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ABSTRACT

In this article, a test statistic for testing exponentiality versus a used better than aged in Laplace transform ordering class of life distribution based on a U-statistic is proposed. Pitman's asymptotic efficiencies of the test are calculated and compared to other tests. The percentiles of this test statistic are tabulated for censored and non-censored data, and the powers of this test are estimated for some famously alternative distributions in reliability, such as the Weibull, Makeham, linear failure rate, and Gamma distributions. Finally, examples in different areas are used as practical applications of the proposed test.

Keywords

used better than aged and used better than aged in expectation classes of life distributions, testing hypothesis, right-censored data, Makeham, Weibull, linear failure rate, Gamma distributions

Introduction

Statistical inferences are used to project the data from the sample to the entire population. Statistical inference based on two main branches, one of them the estimation and the other the testing hypotheses. In general, we do not know the true value (claim) of the population parameters; they must be estimated. However, we do have hypotheses about what the true values (claims) are. The hypothesis actually to be tested is usually given the symbol H_0 and is commonly referred to as the null hypothesis. The other hypothesis, which is assumed to be true when the null hypothesis is false, is referred to as the alternative hypothesis and is often symbolized as H_1. Both the null and alternative hypotheses should be stated before any statistical test of significance is conducted.

In this article, real data are given, and we desire to test H_0, whose data are exponential, versus the alternative hypothesis H_1, whose data are not exponential. To choose between

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H_0 and H_1 or to make a decision we need to define the test statistic. The test statistic is a random variable used to determine how close a specific sample result falls to one of the hypotheses being tested.

In reliability theory, aging life is usually characterized by a nonnegative, continuous random variable $X \geq 0$ representing equipment life with the distribution function F and survival function $\overline{F}(t) = 1 - F(t)$ such that $F(0-) = 0$. One of the most important approaches to the study of aging is based on the concept of residual life. For any random variable X, let $X_t = [X - t | X > t]$, and $t \in \{x : F(x) < 1)$ denote a random variable whose distribution is the same as the conditional distribution of X–t, given that $X > t$ and has the following survival:

$$\overline{F}_t(x) = \begin{cases} \frac{\overline{F}(x+t)}{\overline{F}(t)} & \overline{F}(t) > 0 \\ 0 & \overline{F}(t) = 0 \end{cases}$$

When X is the lifetime of a device which has a finite mean $\mu = E(X) = \int_0^\infty \overline{F}(u) du$, the mean of X_t is called the mean residual life and is given in the following:

$$\mu(t) = E(X_t) = \frac{\int_t^\infty \overline{F}(u) du}{\overline{F}(t)}$$

Furthermore, the hazard rate of X is defined by the following:

$$h(t) = -\frac{d}{dt} ln\, \overline{F}(t) = \frac{f(t)}{\overline{F}(t)}, t \geq 0, \overline{F}(t) > 0$$

where $f(t) = F'(t)$ is the probability density of X, assuming it exists. Note that if $lim_{t \to \infty} h(t) = h(\infty)$ exists and is positive, then we have the following (Willmot and Cai [1]):

$$\mu(\infty) = \lim_{t \to \infty} \mu(t) = \frac{1}{h(\infty)}$$

If X and Y are two random variables with distributions F and G (survivals $\overline{F}$ and $\overline{G}$), respectively, then we say that X is smaller than Y in the following:

(a) Usual stochastic order, denoted by $X \leq_{st} Y$ if

$$\overline{F}(x) \leq \overline{G}(x) \text{ for all } x;$$

(b) Increasing convex order, denoted by $X \leq_{icx} Y$ if

$$\int_x^\infty \overline{F}(u)\, du \leq \int_x^\infty \overline{G}(u)\, du;$$

(c) Increasing concave order, denoted by $X \leq_{icv} Y$ if

$$\int_0^x \overline{F}(u)\, du \leq \int_0^x \overline{G}(u)\, du.$$

Another importing ordering that has come to use in reliability and life testing is the following:

A random variable X is smaller than a random variable Y with respect to the Laplace transform order (denoted by $X \leq_{Lt} Y$) if, and only if, we have the following:

$$\int_0^\infty e^{-sx} dF(x) \geq \int_0^\infty e^{-sx} dG(x), s \geq 0 \tag{1}$$

It is easy to check that Eq 1 is equivalent to the following:

$$\int_0^\infty e^{-sx} \overline{F}(x) dx \leq \int_0^\infty e^{-sx} \overline{G}(x) dx \tag{2}$$

Two classes of life distributions were introduced by Alzaid [2] that are in the used better than aged (UBA) and UBA in expectation (UBAE) classes of life distribution.

Precisely, we have the following definitions:

DEFINITION

The distribution function F is said to be UBA if $0 < \mu(\infty) < \infty$ and for all (see Ahmad [3]):

$$\overline{F}(x+t) \geq \overline{F}(t)e^{-x/\mu(\infty)} \quad x, t \geq 0 \tag{3}$$

DEFINITION

The distribution function F is said to be UBAE if $0 < \mu(\infty) < \infty$; see the following:

$$\mu(t) \geq \mu(\infty) \tag{4}$$

Consider that F is UBA (UBAE) if and only if X_t converges in distribution to a random variable X_A (say) exponentially distributed with failure rate $\frac{1}{\mu}$, and see the following:

$$X_t \leq_{st} X_A, (E(X_t) \leq_{st} E(X_A))$$

According to the aforementioned definitions, we can deduce the following new definition for UBA in the Laplace transform order (UBAL) as follows.

DEFINITION

The distribution function F is said to be UBAL if $0 < \mu(\infty) < \infty$ and for all $x, t \geq 0$; see the following:

$$\int_0^\infty e^{-sx}\overline{\mathrm{F}}(x+t)dx \geq \frac{\mu(\infty)}{1+s\mu(\infty)}\overline{F}(t) \quad s \geq 0, \tag{5}$$

It is obvious that Eq 5 is equivalent to $X_t \leq_{\mathrm{Lt}} X_A$ for all $t \geq 0$.

To introduce the definition of the discrete UBAL, let X be a discrete nonnegative random variable, such that $P(X = k) = p_k, k = 0, 1, 2 \ldots$ Let $\overline{\mathrm{P}}_k = P(X > k), k \geq 1, \overline{\mathrm{P}}_0 = 1$ denote the corresponding survival function.

The discrete nonnegative random variable X is said to be discrete UBAL if, and only if, we have the following:

$$\sum_{k=0}^{\infty} \overline{\mathrm{P}}_{k+i} z^k \geq \overline{\mathrm{P}}_i \sum_{k=0}^{\infty} z^k, \text{for all } 0 \leq z \leq 1 \quad \text{and} \quad i = 0, 1, \ldots.$$

Now $X \leq_{st} X_A \Rightarrow X \leq_{Lt} X_A$.

Then, we have the following implication:

$$\begin{array}{ccccc} \text{Increasing Failure Rate} & \subset & \text{UBA} & \subset & \text{UBAL} \\ & & \cap & & \\ & & \text{UBAE} & & \end{array}$$

Applications, properties, and interpretations of the Laplace transform order in the statistical theory of reliability and economics can be found in Denuit [4], Klefsjö [5], and Ahmed and Kayid [6].

The main objective in this article is to deal with the problem of testing $H_0 : F$, which is exponential, against $H_1 : F$, which is the largest class of life distribution UBAL. The article is organized as follows: in the "Testing Hypothesis Application for Complete Data" section, we give a test statistic based on a U-statistic for complete data. Selected critical values are tabulated for sample sizes 5(5)100 using the Mathematica 8 program (Wolfram Research, Champaign, IL) in the "Monte Carlo Null Distribution Critical Points" section. The Pitman asymptotic efficiency (PAE) for common alternatives is obtained in the "Pitman Asymptotic Relative Efficiency" section. In "The Power of the Proposed Test" section, we also calculate the power estimates for the Weibull, Makeham, Gamma, and linear failure rate (LFR) distributions. A proposed test is presented for right-censored data in the "Test for UBAL in Case for Right-Censored Data" section. In the "Applications" section, we discuss some applications (numerical examples) to show the importance of the proposed test. Finally, we give a conclusion for our work in the "Conclusions" section.

Testing Hypothesis Application for Complete Data

This section is concerned with the construction of the proposed statistic as a U-statistic and discussion of its asymptotic normality.

Here, we hope to test the null hypothesis $H_0 : F$, which is exponential, against $H_1 : F$, which is UBAL and not exponential. Nonparametric testing for classes of life distributions has been considered by many authors (see Mahmoud and Abdul Alim [7]; Mugdadi and Ahmad [8]; Abu-Youssef and Bakr [9–11]; and Abu-Youssef, Mohammed, and Bakr [12,13]).

According to Eq 5, we may use the following as a measure of departure from H_0:

$$\delta(s) = \int_0^\infty \int_0^\infty e^{-su}\overline{F}(u+t)du\,dt - \frac{\mu(\infty)}{1+s\mu(\infty)}\int_0^\infty \overline{F}(t)dt$$

The following theorem is essential for the development of our test statistic.

THEOREM 4

Let X be the UBAL random variable with distribution function F; then, based on the previous technique, use the following:

$$\delta(s) = \frac{\mu}{s(1+s\mu(\infty))} - \frac{1}{s^2}(1-\varphi(s)) \tag{6}$$

where $\varphi(s) = \int_0^\infty e^{-sx}dF(x)$, and $\mu = \int_0^\infty \overline{F}(t)dt$.

Proof

Since

$$\begin{aligned}\delta(s) &= \int_0^\infty \int_0^\infty e^{-su}\overline{F}(u+t)dudt - \frac{\mu(\infty)}{1+s\mu(\infty)}\int_0^\infty \overline{F}(t)dt = \int_0^\infty \int_0^\infty e^{-su}\overline{F}(u+t)dudt - \frac{\mu(\infty)}{1+s\mu(\infty)}\mu \\ &= I - \frac{\mu(\infty)}{1+s\mu(\infty)}\mu\end{aligned}$$

where

$$\begin{aligned}I &= \int_0^\infty \int_0^\infty e^{-su}\overline{F}(u+t)dudt = \int_0^\infty \int_t^\infty e^{-s(x-t)}\overline{F}(x)dxdt = \frac{1}{s}\int_0^\infty (1-e^{-st})\overline{F}(t)dt \\ &= \frac{\mu}{s} - \frac{1}{s^2}(1-\varphi(s))\end{aligned}$$

Hence, the result follows.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution function F. For generality, we assume $\mu(\infty)$ is known and equal one. The empirical estimator $\widehat{\delta}(s)$ of our test statistic can be obtained as follows:

$$\widehat{\delta}_n(s) = \frac{1}{n}\sum_i \left\{ \frac{X_i}{s(1+s)} - \frac{1}{s^2}\left(1 - e^{-sX_i}\right) \right\}$$

To make the test invariant, allow the following:

$$\widehat{\Delta}_n(s) = \frac{\widehat{\delta}_n(s)}{\overline{X}}$$

then

$$\widehat{\Delta}_n(s) = \frac{1}{\overline{X}n}\sum_i \varnothing(X_i)$$

where

$$\varnothing(X_i) = \frac{1}{s}\left\{ \frac{X_i}{(1+s)} - \frac{1}{s}\left(1 - e^{-sX_i}\right) \right\}$$

To find the limiting distribution of $\widehat{\delta}(s)$, we resort to the U-statistic theory (Lee [14]). Set the following:

$$\varnothing(X_1) = \frac{1}{s}\left\{ \frac{X_1}{(1+s)} - \frac{1}{s}\left(1 - e^{-sX_1}\right) \right\}$$

Then, $\widehat{\Delta}_n(s)$ is equivalent to the U-statistic given by the following:

$$U_n = \frac{1}{\binom{n}{1}}\sum_i \varnothing(X_i)$$

The following theorem summarizes the asymptotic normality of $\widehat{\delta}_n(s)$.

THEOREM 5

i. As $n \to \infty$, $(\widehat{\delta}_n(s) - \delta(s))$ is asymptotically normal, with mean 0 and variance $\sigma^2(s)$, where we have the following:

$$\sigma^2(s) = \mathrm{Var}[\widehat{\delta}_n(s)] = E\left(\frac{1}{s}\left\{ \frac{x}{(1+s)} - \frac{1}{s}\left(1 - e^{-sx}\right) \right\}\right)^2$$

ii. Under H_0, the variance is as follows:

$$\sigma_0^2(s) = \frac{1}{(2s+1)(s+1)^3}$$

Proof

i. Using the standard U-statistic theory, Lee [14], we get the following:

$$E\left[\widehat{\delta}_n(s)\right] = E\left(\frac{1}{s}\left\{ \frac{x}{(1+s)} - \frac{1}{s}\left(1 - e^{-sx}\right) \right\}\right) \quad \sigma^2(s) = \mathrm{Var}\left[\widehat{\delta}_n(s)\right] = E\left(\frac{1}{s}\left\{ \frac{x}{(1+s)} - \frac{1}{s}\left(1 - e^{-sx}\right) \right\}\right)^2$$

ii. Under H_0, by direct calculations we have the following:

$$\mu_0 = E\left[\widehat{\delta}_n(s)\right] = \int_0^\infty \left(\frac{1}{s}\left\{ \frac{x}{(1+s)} - \frac{1}{s}\left(1 - e^{-sx}\right) \right\}\right) e^{-x} dx = 0$$

$$\sigma_0^2(s) = \int_0^\infty \left(\frac{1}{s}\left\{ \frac{x}{(1+s)} - \frac{1}{s}\left(1 - e^{-sx}\right) \right\}\right)^2 e^{-x} dx = \frac{2}{(2s+1)(s+1)^3}$$

Monte Carlo Null Distribution Critical Points

Based on 10,000 generated samples from the standard exponential distribution, the Monte Carlo null distribution critical values of our test $\widehat{\delta}_n(s)$ for $s = 2$ and $s = 3$ are simulated and tabulated, where $n = 5(5)100$ in **Table 1**. The Mathematica 8 program is used.

It is clear from **Table 1** and **Fig. 1** that the critical values decrease as the sample size increases, and they increase as the confidence level increases.

Also, our test $\widehat{\delta}_n(s)$ gives higher efficiency whenever s increases.

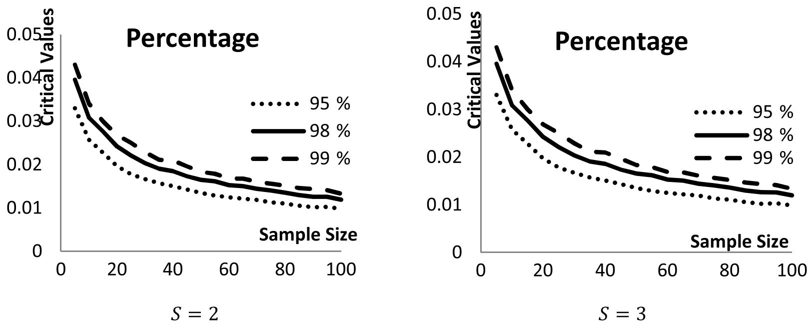
TABLE 1

The upper percentile points of $\widehat{\delta}_n(2)$ and $\widehat{\delta}_n(3)$ with 10,000 replications.

	$S = 2$			$S = 3$		
n	95 %	98 %	99 %	95 %	98 %	99 %
5	0.06085	0.07382	0.08090	0.0330278	0.0395916	0.0430635
10	0.04812	0.05859	0.06557	0.0257983	0.0308428	0.034064
15	0.04127	0.05057	0.05603	0.0227896	0.0276479	0.0300001
20	0.03636	0.04452	0.04940	0.019715	0.0242044	0.026861
25	0.03383	0.04198	0.04584	0.017804	0.022084	0.0250399
30	0.03173	0.03859	0.04342	0.0167156	0.0203496	0.0228828
35	0.02921	0.0359	0.04052	0.0157187	0.0191094	0.0211741
40	0.02748	0.03374	0.03830	0.0150886	0.0185543	0.0209858
45	0.02566	0.03141	0.03503	0.0142948	0.0173694	0.0196203
50	0.02495	0.03065	0.03474	0.0134933	0.016529	0.0183436
55	0.02370	0.02959	0.03310	0.0129006	0.0161976	0.0178974
60	0.02242	0.02773	0.03078	0.0125106	0.0152783	0.0168409
65	0.02207	0.02735	0.03104	0.012161	0.0150751	0.0168118
70	0.02149	0.02634	0.02952	0.0119155	0.0144347	0.016079
75	0.02082	0.02587	0.02882	0.0113417	0.0140585	0.0156581
80	0.02043	0.0252	0.02871	0.0110623	0.0135801	0.0151916
85	0.01971	0.02393	0.02707	0.0105783	0.0129985	0.014651
90	0.01926	0.02401	0.02659	0.0102163	0.0126186	0.0143817
95	0.01840	0.02251	0.02519	0.0102966	0.012609	0.0141236
100	0.01769	0.02236	0.02503	0.00984433	0.0119589	0.0133261

FIG. 1

The relation between sample size and critical values.



Pitman Asymptotic Relative Efficiency

Since the aforementioned test statistic $\widehat{\Delta}(s) = \frac{\delta}{\overline{X}}$ is new and no other tests are known for this class (UBAL), we may compare our test to the other classes. Here we choose the test $\Delta_{\theta,(1)}$ presented by Mugdadi and Ahmad [8], $\delta^{(2)}_{F_n}$ presented by Mahmoud and Abdul Alim [7] for the new better than average failure rate class of life distribution, $\widehat{\Delta}_G$ presented by Abu-Youssef and Bakr [10], $\widehat{\Delta}_k$ presented by Abu-Youssef, Mohammed, and Bakr [12], and $\widehat{\Delta}_{ut}$ presented by Abu-Youssef, Mohammed, and Bakr [13] for the (UBACT) class of life distribution. Then, comparisons are achieved by using Pitman asymptotic relative efficiency (PARE), which is defined as follows:

Let T_{1n} and T_{2n} be two statistics; then PARE of T_{1n}, relative to T_{2n}, is defined by the following:

$$e(T_{1n}, T_{2n}) = \frac{\mu_1^{\backslash}(\theta_0)}{\sigma_1(\theta_0)} \Big/ \frac{\mu_2^{\backslash}(\theta_0)}{\sigma_2(\theta_0)}$$

where $\mu_i^{\backslash}(\theta_0) = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{n_i})\Big|_{\theta\to\theta_0}$ and we have the following:

$$\sigma_i^2(\theta_0) = \lim_{n\to\infty} \text{var}(T_{n_i})$$

Three of the most commonly used alternatives are as follows:

(i) LFR family:

$$\overline{F}_1(x) = e^{-x-\frac{x^2}{2}\theta},\ \theta, x \geq 0 \tag{7}$$

(ii) Makeham family:

$$\overline{F}_2(x) = e^{-x-\theta(x+e^{-x}-1)},\ \theta, x \geq 0 \tag{8}$$

(iii) Weibull family:

$$\overline{F}_3(x) = e^{-x^{\theta}},\ \theta \geq 1, x \geq 0 \tag{9}$$

Note that H_0 (the exponential distribution) is attained at $\theta = 0$ in (i) and (iii) and when $\theta = 1$ in (ii). The PAE of $\delta(s)$ is equal to the following:

$$\text{PAE}(\delta(s)) = \frac{\left|\frac{\partial}{\partial\theta}\delta(s)\right|_{\theta\to\theta_0}}{\sigma_0(s)} = \frac{1}{\sigma_0(s)}\left|\frac{s}{s+1}\int_0^\infty \overline{F}'_{\theta_0}(x)dx - (s+1)\int_0^\infty e^{-sx} d\overline{F}'_{\theta_0}(x)\right|$$

where $\overline{F}'_{\theta_0}(x) = \frac{d}{d\theta}\overline{F}_\theta(u)\Big|_{\theta\to\theta_0}$.

This leads to the following for $s = 2$ and 3:

(i) PAE in the case of the LFR distribution:

$$\text{PAE}(\delta(2)) = \frac{1}{\sigma_0(2)}\left|\frac{-1}{12}\int_0^\infty x^2 e^{-x} dx - \frac{1}{4}\int_0^\infty e^{-sx} d\left(-\frac{x^2}{2}e^{-x}\right)\right| = 2.15$$

$$\mathrm{PAE}(\delta(3)) = \frac{1}{\sigma_0(3)}\left|\frac{-1}{12}\int_0^\infty x^2 e^{-x}dx - \frac{1}{4}\int_0^\infty e^{-sx} d\left(-\frac{x^2}{2}e^{-x}\right)\right| = 1.87$$

(ii) PAE in the case of the Weibull distribution:

$$\mathrm{PAE}(\delta(2)) = \frac{1}{\sigma_0(2)}\left|\frac{-1}{6}\int_0^\infty \mathrm{x}\ln(x)e^{-x}dx - \frac{1}{4}\int_0^\infty e^{-sx} d(-\mathrm{x}\ln(x)e^{-x})\right| = 0.38$$

$$\mathrm{PAE}(\delta(3)) = \frac{1}{\sigma_0(3)}\left|\frac{-1}{6}\int_0^\infty \mathrm{x}\ln(x)e^{-x}dx - \frac{1}{4}\int_0^\infty e^{-sx} d(-\mathrm{x}\ln(x)e^{-x})\right| = 0.32$$

(iii) PAE in the case of the Makeham distribution:

$$\mathrm{PAE}(\delta(2)) = \frac{1}{\sigma_0(2)}\left|\frac{1}{6}\int_0^\infty (1-x-e^{-x})e^{-x}dx - \frac{1}{4}\int_0^\infty e^{-sx} d((1-x-e^{-x})e^{-x})\right| = 1.13$$

$$\mathrm{PAE}(\delta(3)) = \frac{1}{\sigma_0(3)}\left|\frac{1}{6}\int_0^\infty (1-x-e^{-x})e^{-x}dx - \frac{1}{4}\int_0^\infty e^{-sx} d((1-x-e^{-x})e^{-x})\right| = 0.97$$

Direct calculations of the PAE of $\delta(3), \Delta_{\theta,(1)}, \delta_{F_n}^{(2)}, \widehat{\Delta}_{ut}, \widehat{\Delta}_k,$ and $\widehat{\Delta}_G$ are summarized in Table 2; the efficiencies in the table clearly show our U-statistic $\delta(s)$ performs well for $\mathrm{F}_1, \mathrm{F}_2,$ and F_3.

In Table 3, we give the PAREs of $\delta(3)$ with respect to $\widehat{\Delta}_{\theta,(1)}, \delta_{F_n}^{(2)}, \widehat{\Delta}_{ut}, \widehat{\Delta}_k,$ and $\widehat{\Delta}_G$, whose PAE are mentioned in Table 2.

It is clear from Table 3 that the statistic $\delta(3)$ is more efficient than $\Delta_{\theta,(1)}, \delta_{F_n}^{(2)}, \widehat{\Delta}_{ut}, \widehat{\Delta}_k,$ and $\widehat{\Delta}_G$ for all the cases mentioned and gives higher efficiency whenever s is decreased. Hence our test, which deals the much larger UBA, is better and also simpler.

TABLE 2
PAE of $\delta(3), \Delta_{\theta,(1)}, \delta_{F_n}^{(2)}, \widehat{\Delta}_{ut}, \widehat{\Delta}_k,$ and $\widehat{\Delta}_G$.

Distribution	$\delta(3)$	$\Delta_{\theta,(1)}$	$\delta_{F_n}^{(2)}$	$\widehat{\Delta}_{ut}$	$\widehat{\Delta}_k$	$\widehat{\Delta}_G$
LFR	1.87	0.408	0.217	0.748	0.776	1.496
Makeham	0.97	0.0395	0.144	0.248	0.255	0.495
Weibull	0.32	0.170	0.050	–	–	–

TABLE 3
PARE of $\delta(3)$, with respect to $\Delta_{\theta,(1)}, \delta_{F_n}^{(2)}, \widehat{\Delta}_{ut}, \widehat{\Delta}_k,$ and $\widehat{\Delta}_G$.

Distribution	$e(\delta(3), \Delta_{\theta,(1)})$	$e\left(\delta(3), \delta_{F_n}^{(2)}\right)$	$e(\delta(3), \widehat{\Delta}_{ut})$	$e(\delta(3), \widehat{\Delta}_k)$	$e(\delta(3), \widehat{\Delta}_G)$
LFR	4.58	8.62	2.5	2.41	1.25
Makeham	24.56	6.74	3.91	3.80	1.96
Weibull	1.88	6.4	–	–	–

TABLE 4
Power estimate for our test $\widehat{\Delta}(2)$.

n	θ	Weibull	LFR	Gamma	Makeham
10	2	0.9936	0.0183	0.5882	0.1451
	3	0.9999	0.1130	0.9832	0.9964
	4	1	0.3508	1	1
20	2	1	0.0656	0.8622	0.9982
	3	1	0.2318	0.9998	1
	4	1	0.4818	1	1
30	2	1	0.2658	0.9560	0.9988
	3	1	0.4888	1	1
	4	1	0.6808	1	1

The Power of the Proposed Test

The power of the proposed test at a significance level $\alpha = 0.05$, with respect to the alternatives F_1, F_2 and F_3, is calculated based on simulation data. In this simulation, 10,000 samples were generated with the Mathematica 8 program. Table 4 gives the power of the test at different values of parameter $\theta = 2$, 3, and 4, with sizes $n = 10$, 20, and 30.

Test for UBAL in Case for Right Censored Data

In this section, a test statistic is proposed to test:

H_0 ($\overline{F}$ is the exponential distribution with mean μ) versus H_1 ($\overline{F}$ is UBAL and not an exponential distribution), with randomly right-censored data.

It is known that censored data are usually the only information available in a life-testing model or in a clinical study in which patients may be lost (censored) before the completion of a study. We can describe the experimental situation as follows. Suppose n units are put on a test, and $X_1, X_2, \ldots, X_n$ denote their true lifetime. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) according to a continuous life distribution F.

Let $Y_1, Y_2, \ldots, Y_n$ be (i.i.d.) according to a continuous life distribution G. Also, we assume that Xs and Ys are independent. In the randomly right-censored model, we observe the pairs $(Z_i, \delta_i), i = 1, \ldots, n$, where $Z_i = \min(X_i, Y_i)$, and we assume $\mu(\infty)$ is known and equal to one. See the following:

$$\delta_i = \begin{cases} 1 & if \quad Z_i = X_i \,(i\text{th observation is uncensored}) \\ 0 & if \quad Z_i = Y_i \,(i\text{th observation is censored}) \end{cases}$$

Let $Z_{(0)} < Z_{(1)} < \ldots < Z_{(n)}$ denote the order of Z' s and δ_i be the δ corresponding to $Z_{(i)}$, respectively. Use the Kaplan and Meier estimator in the case of censored data $(Z_i, \delta_i), i = 1, \ldots, n$ as follows:

$$\delta_c(s) = \frac{1}{s(1+s)} \sum_{j=1}^{l} \prod_{k=1}^{j-1} C_k^{\delta_k} (Z_{(j)} - Z_{(j-1)}) - \frac{1}{s^2} \left(1 - \sum_{m=1}^{l} e^{-sZ_{(m)}} \left[\prod_{p=1}^{m-2} C_p^{\delta_p} - \prod_{p=1}^{m-1} C_p^{\delta_p} \right] \right),$$

where $\widehat{\mu} = \sum_{j=1}^{l} \prod_{k=1}^{j-1} C_k^{\delta_k} (Z_{(j)} - Z_{(j-1)}), \varnothing(s) = \int_0^\infty e^{-sx} \mathrm{dF}(x)$,

$$\widehat{\varnothing}(s) = \sum_{m=1}^{l} e^{-sZ_{(m)}} \left(\prod_{p=1}^{m-2} C_p^{\delta_p} - \prod_{p=1}^{m-1} C_p^{\delta_p} \right) dF_n(Z_i) = \prod_{q=1}^{j-2} C_i^{\delta_i} - \prod_{q=1}^{j-1} C_i^{\delta_i}, \overline{F}_n(t) = \prod_{m<t} C_m^{\delta_m},$$

$$C_m = \frac{n-m}{n-m+1}, t \in [0, z_{(m)}].$$

TABLE 5

The upper percentile points of $\delta(2)$.

n	95 %	98 %	99 %
5	0.11946	0.16961	0.22646
10	0.08251	0.10732	0.12622
15	0.06748	0.08496	0.09439
20	0.05503	0.06574	0.07675
25	0.05225	0.06322	0.06987
30	0.04390	0.05348	0.06051
35	0.04300	0.05266	0.06006
39	0.04155	0.04931	0.05303
40	0.03985	0.04841	0.05128
45	0.03814	0.04747	0.05333
50	0.03669	0.04462	0.04844

FIG. 2

Relation between sample size and critical values.

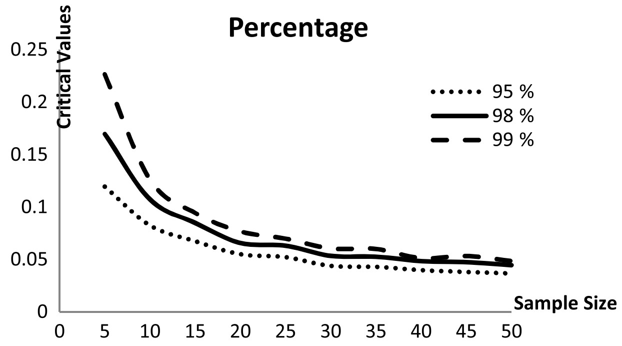


Table 5 gives the critical values of $\delta_c(2)$ for sample sizes $n = 5(5)\ 50,\ (39)$.

From Table 5 and Fig. 2, the critical values decrease as the sample size increases, and they increase as the confidence level increases.

Applications

Here, we introduce some real examples to elucidate the applications of our test in the two cases (censored and non-censored data) at 95 % confidence level.

CASE OF COMPLETE DATA

In this section, two examples are presented, considering $s = 2$.

Example 1

Consider the data in Table 6, Abouammoh, Abdulghani, and Qamber [15]; these data represent a set of 40 patients suffering from blood cancer (leukemia) from one of the Ministry of Health hospitals in Saudi Arabia, and the ordered values in years are shown in Table 6.

It was found that $\widehat{\delta}(2) = 0.09$; that is greater than the critical value of Table 1. Then, we conclude that this dataset has a UBAL property and is not exponential.

TABLE 6
Abouammoh data ordered values in years.

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	4.381

TABLE 7
Fisher data differences in plants heights.

4.9	−6.7	0.8	1.6	0.6
2.3	2.8	4.1	1.4	2.9
5.6	2.4	7.5	6.0	−4.8

TABLE 8
Non-censored lifetimes (in days) of 39 liver cancer patients data.

10	14	14	14	14	14	15	17	18
20	20	20	20	20	23	23	24	26
30	30	31	40	49	51	52	60	61
67	71	74	75	87	96	105	107	107
107	116	150						

TABLE 9
Censored lifetimes (in days) of 12 liver cancer patients data.

30	30	30	30	30	60	150	150	150	150	150	185

Example 2

Consider the data in **Table 7** from Ref. Fisher [16] that represent the differences in heights between cross- and self-fertilized plants of the same pair grown together in one pot.

It was found that $\widehat{\delta}_n(2) = 0.02$, which is less than the critical value of **Table 1**. Then, we accept the null hypothesis that states that the dataset has an exponential property.

CASE OF CENSORED DATA

In this section, an example is presented, considering $S = 2$.

Example 3

Consider the data in **Table 8** from Ref. Mahmoud and Abdul Alim [7] that represent 51 liver cancer patients, taken from the Elminia Cancer Center Ministry of Health in Egypt. Of them, 39 represent whole lifetimes (non-censored data) and the others represent censored data. The ordered non-censored lifetimes (in days) are shown in **Table 8**.

The ordered censored data are shown in **Table 9**.

One can calculate $\delta(2) = 1.1 \times 10^{74}$ that is greater than the critical value of **Table 5**. Then we conclude that this dataset has a UBAL property and is not exponential.

Conclusions

In this work, a test statistic for testing exponentiality versus the UBAL class of life distribution based on a U-statistic is proposed. The PAEs of the test are calculated and compared to other tests. The percentiles of this test statistic are tabulated for censored and non-censored data, and the powers of this test are estimated for some famously alternative distributions in reliability, such as Weibull, LFR, and Gamma distributions. Finally, examples in different areas are used as practical applications of the proposed test.

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