Ch 4 p. 165

Lecture (21)

The long run behavior of Markov chain

- **Defn**

A Markov chain Matrix or transition prob. $M\underline{X}$ $P = [P_{ij}]$ is called regular if their exists a matrix P^K, $K > 0$ such that all of its elements (entries) are positive

$$\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & \cdots & N \\
\hline
0 & + & + & + & \cdots & + \\
1 & + & + & + & \cdots & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
N & + & + & + & \cdots & + \\
\hline
 & \pi_0 & \pi_1 & \pi_2 & & \pi_N
\end{array}$$

Theorem p. 168

Let P a regular transition prob. $M\underline{X}$ on the states $0, 1, 2, \cdots, N$ then the limiting dist$\underline{n}$ $\pi = [\pi_0, \pi_1, \pi_2, \cdots, \pi_N]$ is the unique (وحيد) solution of the Eqns

$$\pi_j = \sum_{K=0}^{N} \pi_k P_{kj}, \quad j = 0, 1, 2, \cdots, N$$

$$, \sum_{K=0}^{N} \pi_k = 1$$

Ex p. 170 Text for social class MX

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.40 & 0.50 & 0.10 \\ 1 & 0.05 & 0.70 & 0.25 \\ 2 & 0.05 & 0.50 & 0.45 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$
$$\pi_0 \quad \pi_1 \quad \pi_2$$

Find the limitting distn $\pi = (\overset{?}{\pi_0}, \overset{?}{\pi_1}, \overset{?}{\pi_2})$

$$\pi_j = \sum_{k=0}^{2} \pi_k P_{kj}, \quad j = 0, 1, 2$$

at $j = 0$
$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20}$$

at $j = 1$
$$\pi_1 = \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21}$$

at $j = 2$
$$\pi_2 = \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22}$$

$$\Rightarrow \boxed{\begin{aligned} \pi_0 &= 0.40\pi_0 + 0.05\pi_1 + 0.05\pi_2 \\ \pi_1 &= 0.5\cancel{0}\pi_0 + 0.7\cancel{0}\pi_1 + 0.5\cancel{0}\pi_2 \\ \pi_2 &= 0.10\pi_0 + 0.25\pi_1 + 0.45\pi_2 \end{aligned}}$$

$\times 100$

$\times 10$

$\times 100$

التبسيط .

$$\Rightarrow \boxed{\begin{aligned} 60\pi_0 - 5\pi_1 - 5\pi_2 &= 0 \\ 5\pi_0 - 3\pi_1 + 5\pi_2 &= 0 \\ 10\pi_0 + 25\pi_1 - 55\pi_2 &= 0 \end{aligned}}$$

Simplify

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$$\Rightarrow \quad 12\pi_0 - \pi_1 - \pi_2 = 0 \quad ①$$

$$5\pi_0 - 3\pi_1 + 5\pi_2 = 0 \quad ②$$

$$2\pi_0 + 5\pi_1 - 11\pi_2 = 0 \quad ③$$

$$\& \because \pi_0 + \pi_1 + \pi_2 = 1 \quad ④$$

Then Solving

$$12\pi_0 - \pi_1 - \pi_2 = 0 \quad (I)$$

$$5\pi_0 - 3\pi_1 + 5\pi_2 = 0 \quad (II)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (III)$$

By using Cramer Rule

the Soln is $\pi_0 = \dfrac{\Delta_0}{\Delta}$, $\pi_1 = \dfrac{\Delta_1}{\Delta}$, $\pi_2 = \dfrac{\Delta_2}{\Delta}$

where $\Delta = \begin{vmatrix} 12 & -1 & -1 \\ 5 & -3 & 5 \\ 1 & 1 & 1 \end{vmatrix}$

$$\Delta = 12(-3-5) + 1\cancel{(5-5)} - 1(5+3)$$

$$\Delta = -96 - 8 = -104$$

$$, \Delta_0 = \begin{vmatrix} 0 & -1 & -1 \\ 0 & -3 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -8 \quad , \Delta_1 = \begin{vmatrix} 12 & 0 & -1 \\ 5 & 0 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -65$$

$$, \Delta_2 = \begin{vmatrix} 12 & -1 & 0 \\ 5 & -3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -31$$

$$\Rightarrow \pi_0 = \frac{-8}{-104} = \frac{1}{13}, \quad \pi_1 = \frac{-65}{-104} = \frac{5}{8}, \quad \pi_2 = \frac{-31}{-104} = \frac{31}{104}$$

$\therefore$ The limitting distn is $\pi = \left(\frac{1}{13}, \frac{5}{8}, \frac{31}{104}\right)$

$\pi = (0.0769, 0.625, 0.2981)$

lower class, Middle class, upper class

- as shown before in p.166

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