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Lecture (7)+

Normal & Log-Normal distributions

Theorem

Suppose $X \sim N(\mu, \sigma^2)$ and let $Z = \frac{X-\mu}{\sigma}$

then $Z \sim N(0,1)$

proof

let F_Z denote the cdF of Z

then, $F_Z(z) = P\left(\frac{X-\mu}{\sigma} \leq z\right)$

$= P(X \leq \sigma z + \mu)$

$= F(\sigma z + \mu)$

$$\therefore F_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\sigma z+\mu} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt$$

Use $y = \frac{t-\mu}{\sigma}$, where $y: -\infty \to z$, and $dt = \sigma\, dy$

$$\Rightarrow F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\, dy = \Phi(z)$$

$\therefore Z \sim N(0,1)$ #

Note that Any normal distn can be transformed to the standard normal distn by subtracting the mean μ from the r.v. and dividing by standard deviation σ.

Corollary

Let $X \sim N(\mu, \sigma^2)$. For $a, b, x \in \mathbb{R}$,

(i) $P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

(ii) $P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

proof If $X \sim N(\mu, \sigma^2)$, then

2:

(i)

$$\because P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$\therefore P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where $Z = \frac{X-\mu}{\sigma}$ is the standard normal random Variable, and $\Phi(z)$ is the distribution function (CdF).

(ii)

$$\because P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

$$\therefore P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

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Corollary $\Phi(-x) = 1 - \Phi(x), \ \forall x \in \mathbb{R}$

Example The IQ (intelligence quotient) test of a randomly selected individual is often supposed to follow a normal distribution with mean 100 and standard deviation 15. Find the probability that an individual has an IQ (a) above 140 and (b) between 120 and 130 and (c) find a value x such that 99% of the population has IQ at least x.

Ans:

We have $X \sim N(100, 15^2)$ and get for (a)

$$\because P(X > 140) = 1 - P(X \le 140)$$

$$= 1 - \Phi\left(\frac{140-100}{15}\right)$$

$$\therefore P(X > 140) = 1 - \Phi(2.67) \approx 0.004$$

$1 - 0.996 = 0.004$

(by using Table A.1)

For (b), we get

$$P(120 \le X \le 130) = \Phi\left(\frac{130-100}{15}\right) - \Phi\left(\frac{120-100}{15}\right)$$

$$= \Phi(2) - \Phi(1.33) \quad \text{(by using Table A.1)}$$

$$= 0.977 - 0.908 = 0.069 \simeq 0.07$$

For the last part, we need to find x such that $P(X > x) = 0.99$

$$P(X > x) = 1 - \Phi\left(\frac{x-100}{15}\right)$$

$$\therefore P(X > x) = \Phi\left(\frac{100-x}{15}\right) = 0.99, \quad \text{where } \Phi(-x) = 1 - \Phi(x)$$

and Table A.2 gives

$$\frac{100-x}{15} = 2.33$$

which gives $x \simeq 65$.

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The lognormal distribution

Def^n: Let $X \sim N(\mu, \sigma^2)$ and let $Y = e^X$, then Y is said to have a lognormal distribution with parameters μ and σ^2 (i.e $\ln Y = X$ is normally distributed)

Note that

① The parameters μ and σ^2 are the mean and variance of the underlying normal r.v. X, not of the lognormal r.v. Y.

② For a lognormal r.v. Y with parameters μ and σ^2, then Y has the pdf

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma y} e^{-(\ln y - \mu)^2/2\sigma^2}, \quad y > 0, \ \sigma > 0$$

see p.28 Textbook

where "$\ln y$" stands for the natural logarithm of y.

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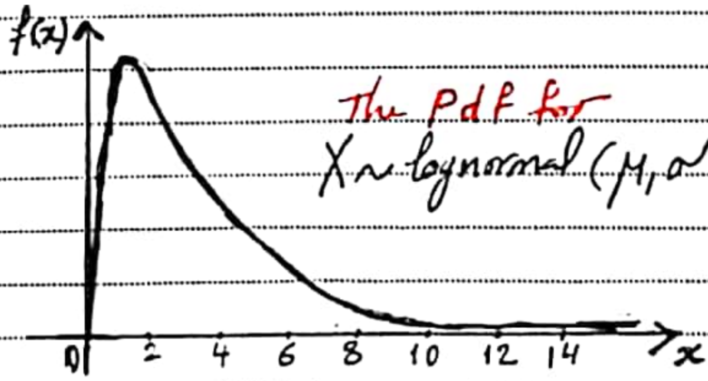
Conversely if $X \sim \text{lognormal}(\mu, \sigma^2)$, then X has the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-(\ln x - \mu)^2/2\sigma^2}$$

and $Y \sim \text{Normal}(\mu, \sigma^2)$. (i.e $\ln X = Y$ is normally distributed)



③ By computing the usual integrals, we can find the mean and variance of the lognormal distribution. For $X \sim \text{lognormal}(\mu, \sigma^2)$

$$E(X) = e^{\mu + \sigma^2/2} \quad \text{and} \quad Var(X) = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right).$$

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Example: Suppose that the price X of a particular stock at closing has a lognormal distribution with $E[X]=20$ dollars and $Var[X]=4$. What is the probability that the price exceeds \$22?

Ans:

First, we find μ and σ^2 By using the following

$$E[X]=e^{\mu+\sigma^2/2}=20 \quad (1)$$

and

$$Var[X]=e^{2\mu+\sigma^2}(e^{\sigma^2}-1)=4 \quad (2)$$

$$(1) \Rightarrow \mu+\sigma^2/2=\ln 20 \quad \therefore 2\mu+\sigma^2=2\ln 20=\ln 400 \quad (3)$$

Substitute (3) in (2), we get

$$e^{\ln 400}(e^{\sigma^2}-1)=4$$

$$e^{\ln a}=a,\ a>0$$

$$\therefore 400(e^{\sigma^2}-1)=4$$

$$e^{\sigma^2}-1=0.01$$

$$e^{\sigma^2}=1.01$$

$$\therefore \sigma^2=\ln(1.01)\approx 0.01$$

and $\mu=\ln 20-\dfrac{\ln(1.01)}{2}\approx 3$

Finally, since $\ln X$ is normally distributed, we can use the Standard Normal distn Table (Table A.1), to get

$$P(X>22)=1-P(X\leq 22)$$

$$=1-P(\ln X\leq \ln 22)$$

$$=1-\Phi\left(\frac{\ln 22-3}{\sqrt{0.01}}\right)$$

$$=1-\Phi(0.91)$$

$$\therefore P(X>22)=1-0.819\approx 0.18$$

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Table A.1 Values of the cdf $\Phi(x)$ of the standard normal distribution [e.g., $\Phi(1.41) = 0.921$]

x	0	1	2	3	4	5	6	7	8	9
0.0	.500	.504	.508	.512	.516	.520	.524	.528	.532	.536
0.1	.540	.544	.548	.552	.556	.560	.564	.568	.571	.575
0.2	.579	.583	.587	.591	.595	.599	.603	.606	.610	.614
0.3	.618	.622	.626	.629	.633	.637	.641	.644	.648	.652
0.4	.655	.659	.663	.666	.670	.674	.677	.681	.684	.688
0.5	.692	.695	.698	.702	.705	.709	.712	.716	.719	.722
0.6	.726	.729	.732	.736	.739	.742	.745	.749	.752	.755
0.7	.758	.761	.764	.767	.770	.773	.776	.779	.782	.785
0.8	.788	.791	.794	.797	.800	.802	.805	.808	.811	.813
0.9	.816	.819	.821	.824	.826	.829	.832	.834	.836	.839
1.0	.841	.844	.846	.848	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.892	.894	.896	.898	.900	.902
1.3	.903	.905	.907	.908	.910	.912	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.926	.928	.929	.931	.932
1.5	.933	.934	.936	.937	.938	.939	.941	.942	.943	.944
1.6	.945	.946	.947	.948	.950	.951	.952	.952	.9545	.954
1.7	.955	.956	.957	.958	.959	.960	.961	.962	.962	.963
1.8	.964	.965	.966	.966	.967	.968	.969	.969	.970	.971
1.9	.971	.972	.973	.973	.974	.974	.975	.976	.976	.977
2.0	.977	.978	.978	.979	.979	.980	.980	.981	.981	.982
2.1	.982	.983	.983	.983	.984	.984	.985	.985	.985	.986
2.2	.986	.986	.987	.987	.988	.988	.988	.988	.989	.989
2.3	.989	.990	.990	.990	.990	.991	.991	.991	.991	.992
2.4	.992	.992	.992	.992	.993	.993	.993	.993	.993	.994
2.5	.994	.994	.994	.994	.995	.995	.995	.995	.995	.995
2.6	.995	.996	.996	.996	.996	.996	.996	.996	.996	.996
2.7	.996	.997	.997	.997	.997	.997	.997	.997	.997	.997
2.8	.997	.998	.998	.998	.998	.998	.998	.998	.998	.998
2.9	.998	.998	.998	.998	.998	.998	.998	.998	.999	.999

Table A.2 Values of $\Phi(x)$ commonly used in confidence intervals and tests, and the corresponding x values

$\Phi(x)$	0.90	0.95	0.975	0.99	0.995
x	1.28	1.64	1.96	2.33	2.58