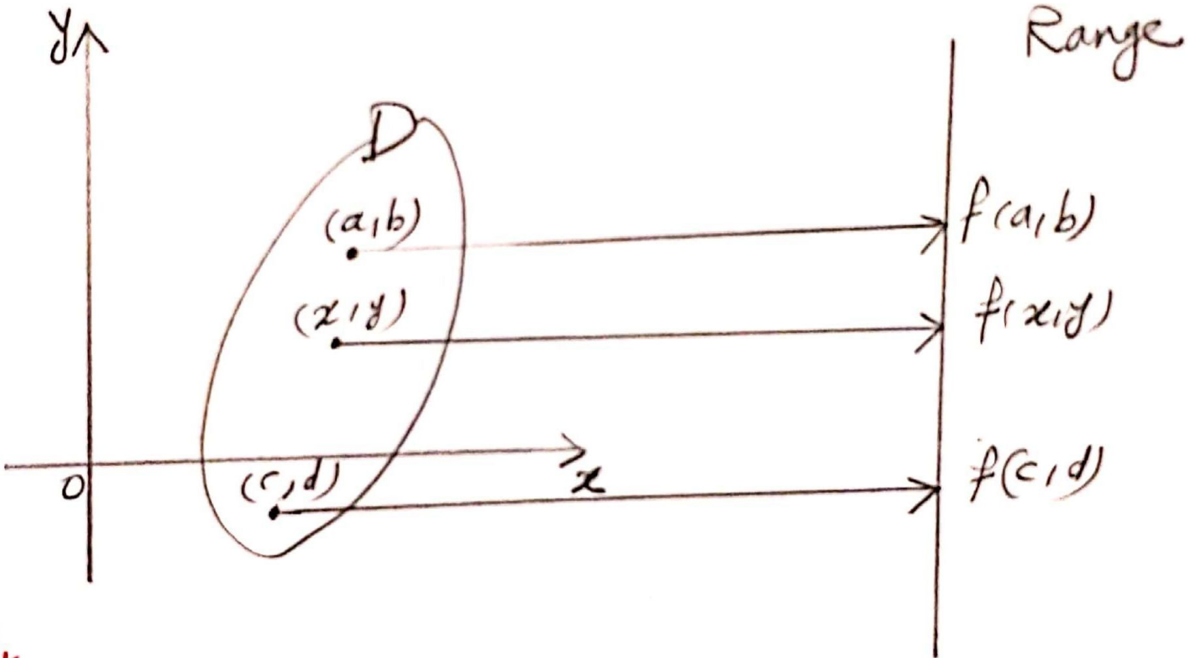
Ch 12: Functions of Several Variables and Differentiation

Def$\underline{n}$

A function of two Variables $f: R^2 \rightarrow R$ is a correspondence such that for every point $(x,y) \in D$, $D \subset R^2$ their exist only a real number $f(x,y) \in R$. D is called the domain and the set of all real numbers $f(x,y)$ where $(x,y) \in D$ is called the range.

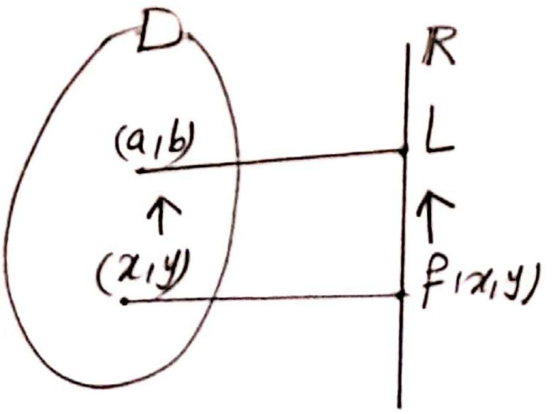


* **Limit**

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

This means that

$f(x,y) \rightarrow L$ as $(x,y) \rightarrow (a,b)$



* **Continuity**

f is continuous at $(a,b) \in D$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

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Notes 1) All polynomial functions are continuous in xy-plane

2) Rational fns are continuous except at points where the denominator is zero

Ex Show that the function $f(x,y) = \dfrac{x+2y}{x^2+y}$ is continuous at the point $(1,4)$

Ans:

$$f(1,4) = \frac{1+2(4)}{1+4} = \frac{9}{5} \quad ①$$

$$\lim_{(x,y)\to(1,4)} \frac{x+2y}{x^2+y} = \lim_{(x,y)\to(1,4)} \frac{1+2(4)}{1+4} = \frac{9}{5} \quad ②$$

(limit exists)

$\therefore$ ①, ② $\Rightarrow \displaystyle\lim_{(x,y)\to(1,4)} \frac{x+2y}{x^2+y} = f(1,4)$

$\therefore$ $f(x,y)$ is continuous at the point $(1,4)$.

Partial Derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation

$$f_x(x,y) = \frac{\partial f}{\partial x} = D_x f$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = D_y f$$

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Ex) Find First partial Derivatives for the fn

$$f(x,y,z) = x^2y^3z^4 + 2x - 5yz + 7$$

Ans:

$$f_x = \frac{\partial f}{\partial x} = 2xy^3z^4 + 2$$

$$f_y = \frac{\partial f}{\partial y} = 3x^2y^2z^4 - 5z$$

$$f_z = \frac{\partial f}{\partial z} = 4x^2y^3z^3 - 5y$$

* Second partial Derivatives

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Note that: $\boxed{f_{xy} = f_{yx}}$ where f, f_x, f_y, f_{xy} and f_{yx} are continuous fns

Ex) Find the second partial derivatives of f

if $f(x,y) = xy^4 - 2x^2y^3 + 4x^2 - 3y + 5$

Ans: $f_x = \frac{\partial f}{\partial x} = y^4 - 4xy^3 + 8x$

$f_y = \frac{\partial f}{\partial y} = 4xy^3 - 6x^2y^2 - 3$

$f_{xy} = \frac{\partial f_y}{\partial x} = 4y^3 - 12xy^2$

, $f_{yx} = \frac{\partial f_x}{\partial y} = 4y^3 - 12xy^2$

Note that $f_{xy} = f_{yx}$

$f_{xx} = \frac{\partial f_x}{\partial x} = -4y^3 + 8$

$f_{yy} = \frac{\partial f_y}{\partial y} = 12xy^2 - 12x^2y$

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* Partial Derivatives & Increments and Differentials

Defn) If $w = f(x,y)$ then

(1) the increment is $\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y)$

(2) the total differential for the dependent variable w

is $dw = f_x(x,y)\,dx + f_y(x,y)\,dy$

or $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$