Curvature

Lecture 26

(5) The Curvature of the curve C at $P(x, y)$

where $x = f(t)$ and $y = g(t)$

(Parametric Eqns of the curve C)

is $$K = \frac{|f'(t)\, g''(t) - g'(t)\, f''(t)|}{[(f'(t))^2 + (g'(t))^2]^{3/2}}$$

(6) The curvature of the curve C where,

$y = f(x)$ at $P(x, y)$ is

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

(7) The Center of the curvature is (h, k), where

$$h = x - \frac{y'[1 + (y')^2]}{y''}$$

$$k = y + \frac{1 + (y')^2}{y''}$$

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Ex Find the curvature, radius of curvature and center of the curve $y = x^4$ at point $p(1,1)$

Ans:

* The curvature

$$K = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

$$y = x^4 \,, \quad y' = 4x^3 \,, \quad y'' = 12x^2$$

at $p(1,1)$

$$y = 1 \,, \quad y' = 4 \,, \quad y'' = 12$$

$$\therefore K = \frac{12}{(17)^{3/2}} \simeq 0.1712$$

* the radius of the curvature is

$$\rho = \frac{1}{K} = \frac{(17)^{3/2}}{12} \simeq 5.8411$$

* the center of curvature is given by

$$h = x - \frac{y'\left[1+(y')^2\right]}{y''}$$

$$k = y + \frac{1+(y')^2}{y''}$$

$$\therefore h = 1 - \frac{4\,[1+16]}{12} = \frac{-14}{3}$$

$$k = 1 + \frac{1+16}{12} = \frac{29}{12}$$

$\therefore$ The center of curvature is $\left(\frac{-14}{3}, \frac{29}{12}\right)$

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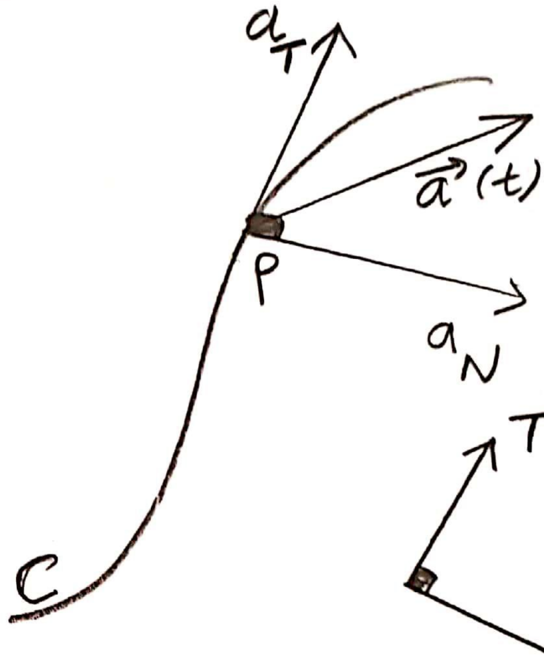
• Tangential and Normal Components of Acceleration

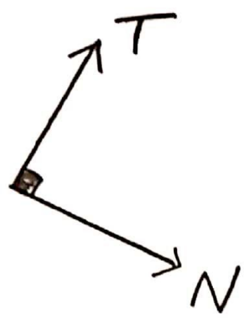
The acceleration of a moving particle on the curve C at time t is given by



$$\vec{a}(t) = a_T \vec{T} + a_N \vec{N}$$

, Its magnitude is

$$\|\vec{a}\| = \sqrt{a_T^2 + a_N^2}$$



where

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

Tangential Comp. of acceleration

$$, \quad a_N = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

Normal Comp. of acceleration

where $a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$

and the Curvature is

$$K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{1}{\|\vec{r}'(t)\|^2} a_N$$

Ex The position vector of a moving particle at time t is $\vec{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$, Find the tangential and normal Components of acceleration and curvature at time t.

Ans: $\vec{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$

$$\vec{r}'(t) = \langle -4\sin t, 9\cos t, 1 \rangle$$

$$\vec{r}''(t) = \langle -4\cos t, -9\sin t, 0 \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{16\sin^2 t + 81\cos^2 t + 1}$$

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}'(t)\|}$$

$$a_T = \frac{16\sin t\cos t - 81\sin t\cos t + 0}{\sqrt{16\sin^2 t + 81\cos^2 t + 1}}$$

$$a_T = \frac{-65\sin t\cos t}{\sqrt{16\sin^2 t + 81\cos^2 t + 1}}$$

$$a_N = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4\sin t & 9\cos t & 1 \\ -4\cos t & -9\sin t & 0 \end{vmatrix}$$

$$= 9\sin t\,\vec{i} - 4\cos t\,\vec{j} + (36\sin^2 t + 36\cos^2 t)\,\vec{k}$$

$$= 9\sin t\,\vec{i} - 4\cos t\,\vec{j} + 36\vec{k}$$

$$\therefore a_N = \sqrt{\frac{81\sin^2 t + 16\cos^2 t + 1296}{16\sin^2 t + 81\cos^2 t + 1}}$$

$$K = \frac{1}{\|\vec{r}'(t)\|^2} a_N$$

$$\therefore K = \frac{\sqrt{81\sin^2 t + 16\cos^2 t + 1296}}{(16\sin^2 t + 81\cos^2 t + 1)^{3/2}}$$