Lecture ㉛

Lagrange Multipliers

Let f and g are two functions in three variables

To find the maximum or minimum value of $f(x,y,z)$

[illegible] القيد

Subject to Constraint $g(x,y,z)=0$

* Do the following

1 Solve the Eqns $\nabla f = \lambda \nabla g$, $g(x,y,z)=0$

λ is called Lagrange Multiplier

to get all solutions $(x_1, y_1, z_1, \lambda_1)$, $(x_2, y_2, z_2, \lambda_2)$, ...

2 Discard λ, and find x, y and z

Remember

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

grad f

3 Evaluate $f(x,y,z)$ for each solution,

The Largest value yields the max. value

and the Smallest value yields the min value

Ex1 Use Lagrange Multipliers to find greatest and Smallest values of $f(x,y,z) = x+y+z$ subject to Constraint $\boxed{x^2+y^2+z^2=25}$

Ans:

(pb) $\Rightarrow$ $\boxed{\begin{array}{l} \text{max or min } f(x,y,z) = x+y+z \\ \text{s.t } x^2+y^2+z^2 = 25 \end{array}}$

$\boxed{1}$ Solve the Eqns

$$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \quad ① \\ , g(x,y,z) = x^2+y^2+z^2-25=0 \quad ② \end{array}}$$

$$\nabla f = \langle 1,1,1\rangle, \ \nabla g = \langle 2x, 2y, 2z\rangle$$

$$① \Rightarrow \langle 1,1,1\rangle = \lambda \langle 2x, 2y, 2z\rangle$$

$$\left.\begin{array}{lll} \Rightarrow 1 = 2\lambda x & \therefore & x = \frac{1}{2\lambda} \\ 1 = 2\lambda y & \therefore & y = \frac{1}{2\lambda} \\ 1 = 2\lambda z & \therefore & z = \frac{1}{2\lambda} \end{array}\right] ③$$

Substitute ③ in ②, We get

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 - 25 = 0$$

$$\therefore \frac{3}{4\lambda^2} = 25 \Rightarrow \lambda^2 = \frac{3}{100} \quad \therefore \lambda = \pm \frac{\sqrt{3}}{10}$$

أهمل

$\boxed{2}$ Discard λ and find x, y, z from ③,

$$\Rightarrow x = \pm \frac{5}{\sqrt{3}}, \ y = \pm \frac{5}{\sqrt{3}}, \ z = \pm \frac{5}{\sqrt{3}}$$

$\boxed{3}$ Find the values of $f(x,y,z)$

$$f\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right) = \frac{5}{\sqrt{3}} + \frac{5}{\sqrt{3}} + \frac{5}{\sqrt{3}} = \frac{15}{\sqrt{3}} = 5\sqrt{3}$$

which is the greatest value of f

$$, f\left(\frac{-5}{\sqrt{3}}, \frac{-5}{\sqrt{3}}, \frac{-5}{\sqrt{3}}\right) = \frac{-5}{\sqrt{3}} - \frac{5}{\sqrt{3}} - \frac{5}{\sqrt{3}} = -5\sqrt{3}$$

which is the smallest value of f

$\therefore$ the max. value of f is $5\sqrt{3}$ and its min. value is $-5\sqrt{3}$.

#

EX 2

If $f(x,y,z) = 4x^2 + y^2 + 5z^2$, find the point on the plane $2x + 3y + 4z = 12$ at which $f(x,y,z)$ has its least value.

Ans (pb) ⇒ $\min f(x,y,z) = 4x^2 + y^2 + 5z^2$

s.t $2x + 3y + 4z = 12$

1 Solve the Eqns

$$\nabla f = \lambda \nabla g \quad ①$$

$$g(x,y,z) = 2x + 3y + 4z - 12 = 0 \quad ②$$

, $g(x,y,z) = 2x + 3y + 4z - 12 = 0$

$\because \nabla f = \langle 8x, 2y, 10z \rangle$, $\nabla g = \langle 2, 3, 4 \rangle$

$① \Rightarrow \langle 8x, 2y, 10z \rangle = \lambda \langle 2, 3, 4 \rangle$

$$\therefore \boxed{\begin{aligned} 8x &= 2\lambda \\ 2y &= 3\lambda \\ 10z &= 4\lambda \end{aligned}} \Rightarrow \boxed{\begin{aligned} x &= \tfrac{1}{4}\lambda \\ y &= \tfrac{3}{2}\lambda \\ z &= \tfrac{2}{5}\lambda \end{aligned}} \quad ③$$

Subs. ③ in ②

$$\Rightarrow \tfrac{1}{2}\lambda + \tfrac{9}{2}\lambda + \tfrac{8}{5}\lambda - 12 = 0$$

$$\tfrac{66}{10}\lambda = 12 \quad \therefore \lambda = \boxed{\tfrac{20}{11}}$$

$$\frac{\overset{2}{\cancel{12}} \times 10}{\underset{11}{\cancel{66}}}$$

2 Discard λ and find x, y, z from Eq. ③

$$③ \Rightarrow \begin{aligned} x &= \tfrac{1}{4}\left(\tfrac{20}{11}\right) = \tfrac{5}{11} \\ y &= \tfrac{3}{2}\left(\tfrac{20}{11}\right) = \tfrac{30}{11} \\ z &= \tfrac{2}{5}\left(\tfrac{20}{11}\right) = \tfrac{8}{11} \end{aligned}$$

$\boxed{3}$ Find the values of $f(x,y,z)$ on the critical points

There's only one critical point which is

$$\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)$$

$$\Rightarrow f\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)$$

$$= 4\left(\frac{5}{11}\right)^2 + \left(\frac{30}{11}\right)^2 + 5\left(\frac{8}{11}\right)^2$$

$$= \frac{120}{11} \simeq 10.91$$

which is the min. value of function f occurs at $p\left(\frac{5}{11}, \frac{30}{11}, \frac{8}{11}\right)$. #

✓