

Solution Key:

King Saud University
College of Sciences
Department of Mathematics

Math-244 (Linear Algebra); Final Exam; Semester 442

Max. Marks: 40

Time: 3 hours

Question 1 [Marks: 5+5]:

I. Choose the correct answer:

- (i) Let B and C be ordered bases of a vector space V with transition matrix ${}_C P_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. If

the coordinate vector $[v]_C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ then the coordinate vector $[v]_B$ is:

- (a) (1,3,6) (b) (6,3,1) (c) ✓ (1,1,1) (d) (1,1,2).
- (ii) The dimension of the column space $col(A^t)$ of $A = \begin{bmatrix} 2 & 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is:
- (a) 1 (b) ✓ 2 (c) 4 (d) 5.
- (iii) If $U = \begin{bmatrix} -1 & 3 \\ y & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 5 & 2y \\ -4 & 1 \end{bmatrix}$ are two orthogonal matrices with respect to the inner product $\langle A, B \rangle = trace(AB^t)$ on the vector space M_2 of 2×2 real matrices, then:
- (a) ✓ $y = 2$ (b) $y = -2$ (c) $y = 0$ (d) $y = 1$.
- (iv) If the inner product on the vector space \mathbf{P}_2 of polynomials with degree ≤ 2 is defined by $\langle p, q \rangle = aa_1 + 2bb_1 + cc_1$, $\forall p = a + bx + cx^2$, $q = a_1 + b_1x + c_1x^2 \in \mathbf{P}_2$ and θ is the angle between the polynomials $1 + x - x^2$ and $2 + x - 2x^2$, then:
- (a) $\cos \theta = \frac{5}{3\sqrt{3}}$ (b) $\cos \theta = \frac{2}{\sqrt{3}}$ (c) ✓ $\cos \theta = \frac{3}{\sqrt{10}}$ (d) $\cos \theta = 1$.
- (v) If $S = \{v_1 = (2,1), v_2 = (1,0)\}$ is a basis for Euclidean space \mathbb{R}^2 and $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear transformation defined by $T(v_1) = (1,5)$ and $T(v_2) = (0,3)$, then $T(4,6)$ is equal to:
- (a) ✓ (6,6) (b) (-8, -22) (c) (-10, -8) (d) (4,23).

II. Determine whether the following statements are true or false; justify your answer.

- (i) If $\{(-3r + 4s, r - s, r, s) : r, s \in \mathbb{R}\}$ is the solution space of homogeneous system $AX = 0$, then $nullity(A) = 2$.
True: $\{(-3, 1, 1, 0), (4, -1, 0, 1)\}$ is a basis for the null space.
- (ii) For any $m \times n$ matrix A , $dim(N(A^t)) + dim(col(A)) = m$.
True: $dim(row(A^t)) = dim(col(A)) \Rightarrow dim(N(A^t)) + dim(row(A^t)) = nullity(A^t) + rank(A^t) = m$.
- (iii) The transformation $T: \mathbb{R} \rightarrow \mathbb{R}$ given by $T(r) = |r|$ is linear.
False: for example, $|-1 + 1| = 0 \neq 2 = |-1| + |1|$. (Or, $|(-1)1| = 1 \neq -1 = -1|1|$).
- (iv) Eigenvalues of any matrix are same as the eigenvalues of its reduced row echelon form.
False: for example, 2 is an eigenvalue of $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ but neither of the two eigenvalues of its RREF is 2.
- (v) If the characteristic polynomial of a matrix A is $q_A(\lambda) = \lambda^2 - 2$, then A is diagonalizable.
True: the quadratic characteristic polynomial gives two different eigenvalues of 2×2 matrix A .

Question 2 [Marks: 2.5+1+2.5]: Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$. Then:

(i) Find a basis for $\text{col}(A)$.

Solution: $\begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (REF). So, $\{(1,-1,0,1), (0,1,2,1), (3,-3,1,4)\}$ is a basis for $\text{col}(A)$.

(ii) Find $\dim(\text{row}(A))$.

Solution: $\dim(\text{row}(A)) = \dim(\text{col}(A)) = 3$ as is clear from the above Part (i).

(iii) Find a basis for the null space $N(A)$.

Solution: The REF of A obtained in Part (i) gives the basis $\{(2,-3,1,0,0), (-1,0,0,1,0)\}$ for $N(A)$.

Question 3 [Marks: 3+3]: Let $E = \{v_1 = (1,1, -4, -3), v_2 = (2,0,2, -2), v_3 = (2, -1,3,2)\}$. Then:

(i) Find a basis B for the vector space $\text{span}(E)$ such that $B \subseteq E$. If $E - B \neq \emptyset$, then express each element of $E - B$ as linear combination of the basic vectors.

Solution: The set E is linearly independent and so $B = E$ is a basis for $\text{span}(E)$; this completes the solution of Part (i).

(ii) Use the basis B (as in Part (i)) to find a basis C for the Euclidean space \mathbb{R}^4 .

Solution: It is easily seen that $\{v_1, v_2, v_3, (1,0,0,0)\}$ being linearly independent is a basis for the space \mathbb{R}^4 .

Question 4: [Marks: 2+4]

a) Let $\{u, v, w\}$ be an orthogonal set of vectors in an inner product space. Then show that:

$$\|u\|^2 + \|v\|^2 + \|w\|^2 = \|u + v + w\|^2.$$

Solution: $\|u + v + w\|^2 = \langle u + v + w, u + v + w \rangle = \|u\|^2 + 2(\langle u, v \rangle + \langle u, w \rangle + \langle v, w \rangle) + \|v\|^2 + \|w\|^2$. Hence, by the given orthogonality of $\{u, v, w\}$, $\|u + v + w\|^2 = \|u\|^2 + \|v\|^2 + \|w\|^2$.

b) Let $A = \{u_1 = (1,1,1), u_2 = (0,1,-1), u_3 = (3,-2,2)\}$. Use the Gram-Schmidt algorithm to obtain an orthonormal set B of vectors such that $\text{span}(B) = \text{span}(A)$.

Solution: Put $v_1 := u_1 = (1,1,1)$, $v_2 := u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = u_2 = (0,1,-1)$, $v_3 := u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = (2, -1, -1)$.

Then, $B := \{w_1 := \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{3}}(1,1,1), w_2 := \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}}(0,1,-1), w_3 := \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{6}}(2, -1, -1)\}$ is as required.

Question 5: [Marks: (2+1.5+2.5) + (2.5+1+2.5)]

a) Let the linear transformation $T: M_2 \rightarrow \mathbb{R}^2$ be defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b)$, $\forall a, b, c, d \in \mathbb{R}$. Then find:

(i) A basis for $\ker(T)$.

Solution: Clearly, $\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} : c, d \in \mathbb{R} \right\} = \text{span}\left(\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}\right)$; hence, $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ being linearly independent in M_2 is a basis for $\ker(T)$.

(ii) $\text{rank}(T)$.

Solution: $\text{rank}(T) = \dim(M_2) - \text{nullity}(T) = 4 - 2 = 2$ from the solution of Part (i).

(iii) The standard matrix $[T]_B^C$, where B and C are the standard bases of M_2 and \mathbb{R}^2 , respectively.

Solution: $[T]_B^C = \left[\begin{matrix} [T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)]_C \\ [T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)]_C \\ [T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)]_C \\ [T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)]_C \end{matrix} \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ y & z & -3 \end{bmatrix}$. Then:

(i) Find the values of x , y and z such that $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -3$ are the eigenvalues of A with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, respectively.

Solution: $\begin{bmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ y & z & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ x+2 \\ y+z-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x+2=1, y+z-3=1$; and $\begin{bmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ y & z & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow z-3=2$.

Hence, $x = -1, y = -1, z = 5$.

(ii) Use the values of x, y and z (as in Part (i)) to show that the matrix A is diagonalizable.

Solution: Since A is a 3×3 matrix having 3 different eigenvalues, it is diagonalizable.

(iii) Find A^5 .

Solution: Since the matrix A is diagonalizable having eigenvalues 1, 2, -3 with corresponding eigen vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ respectively, we get } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ with } P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

satisfying $A = PD P^{-1}$. Hence, $A^5 = P D^5 P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -31 & 32 & 0 \\ -31 & 275 & -243 \end{bmatrix}$.

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