = 75\text{Hz}$. The discrete time signal will be

$$x[n] = x_s(t) = 3\cos\left(100\pi \cdot \frac{n}{75}\right) = 3\cos\left(\frac{4\pi}{3}n\right)$$

$f = 1/3$

It can be written as

$$x[n] = 3\cos\left(2\pi - \frac{2\pi}{3}\right)n = 3\cos\left(-\frac{2\pi}{3}n\right) = 3\cos\left(\frac{2\pi}{3}n\right)$$

Answer

(d) Here the sampling rate is $f_s = 75\text{Hz}$. We defined the normalized / relative frequency as:

$$f = F/f_s \quad \Rightarrow \quad F = f \cdot f_s = \frac{1}{3} \cdot 75 = 25\ \text{Hz}$$

where $f = \frac{1}{3}$ is the frequency of the discrete time signal of part (c). Therefore the signal

$$y_a(t) = 3\cos(2\pi F t) = 3\cos 50\pi t$$

sampled at $F_s = 75$ samples per second yields the identical samples as obtained from $x_a = 3\cos 100\pi t$. Hence $F = 50\text{Hz}$ is an alias of $F = 25\text{Hz}$ sampled at the rate of $f_s = 75\text{ Hz}$ (for sinusoidal signals).

Example 1.4.3 (Proakis's Book)

Consider the analog signal

$$x_a(t) = 3\cos 50\pi t + 100 \sin 300\pi t - \cos 100\pi t$$

What is the Nyquist rate/limit for this signal.

Solution

The frequencies present in the signal are:

$$F_1 = 25\,Hz, \quad F_2 = 150\,Hz, \quad F_3 = 50\,Hz$$

Thus $f_{max} = 150\,Hz$. According to the sampling theorem, the Nyquist rate/limit is:

$$F_N = 2f_{max} = 300\,Hz$$

Note: If we set the sampling rate equal to the Nyquist limit i.e. $f_s = 300\,Hz$, then the signal component $100 \sin 300\pi t$ results in

$$100 \sin 300\pi \frac{n}{300} = 100 \sin(\pi n)$$

As $\sin(\pi n) = 0$ for $n = 0, \pm 1, \pm 2, \cdots$, therefore, we are sampling this component of the signal at its zero-crossing points. This will lead to no contribution to the overall sampled signal from this component.

This situation can be avoided by taking a sampling rate greater than the Nyquist rate.

Example 2.1 (Li Tan's Book)

Suppose that an analog signal is given as:

$$x(t) = 5\cos(2\pi .1000t) \quad \text{for } t \geq 0$$

and is sampled at a rate of 8,000 Hz.

a. Sketch the spectrum of the original signal.

b. Sketch the spectrum of the sampled signal from 0 to 20 kHz.

Solution

(a) Given signal is

$$x(t) = 5\cos(2\pi .1000t) \quad \text{for } t \geq 0$$

Using Euler's formula, it can be written as:

$$x(t) = 5\left(\frac{e^{j2\pi .1000t} + e^{-j2\pi 1000t}}{2}\right)$$

$$\Rightarrow x(t) = 2.5\,e^{j2\pi 1000t} + 2.5\,e^{-j2\pi 1000t}$$

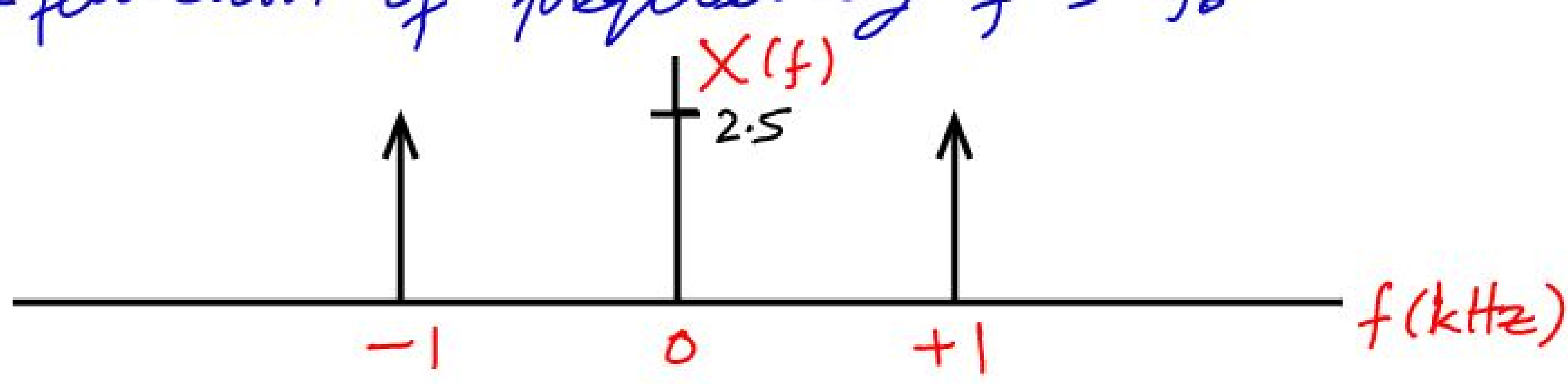
Comparing with the Fourier serie

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t}$$

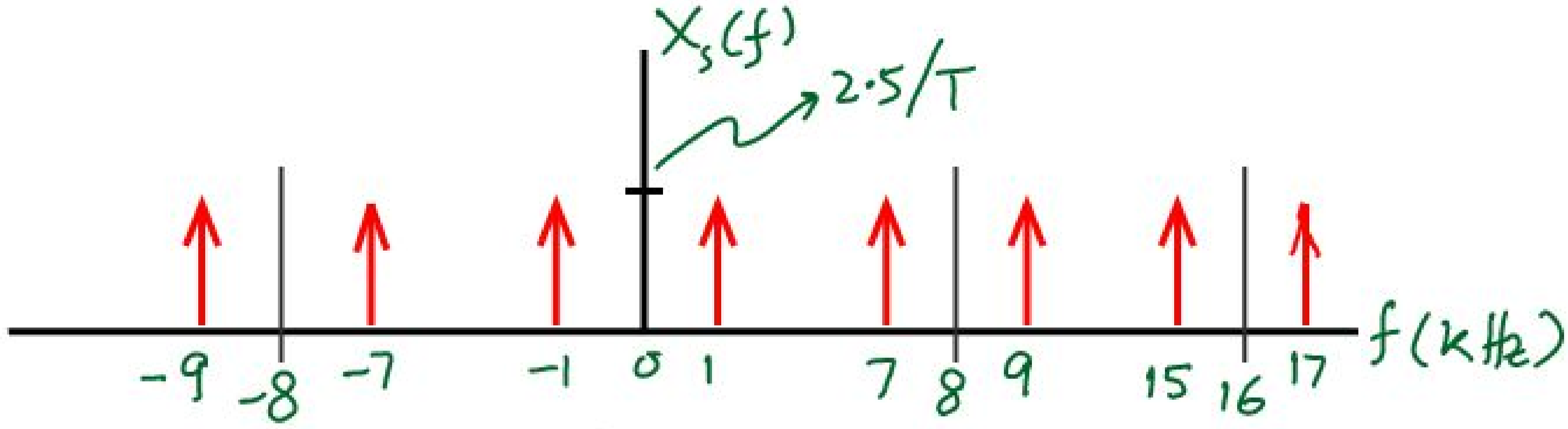
The Fourier coefficients are:

$$a_1 = 2.5 \quad \text{and} \quad a_{-1} = 2.5$$

The spectrum of the original signal is therefore, the plot of the magnitudes of these cofficients as a function of frequency $f = kf_0$.



(b) After sampling at a rate of $f_s = 8000\,\text{Hz}$, the sampled signal spectrum and its replicas are centered at frequencies $f = \pm k f_s$.



It should be noticed that

(1) The spectrum of the sampled signal contains the images of the spectrum of the original signal.

(2) These images repeat at multiples of the sample frequency f_s (8 kHz, in this case).

To reconstruct the original signal spectrum all of the replicas (images) must be removed.

If a sampling rate of $f_s < 2000\,\text{Hz}$ (= 2 × the maximum frequency of the original signal), then the replicas adjoining the original signal spectrum, overlap. The removal of such replicas will become quite difficult in such a case.

Example 2.2 (Li Tan's Book)

Assuming that an analog signal is

$$x(t) = 5\cos(2\pi \cdot 2000t) + 3\cos(2\pi \cdot 3000t), \quad t \geq 0$$

and is sampled at a rate of 8000 Hz,

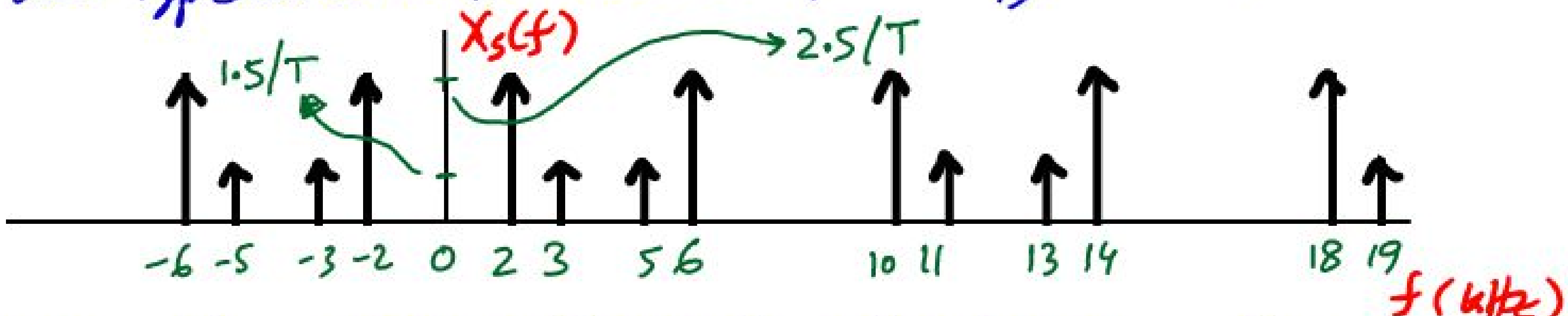
(a) Sketch the spectrum of the sampled signal up to 20 kHz.

(b) Sketch the recovered analog signal spectrum if an ideal lowpass filter with cutoff frequency of 4 KHz is used to filter the sampled signal ($y[n] = x[n]$ in this case) to reconstruct the original signal.

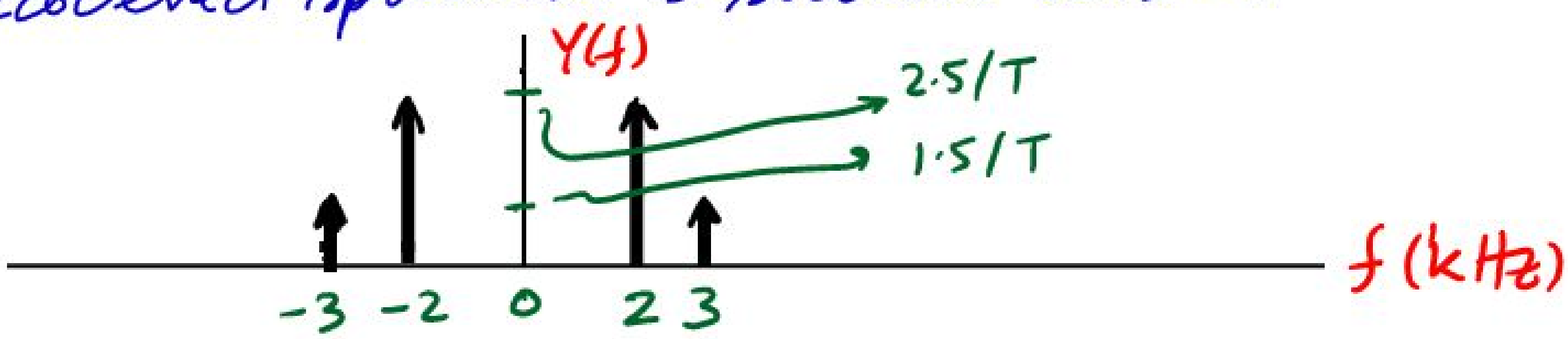
Solution (a) Using Euler's formula

$$x(t) = \frac{3}{2}e^{-j2\pi 3000t} + \frac{5}{2}e^{-j2\pi 2000t} + \frac{5}{2}e^{j2\pi 2000t} + \frac{3}{2}e^{j2\pi 3000t}$$

The spectrum for the sampled signal is:



(b) Based on the spectrum in (a), the sampling theorem is satisfied. We can recover the original spectrum by using a lowpass construction filter. The filter has frequency cut off at -4 Hz and 4 Hz. The recovered spectrum is shown below:



Example 2.3 (Li Tan's Book)

Given an analog signal

$x(t) = 5\cos(2\pi \cdot 2000t) + \cos(2\pi \cdot 5000t)$ for $t \geq 0$

which is sampled at a rate of 8000 Hz.

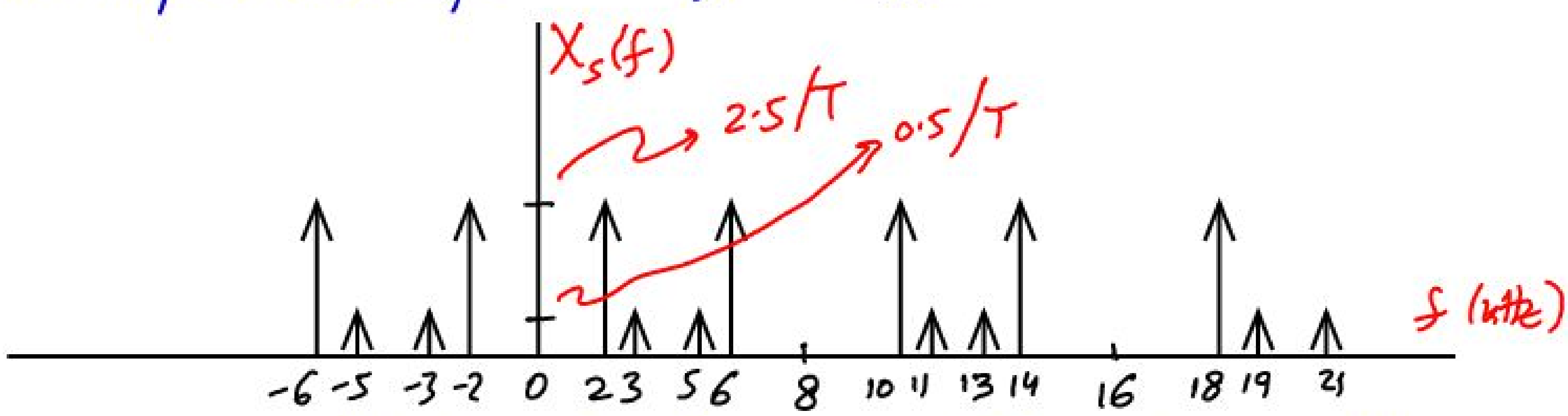
(a) Sketch the spectrum of the sampled signal upto 20kHz.

(b) Sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4kHz is used to recover the original signal, ($y[n] = x[n]$ in this case).

Solution (a) The original signal can be written as

$$x(t) = \frac{1}{2}\, e_1^{-j2\pi5000.t} + \frac{5}{2}\, e^{-j2\pi2000.t} + \frac{5}{2}\, e^{j2\pi2000.t} + \frac{1}{2}\, e^{j2\pi5000.t}$$

The spectrum of the signal ($f_s = 8\,kHz$) is:

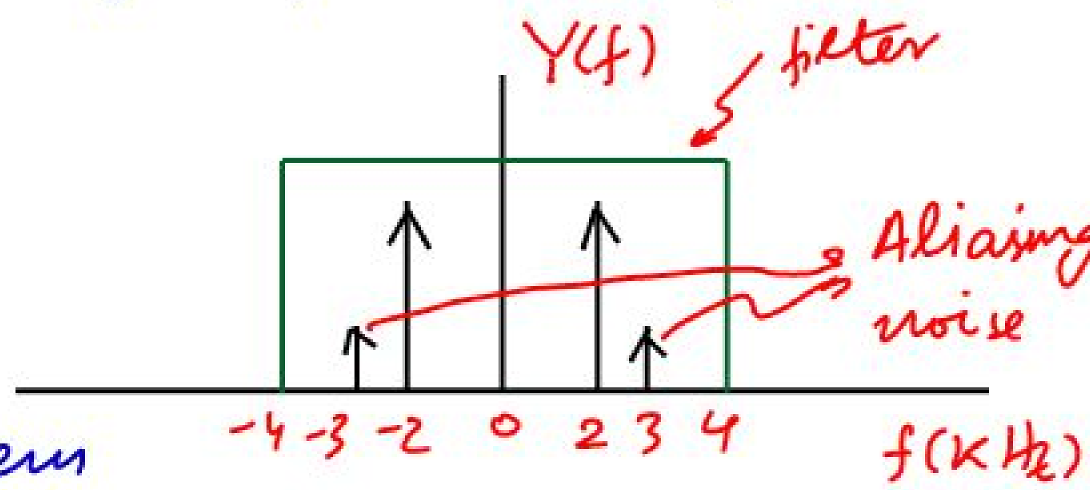


(b) After applying the lowpass filter, the spectrum obtained is:

In this case $f_s = 8\,kHz$,

while $f_{max} = 5\,kHz$,

so the sampling theorem



is not satisfied ($8 < 2\times5 = 10$). Therefore, an aliasing noise has appeared in the reconstructed signal spectrum.