King Saud University,
College of Sciences
Mathematical Department.

Mid-Term2 /S1/2019
Full Mark:25. Time 1H30mn
Math 204 14/11/2019

Question 1[4,4] **a)** Find the largest local interval for which the following initial value problem has a unique solution

$$\begin{cases} (x-1)^3 y'' + y' \ln(3-x) + \frac{1}{\sqrt{x}} y = e^x \\ y(2) = 0, \ y'(2) = 1. \end{cases}$$

b) By using the method of undetermined coefficients, find only the form of the particular solution of the differential equation

$$y''' - y'' - 4y' + 4y = -3xe^x + 5e^{-2x} + \sin(2x).$$

Question 2 [3,5]. **a)** Determine a homogeneous differential equation with constant coefficients having the set of fundamental solutions

$$\{2, \ e^{-x}, \ 3x, \ 5\sin x\}, \cos x\ \}$$

$$\{2, e^{-x}, 3x, 5\sin x, \cos x\}$$

b) If $y_1 = e^{-2x}$ is a solution of the differential equation

$$(1+2x)y'' + 4xy' - 4y = 0, \quad x > -\frac{1}{2},$$

then find its general solution..

Question 3 [5] Find the general solution of the differential equation

$$x^2 y'' - 2y = \frac{1}{x}, \qquad x > 0.$$

Question 4 [4] Show wether the functions $f_1(x) = \sin x$, $f_2(x) = \cos x$, $f_3(x) = \sin(x-5)$ are linearly dependent or independent on $\mathbb{R}$.

Complete Solutions of Mid-two

M 204, First Semester 1441H.

Question 1

(a) $\begin{cases} (x-1)^3 y'' + y' \ln(3-x) + \dfrac{1}{\sqrt{x}} y = e^x \\ y(2)=0, \quad y'(2)=1 \end{cases}$

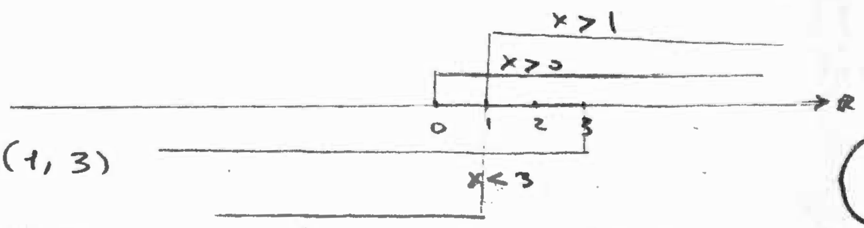
4

$a_2(x) = (x-1)^3$ is Continuous on $\mathbb{R}$ and $a_2(x) \neq 0 \ \forall x \neq 1$ 1

$a_1(x) = \ln(3-x)$ is Continuous on $x<3$

$a_0(x) = \dfrac{1}{\sqrt{x}}$ is Continuous on $x>0$

$g(x) = e^x$ is Continuous on $\mathbb{R}$ 1



As $2 \in (1, 3)$ 2

Then the longest interval for which the IVP has a unique Solution is $\boxed{I = (1,3)}$

(b) $y''' - y'' - 4y' + 4y = -3xe^x + 5e^{-2x} + \sin(2x)$ 4

$y''' - y'' - 4y' + 4y = 0 \Rightarrow m^3 - m^2 - 4m + 4 = 0$ 1

$m^2(m-1) - 4(m-1) = 0$

$(m^2-4)(m-1) = (m+2)(m-2)(m-1) = 0$ 1

$m = -2, 2, 1$

Then $\boxed{y_p = x(Ax+B)e^x + Cxe^{-2x} + D\sin(2x) + E\cos(2x)}$ 2

Question 2

(a) $\{2, e^{-x}, 3x, 5\sin x\}, \cos x$ 3

$y = c_1(2)e^{0x} + c_2(3x)e^{0x} + c_3 e^{-x} + c_4(5\sin x) + c_5 \cos x$

Then $m = 0, 0, \quad m = -1, \quad m^2+1 = 0,$ 1

So we have the characteristic equation: $m^2(m+1)(m^2+1) = 0$

$(m^3+m^2)(m^2+1) = m^5 + m^3 + m^4 + m^2 = 0$, and then

$\boxed{y^{(5)} + y^{(4)} + y''' + y'' = 0}$ is the D.E 2

(b) $(1+2x)y'' + 4xy' - 4y = 0$; $x > \frac{-1}{2}$, $y_1 = e^{-2x}$ is a solution of the D.E

$y'' + 4\frac{x}{1+2x}y' - \frac{4}{1+2x}y = 0$, $P(x) = \frac{4x}{1+2x} = 2\frac{1+2x-1}{1+2x}$

$P(x) = 2\left(1 - \frac{1}{1+2x}\right) = 2 - \frac{2}{1+2x}$ 1

5 $e^{-\int P(x)dx} = e^{-2x} \cdot e^{+\ln(1+2x)} = e^{-2x}(1+2x)$ ✓

$y_2 = y_1 \int \frac{e^{\int -P(x)dx}}{y_1^2} dx = y_1 \int \frac{e^{-2x}(1+2x)}{e^{-4x}} dx$ ✓ 1

✓ $y_2 = e^{-2x}\left[\int e^{2x}(1+2x)dx\right]$ $u = 1+2x$, $dv = e^{2x}dx$
$du = 2dx$, $v = \frac{1}{2}e^{2x}$

$= e^{-2x}\left[\frac{1}{2}(1+2x)e^{2x} - \int \frac{1}{2}e^{2x} 2\,dx\right]$ 1

$= e^{-2x}\left[\frac{1}{2}(1+2x)e^{2x} - \frac{1}{2}e^{2x}\right]$

✓ $= \frac{1}{2}(2x+1) - \frac{1}{2} = x$ $\boxed{y_2 = x}$ 2

So the G. solution of the D.E is $\boxed{y = c_1 e^{-2x} + c_2 x}$

Question 3 $x^2 y'' - 2y = \frac{1}{x}$; $x > 0$ $x^2 u_1' + x^{-1} u_2' = 0$
$x u_1 \quad u_2' = \frac{1}{x^3}$

Q3 i) $x^2 y'' - 2y = 0$, $y = x^m$

$m(m-1) - 2 = 0 \Rightarrow m^2 - m - 2 = (m-2)(m+1) = 0$

$m = 2$, $m = -1$ 1

8 Then $y_c = c_1 x^2 + c_2 x^{-1}$

$W = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -3 \neq 0$ $W_1 = \begin{vmatrix} 0 & x^{-1} \\ x^{-3} & -x^{-2} \end{vmatrix} = -x^{-4}$ $W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & x^{-3} \end{vmatrix} = x^{-1}$ 1

$u_1' = \frac{W_1}{W} = \frac{+1}{3}x^{-4}$, $u_1 = \frac{+1}{3}\int x^{-4}dx = \frac{-1}{9}x^{-3}$ where $y = x^2$ and $y_1 = x^{-1}$

$u_2' = \frac{W_2}{W} = \frac{-1}{3}x^{-1}$, $u_2 = \frac{-1}{3}\int \frac{dx}{x} = \frac{-1}{3}\ln x$ 1

$$y_p = y_1 u_1 + y_2 u_2 = x^2\left(\frac{-1}{9}x^{-3}\right) + x^{-1}\left(\frac{-1}{3}\ln x\right)$$

$$\boxed{y_p = \frac{-1}{9}x^{-1} + \frac{1}{3}x^{-1}\ln x}$$

(2)

Then the G. solution is

$$y_G = y_c + y_p = c_1 x^2 + c_2 x^{-1} - \frac{1}{9}x^{-1} - \frac{1}{3}x^{-1}\ln x$$

or

$$y_G = c_1 x^2 + \left(c_2 - \frac{1}{9}\right)x^{-1} - \frac{1}{3}x^{-1}\ln x$$

$$\boxed{y_G = c_1 x^2 + c_3 x^{-1} - \frac{1}{3}x^{-1}\ln x \text{ ;}} \quad \text{where } c_3 = c_2 - \frac{1}{9}$$

Q4 (4)

$f(x) = \sin x$, $f_2 = \cos x$, $f_3(x) = \sin(x-5)$

$f_3(x) = \sin(x-5) = \sin x \cos 5 - \sin 5 \cos x = (\cos 5) f_1(x) + (-\sin 5) f_2(x)$ (1)

$\underbrace{(\cos 5)}_{c_1} f_1(x) + \underbrace{(-\sin 5)}_{c_2} f_2(x) + \underbrace{(-1)}_{c_3} f_3(x) = 0$ for all $x \in \mathbb{R}$ (1)

$(c_1 = \cos 5;\ c_2 = -\sin 5,\ c_3 = -1) \neq (0, 0, 0)$

Then f_1, f_2 and f_3 are linearly dependant on $\mathbb{R}$ (1)
