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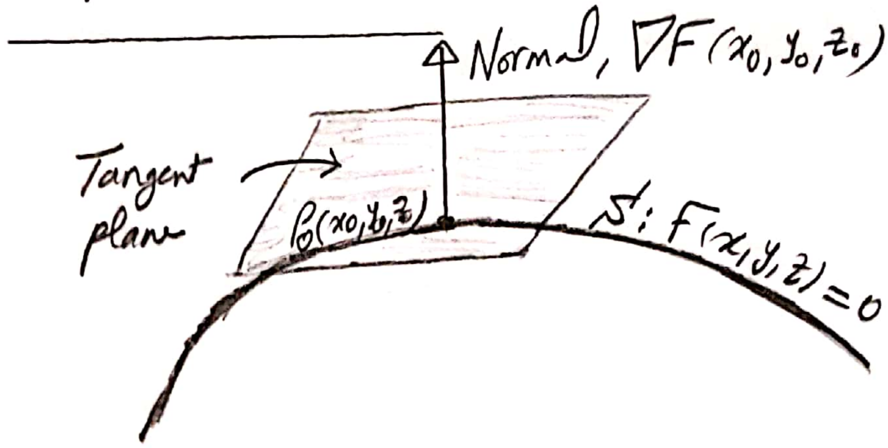
Lecture 30

Tangential planes and normal lines

In the opp. Fig.



∇F is the outward normal vector to the tangent plane for the surface $S: F(x,y,z) = 0$ at the point $P_0(x_0, y_0, z_0)$

The Eq$^{\underline{n}}$ of tangent plane is

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$$

where $\nabla F = \langle F_x, F_y, F_z \rangle$

and the parametric eqns of normal line to surface S are given by:

$$x = x_0 + F_x t$$
$$y = y_0 + F_y t$$
$$z = z_0 + F_z t$$

Ex Find the eqn of tangent plane and normal line to the graph of $x^3 - 2xy + z^3 + 7y + 6 = 0$ at point $P(1, 4, -3)$

Ans $\nabla F = \frac{\partial F}{\partial x} \vec{i} + \frac{\partial F}{\partial y} \vec{j} + \frac{\partial F}{\partial z} \vec{k}$

$\nabla F = \langle 3x^2 - 2y, -2x + 7, 3z^2 \rangle \Rightarrow \boxed{\nabla F(1, 4, -3) = \langle -5, 5, 27 \rangle}$

$\therefore$ Eq$^{\underline{n}}$ of tangent plane is $-5(x-1) + 5(y-4) + 27(z+3) = 0$

$\Rightarrow \boxed{5x - 5y - 27z = 66}$

& parametric Eqns of Normal line are

$\boxed{x = 1 - 5t, \; y = 4 + 5t, \; z = -3 + 27t}$

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Extrema of functions of several variables.

To find local extrema of the fn f of two variables

Step ① Find the critical points that of the form (a,b) by solving $f_x(x,y)=0$ and $f_y(x,y)=0$

critical pts

Step ② Find the Discriminant D of f

Discriminant

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$D = f_{xx} f_{yy} - (f_{xy})^2$, and then find D for each critical point (a,b).

Step ③ (i) If $D>0$ and $f_{xx}(a,b)>0$, then f has a local minimum at (a,b)

Conclusion

(ii) If $D>0$ and $f_{xx}(a,b)<0$, then f has a local maximum at (a,b)

(iii) If $D<0$ then $(a,b,f(a,b))$ is a saddle point

(V) If $D=0$ then the test fails.

Ex If $f(x,y)=\frac{1}{3}x^3+\frac{4}{3}y^3-x^2-3x-4y-3$, Find the local extrema and saddle points of f

Ans

$f_x = \boxed{x^2-2x-3}$ ①, $\boxed{f_y = 4y^2-4}$ ②

step ① To find critical points, solve $f_x=0$, $f_y=0$

① $\Rightarrow (x-3)(x+1)=0 \Rightarrow \boxed{x=3,\ x=-1}$

② $\Rightarrow 4(y^2-1)=0 \Rightarrow y^2-1=0 \Rightarrow (y-1)(y+1)=0 \Rightarrow \boxed{y=1,\ y=-1}$

$\therefore$ The critical points are $(3,1), (-1,-1), (3,-1)$ and $(-1,1)$.

Step ②: Discriminant D

$\boxed{f_{xx} = 2x-2}$, $f_{yy} = 8y$, $f_{xy} = 0$, $f_{yx} = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x-2 & 0 \\ 0 & 8y \end{vmatrix}$$

$$= 16(x-1)y = \boxed{16y(x-1)}$$

Step ③: Conclusion

$f(x,y) = \frac{1}{3}x^3 + \frac{4}{3}y^3 - x^2 - 3x - 4y - 3$

Critical points	Value of D	Value of f_{xx}	Conclusion
$(3,1)$	$D(3,1) = 32 > 0$	$f_{xx}(3,1) = 4 > 0 \Rightarrow$	$f(3,1) = \frac{-44}{3}$ is a local minimum
$(-1,-1)$	$D(-1,-1) = 32 > 0$	$f_{xx}(-1,-1) = -4 < 0 \Rightarrow$	$f(-1,-1) = \frac{4}{3}$ is a local maximum
$(3,-1)$	$D(3,-1) = -32 < 0$	Not required $\Longrightarrow$	$(3,-1,f(3,-1))$ is a saddle point
$(-1,1)$	$D(-1,1) = -32$	Not required $\Longrightarrow$	$(-1,1,f(-1,1))$ is a saddle point