1// Lecture (11)++

- Ch 2 Sec 2.5 p. 71: Martingales

Definition: A stochastic process $\{X_n : n = 0, 1, \dots\}$ is a martingale if for $n = 0, 1, \dots$

(a) $E[|X_n|] < \infty$

(b) $E[X_{n+1} | X_0, \dots, X_n] = X_n$

* See also some properties for martingale p. 72 and revise all properties p. 50 Textbook.

Note that:

① The law of total probability for conditional expectation can be written as

$$E[g(X)] = \sum_y E[g(X) | Y = y] P_Y(y)$$

$$\therefore E[g(X)] = E[E[g(X)|Y]]$$

For $g(X) = X$, we have $E(X) = \sum_y E[X | Y = y] P_Y(y)$

$$\therefore E(X) = E[E[X|Y]]$$

② $E[g(X)|Y = y] = E[g(X)]$ if X and Y are independent

③ $E[g(X)h(Y)|Y = y] = h(y)E[g(X)|Y = y]$, h is determined by y.

Remarks (i) If $g(X) = 1$ then $E[h(Y)|Y = y] = h(y)$

(ii) If Z is determined by $X_0, \dots, X_n$, then

$$E[Z | X_0, \dots, X_n] = Z$$

④ $E[g(X)h(Y)] = \sum_y E[g(X)h(Y) | Y=y] P_Y(y)$

... by using ①

$= \sum_y h(y) E[g(X) | Y=y] P_Y(y)$

... by using ③

$\therefore E[g(X)h(Y)] = E[h(Y)E[g(X)|Y]]$

* pb 2.5.1 p. 77

Use the law of total probability for Conditional Expectations

$E[E\{X|Y,Z\}|Z] \overset{①}{=} E[X|Z]$ to show

$E[X_{n+2}|X_0,\dots,X_n] = E[E\{X_{n+2}|X_0,\dots,X_{n+1}\}|X_0,\dots,X_n]$. ②

Conclude that when X_n is a martingale,

$E[X_{n+2}|X_0,\dots,X_n] = X_n$ ③

Answer:

Let $Z = X_0,\dots,X_n$ and $Y = X_{n+1}$ in R.H.S of ②

$\therefore E[E\{X_{n+2}|X_0,\dots,X_n,X_{n+1}\}|X_0,\dots,X_n]$ R.H.S of ②

$= E[E\{X_{n+2}|Z,Y\}|Z] = E[X_{n+2}|Z]$ ... by using ①

$= E[X_{n+2}|X_0,\dots,X_n]$ L.H.S of ②

thus formula ② is already proved.

Now, if X_n is a martingale, then for $n \geq 0$

$E[X_{n+1}|X_0,\dots,X_n] = X_n$ ④, So $E[X_{n+2}|X_0,\dots,X_{n+1}] = X_{n+1}$ ⑤

Substitute ⑤ in the R.H.S of ②, we get

$E[X_{n+2}|X_0,\dots,X_n] = E[X_{n+1}|X_0,\dots,X_n] = X_n$

--- by using ④

Thus formula ③ is also proved. #

* Pb 2.5.2 p.78

Let $U_1, U_2, \ldots$ be independent random variables each uniformly distributed over the interval $(0, 1]$. Show that $X_0 = 1$ and $X_n = 2^n U_1 \ldots U_n$ for $n = 1, 2, \ldots$ defines a martingale.

Answer:

To prove that X_n is a martingale, we have to prove that:

$\boxed{1}$ $E[|X_n|] < \infty$

$\boxed{2}$ $E[X_{n+1} | X_0, \ldots, X_n] = X_n$

For Condition $\boxed{1}$,

as X_n is a non-negative random variable

$$E[|X_n|] = E[X_n]$$

$$= E[2^n U_1 U_2 \ldots U_n]$$

$$= 2^n E[U_1] E[U_2] \ldots E[U_n]$$

where U_i, $i = 1, 2, \ldots, n$ are indep. r.v's

$$= 2^n \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}}$$

$$\therefore E[|X_n|] = \frac{2^n}{2^n} = 1 < \infty \quad (1)$$

(finite)

Remember

If $U \sim$ uniform (a, b) then $E(U) = \frac{1}{2}(a+b)$

i.e $E(U_i) = \frac{1}{2}(0+1) = \frac{1}{2}$ for $U_i \sim$ uniform $(0, 1)$

For Condition $\boxed{2}$,

$$E[X_{n+1} | X_0, \ldots, X_n]$$

$$= E[2^{n+1} U_1 \ldots U_n U_{n+1} | X_0, \ldots, X_n]$$

$$= \underbrace{2^n U_1 \ldots U_n}_{\text{This product}} E[2 U_{n+1} | X_0, \ldots, X_n]$$

is out because it's determined by $X_0, \ldots, X_n$.

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$\therefore E[X_{n+1} | X_0, \dots, X_n]$

$= 2^n U_1 \cdots U_n \cdot 2 E[U_{n+1}]$ because U_{n+1} is independant of $X_0, \dots, X_n$

$= 2^n U_1 \cdots U_n \cdot \cancel{2} \cdot \frac{1}{\cancel{2}}$ where $E[U_i] = \frac{1}{2}$, $i = 1, 2, \dots, n$

$= X_n$, $n = 1, 2, \dots$ (2)

From (1) and (2) we can deduce that X_n, $n = 0, 1, 2, \dots$ where $X_0 = 1$ defines a martingale.

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- Not that:

We can deduce from pb 2.5.1 that the martingale equality (b) (see the definition of a martingale) can be extended to future times to be in the form

$$E[X_m | X_0, \dots, X_n] = X_n \text{ for } m \geq n.$$
