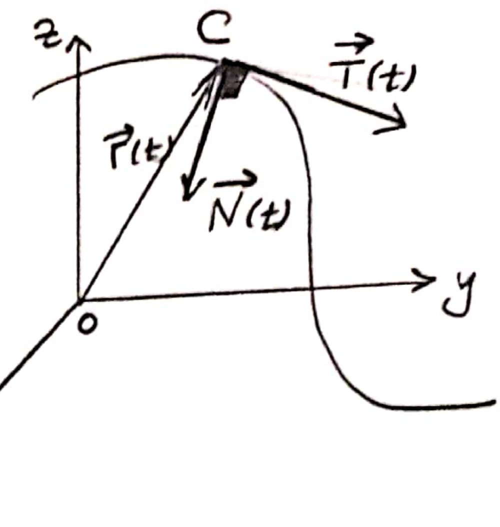
Lecture (25)

Curvature



① The tangent unit vector to C
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at point P is

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$$

where $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

② The principal normal unit vector
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$$\boxed{\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}}$$

③ The curvature of the curve C where the curve C is
الانحناء (التقوس)
represented by

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

is

$$\boxed{K = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}}, \quad \|\vec{r}\,'(t)\| \neq 0$$

Kappa

④ The radius of curvature is

$$\rho = \frac{1}{K}$$

Ex ① Find $\vec{T}(t)$, $\vec{N}(t)$, K and ρ at $t=1$ for the curve given by $\vec{r}(t) = 2t\,\vec{i} + t^2\vec{j} - \frac{1}{3}t^3\vec{k}$

Ans:

$$\vec{r}\,'(t) = 2\vec{i} + 2t\vec{j} - t^2\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = t^2+2 \quad \text{(i)}$$

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* The tangent unit vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\boxed{\vec{T}(t) = \frac{2\vec{i} + 2t\vec{j} - t^2\vec{k}}{t^2+2}} \quad \text{①}$$

at t=1 $\vec{T}(1) = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3} = \frac{1}{3}\langle 2, 2, -1 \rangle$

* the principal normal unit vector

$$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}} \quad \text{②}$$

① ⇒ $\vec{T}'(t) = \frac{(t^2+2)(2\vec{j} - 2t\vec{k}) - 2t(2\vec{i} + 2t\vec{j} - t^2\vec{k})}{(t^2+2)^2}$ ②'

$$\vec{T}'(t) = \frac{-4t\vec{i} + (2t^2 + 4 - 4t^2)\vec{j} + (\cancel{-2t^3} - 4t + \cancel{2t^3})\vec{k}}{(t^2+2)^2}$$

$$\boxed{\vec{T}'(t) = \frac{-4t\vec{i} + (4-2t^2)\vec{j} - 4t\vec{k}}{(t^2+2)^2}} \quad \text{③}$$

$$\|\vec{T}'(t)\| = \sqrt{\frac{16t^2 + (4-2t^2)^2 + 16t^2}{(t^2+2)^4}}$$

$$\|\vec{T}'(t)\| = \sqrt{\frac{4t^4 + 16t^2 + 16}{(t^2+2)^4}} = \frac{(2t^2+4)}{(t^2+2)^2}$$

$$\|\vec{T}'(t)\| = \frac{2\cancel{(t^2+2)}}{(t^2+2)^{\cancel{2}}} = \frac{2}{t^2+2} \quad \text{(ii)}$$

* Subs. ③ and (ii)
in ② ⇒ $\vec{N}(t) = \frac{-4t\vec{i} + (4-2t^2)\vec{j} - 4t\vec{k}}{2(t^2+2)}$

$$\vec{N}(t) = \frac{-2t\vec{i} + (2-t^2)\vec{j} - 2t\vec{k}}{t^2+2} \quad \text{④}$$

at t=1 $\vec{N}(1) = \frac{-1}{3}\langle 2, -1, 2 \rangle$

3// (1), (4) $\Rightarrow$

$$\vec{T}(1)\cdot\vec{N}(1) = \frac{1}{3}\langle 2, 2, -1\rangle \cdot \frac{-1}{3}\langle 2, -1, 2\rangle$$

$$\vec{T}(1)\cdot\vec{N}(1) = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9} = 0$$

Clearly, $\vec{T}(t)\cdot\vec{N}(t) = 0 \Rightarrow \vec{T}(t) \perp \vec{N}(t)$

* The curvature K

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

(i), (ii) $\Rightarrow$

$$K = \frac{2}{t^2+2}\left(\frac{1}{t^2+2}\right)$$

$$K = \frac{2}{(t^2+2)^2}$$

at $t=1$ $K = \frac{2}{9}$, and radius of curvature is

$$\rho = \frac{9}{2} \quad \#$$

Note

You can get $\vec{N}(1)$ directly by using (2)' as follows.

$$T'(1) = \frac{3(2\vec{j} - 2\vec{k}) - 2(2\vec{i} + 2\vec{j} - \vec{k})}{9}$$

$$\therefore T'(1) = \frac{-4\vec{i} + 2\vec{j} - 4\vec{k}}{9} = \frac{-2}{9}\langle 2, -1, 2\rangle$$

$$\Rightarrow \|T'(1)\| = \frac{2}{9}\sqrt{4+1+4} = \frac{6}{9} = \frac{2}{3}$$

$$\therefore \vec{N}(1) = \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{-2/9\langle 2, -1, 2\rangle}{2/3}$$

$$\therefore \vec{N}(1) = \frac{-1}{3}\langle 2, -1, 2\rangle \quad \#$$