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Lecture (6): properties of Matrices

* Some props of Matrices

Let A, B and C are three matrices, their sizes are chosen such that the indicated operations can be performed, then

MX → Matrix

1. $A+B=B+A$ Commutative prop. for MX addition
2. $A+(B+C)=(A+B)+C$ Associative " " "
3. $A(BC)=(AB)C$ " " for MX multiplication
4. $A(B+C)=AB+AC$ left distribution prop.
5. $(B+C)A=BA+CA$ right "

Example

If $A=\begin{bmatrix}1&2\\3&4\\0&1\end{bmatrix}$, $B=\begin{bmatrix}4&3\\2&1\end{bmatrix}$ $C=\begin{bmatrix}1&0\\2&3\end{bmatrix}$

verify from prop (3) and (1)

then $AB=\begin{bmatrix}1&2\\3&4\\0&1\end{bmatrix}\begin{bmatrix}4&3\\2&1\end{bmatrix}=\begin{bmatrix}8&5\\20&13\\2&1\end{bmatrix}$

$(AB)C=\begin{bmatrix}8&5\\20&13\\2&1\end{bmatrix}_{3\times 2}\begin{bmatrix}1&0\\2&3\end{bmatrix}_{2\times 2}=\begin{bmatrix}18&15\\46&39\\4&3\end{bmatrix} \rightarrow (1)$

$BC=\begin{bmatrix}4&3\\2&1\end{bmatrix}\begin{bmatrix}1&0\\2&3\end{bmatrix}=\begin{bmatrix}10&9\\4&3\end{bmatrix}$

$A(BC)=\begin{bmatrix}1&2\\3&4\\0&1\end{bmatrix}_{3\times 2}\begin{bmatrix}10&9\\4&3\end{bmatrix}_{2\times 2}=\begin{bmatrix}18&15\\46&39\\4&3\end{bmatrix} \rightarrow (2)$

Also, note that: (1), (2) ⇒ $(AB)C=A(BC)$ prop (3)

$B+C=C+B=\begin{bmatrix}5&3\\4&4\end{bmatrix}$ prop (1)

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Def$\underline{n}$: Zero Matrix

It's a matrix whose entries are all zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [0]$$

we denote it by 0

If A and 0 are matrices with the same size, then

$$\boxed{A + 0 = 0 + A = A}$$

Def$\underline{n}$: Identity Matrix

It's a square matrix with 1's on the main diagonal and zeros elsewhere

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If A is any $m \times n$ matrix, then

$$A I_n = A \quad \text{and} \quad I_m A = A$$

Example

If $A = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 5 & 3 \end{bmatrix}_{2\times 3}$ then

$$A I_3 = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 7 \\ -1 & 5 & 3 \end{bmatrix} = A$$

Also, $I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 7 \\ -1 & 5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 7 \\ -1 & 5 & 3 \end{bmatrix} = A$$

Defn: Inverse of a Matrix

If A be a square matrix, then A is invertible (nonsingular) if their exists a matrix B of the same size of A such that $\boxed{AB = BA = I}$.

- In this case B is called the inverse of A, and we can denote it by A^{-1}!

theorem: Inverse of 2x2 Mx

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff

$\det(A) = ad - bc$ where $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Not that $\det(A)$ means the determinant (محدد) of A (المصفوفة)

Example ①

Determine whether $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ is invertible.

If so, find its inverse.

$$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\det A = 6(2) - 1(5) = 7$$

$\because \det A \neq 0$

$\therefore A$ is invertible and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{bmatrix}$$

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Prop. $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Example ② Consider the two matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

Show that: $(AB)^{-1} = B^{-1}A^{-1}$

Ans: $AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \Rightarrow (AB)^{-1} = \frac{1}{2}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -3 \\ -9/2 & 7/2 \end{bmatrix} \quad ①$$

$$, B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 3/2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ \frac{-9}{2} & \frac{7}{2} \end{bmatrix} \quad ②$$

$\therefore$ ①, ② $\Rightarrow \boxed{(AB)^{-1} = B^{-1}A^{-1}}$

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