Lecture (12)

properties of Determinantial function & Inverse of a Matrix by using Cofactor Method

* **Props**

If A and B are $n\times n$ square Matrices then

(1) $\det(KA) = K^n \det(A)$

(2) $\det(A+B) \neq \det A + \det B$

(3) $\det(AB) = \det A \cdot \det B$

(4) $\det(A^n) = (\det A)^n$

(5) $\det(A^{-1}) = \dfrac{1}{\det(A)}$, $\det(A) \neq 0$

(6) A square Matrix A is invertible iff $\det(A) \neq 0$

(7) $\det(A^T) = \det(A)$

Examples

For the Matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -1-6 = -7$

① $\det(2A) = \begin{vmatrix} 2 & 6 \\ 4 & -2 \end{vmatrix} = 2^2 \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 2^2 \det(A)$

② If $B = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 5 & 1 \end{bmatrix}$

$\det(B) = -8$

$\left.\begin{array}{l} \det(A+B) = \begin{vmatrix} 0 & 5 \\ 5 & 1 \end{vmatrix} = 0-25 = -25 \\ \underline{\text{But}}\ \det(A)+\det(B) = -7-8 = -15 \end{array}\right\}$ Note that $\det(A+B) \neq \det A + \det B$

③ $AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ -5 & 2 \end{bmatrix}$

$\det(AB) = \begin{vmatrix} 8 & 8 \\ -5 & 2 \end{vmatrix} = 16+40 = 56$ & $\det(A)\det(B) = -7(-8) = 56$

i.e $\det(AB) = \det(A)\det(B)$

④ For $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\det(A^2) = \begin{vmatrix} 7 & 0 \\ 0 & 7 \end{vmatrix} = 49$

and also $(\det A)^2 = (-7)^2 = 49$

$\therefore \det(A^2) = (\det A)^2$

⑤ For $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}_{2\times 2} \Rightarrow A^{-1} = \frac{1}{-7}\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}$

$\det(A^{-1}) = \left(\frac{-1}{7}\right)^2 \begin{vmatrix} -1 & -3 \\ -2 & 1 \end{vmatrix}$ prop. 1

$\det(A^{-1}) = \frac{1}{49}[-1-6] = \frac{-1}{7}$

$\therefore \det(A^{-1}) = \frac{1}{-7} = \frac{1}{\det(A)}$

⑥ $\det(A) \neq 0 \Leftrightarrow A$ is invertible — See also, Lecture ⑪

⑦ For $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

$\because \det(A) = -7 \quad , \det(A^T) = -7$

$\therefore \det(A) = \det(A^T)$ #

EX If A is a 3×3 Matrix and $\det(A) = 2$
find $\det(4A)$, $\det(4A^{-1})$ and $\det(A^4)$

Ans $\det(4A) = 4^3 \det(A) = 64(2) = 128$

$\det(4A^{-1}) = 4^3 \det(A^{-1}) = 64 \frac{1}{\det(A)} = \frac{64}{2} = 32$

$\det(A^4) = (\det A)^4 = 2^4 = 16$

• Inverse of a Matrix by using Cofactor Method

Defn

If A is an invertible Matrix, $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \left[\mathrm{adj}(A) \right]$$

where $\mathrm{adj}(A) = C^T$, C is the Matrix of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Ex Find A^{-1} of a Matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

Ans

$$C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix} \Rightarrow \mathrm{adj}(A) = C^T = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

The inverse of Matrix A is

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix} = 2(-12) + \cancel{0(-4)} + 3(6)$$

$$= -24 + 18 = -6$$

$$\therefore A^{-1} = \frac{1}{-6} \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}$$

\#