King Saud University
College of Sciences
Department of Mathematics
MATH-244 (Linear Algebra); Final Exam; Semester 441

Max. Marks: 40 **Time: 3 hours**

Name: **ID:** **Section:** **Signature:**

Note: ***Attempt all the five questions. Scientific calculators are not allowed***!

Question 1 [Marks: 5+5]**:**

I. Choose the correct answer:

(i) If W is the subspace $\{(a,b,c,d) \in \mathbb{R}^4 : b = a - c\}$ of Euclidean space $\mathbb{R}^4$, then $\dim(W)$ is:
a) 1 b) 2 c) 3 d) 4.

(ii) If $rank(A) = 3$ where A is a matrix of size 5×9, then $nullity(A^T)$ is:
a) 1 b) 2 c) 3 d) 6.

(iii) If θ is the angle between the matrices $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$ with respect to the inner product $< A, B > = trace\,(AB^{\mathrm{T}})$, then $\cos\theta$ is:
a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) 0.

(iv) The value of k for which the vectors $\boldsymbol{u} := (u_1 = 2, u_2 = -4)$ and $\boldsymbol{v} := (v_1 = 1, v_2 = 3)$ in $\mathbb{R}^2$ are orthogonal with respect to the inner product $\langle \boldsymbol{u},\boldsymbol{v} \rangle = 2u_1v_1 + ku_2v_2$ is:
a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) $\frac{1}{3}$.

(v) If $B = \{(2,1), (-3,4)\}$ and $C = \{(1,1), (0,3)\}$ are bases of $\mathbb{R}^2$, then the transition matrix ${}_{B}P_{C}$ from C to B is:
a) $\begin{bmatrix} 7/11 & 1/11 \\ 9/11 & 6/11 \end{bmatrix}$ b) $\begin{bmatrix} 7/11 & 9/11 \\ 1/11 & 6/11 \end{bmatrix}$ c) $\begin{bmatrix} 7/11 & 9/11 \\ 6/11 & 1/11 \end{bmatrix}$ d) $\begin{bmatrix} 9/11 & 7/11 \\ 1/11 & 6/11 \end{bmatrix}$.

II. Determine whether the following statements are true or false; justify your answer.

(i) If $A, B \in M_n(\mathbb{R})$, then $det\left(A^{\mathrm{T}}B\right) = det\left(B^{\mathrm{T}}A\right)$.

(ii) A basis for solution space of the following linear system is $\{(4,1,0,0), (-3,0,1,0)\}$:
$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 3x_4 = 0.$$

(iii) If $W = \{A \in M_2(\mathbb{R}) : A \text{ is singular}\}$, then W is vector subspace of $M_2(\mathbb{R})$.

(iv) If u, v and w are vectors in an inner product space such that $\langle u,v \rangle = 3$, $\langle v,w \rangle = -5$, $\langle u,w \rangle = -1$ and $\|u\| = 2$, then $\langle u - 2w, 3u + v \rangle = 25$.

(v) If the characteristic polynomial of 2×2 matrix A is $q_A(\lambda) = \lambda^2 - 1$, then A is diagonalizable.

Question 2 [Marks: 2+2+2]**:** Consider the matrices $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & -2 \end{bmatrix}$. Then:

a) Find A^{-1} by the elementary matrix method.
b) Show that $nullity(A) \neq nullity(B)$.
c) Find a basis for the null space $N(B)$.

Question 3 [Marks: 3+3]**:**

a) Find the values of x so that the set $\{(1,-2,x),(1,-x,2),(1,-4,2x)\}$ is linearly independent in the Euclidean space $\mathbb{R}^3$.
b) Let $\boldsymbol{F} := span(\{(1,-1,0,1),(0,1,0,-1),(-1,2,0,-1)\})$ in $\mathbb{R}^4$. Find a basis for $\boldsymbol{F}$ and show that $(0,1,0,0) \in \boldsymbol{F}$.

Question 4: [Marks: 2+4]

a) Let u and v be any two vectors in an inner product space. Show that:
$$2(\|u\|^2 + \|v\|^2) = \|u+v\|^2 + \|u-v\|^2.$$
b) Let the set $B := \{\boldsymbol{u}_1 = (1,0,0), \boldsymbol{u}_2 = (3,1,-1), \boldsymbol{u}_3 = (0,3,1)\}$ be linearly independent in the Euclidean inner product space $\mathbb{R}^3$. Construct an orthonormal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on B.

Question 5: [Marks: (4+2) + (2+2+2)]

a) Let $\boldsymbol{B} = \{(1,1,0),(0,1,1),(1,0,1)\}$ be a basis for $\mathbb{R}^3$, $\boldsymbol{C} = \{x+1, x-1, x^2+1\}$ be a basis for P_2 (the vector space of all real polynomials in variable x of degree ≤ 2 . Let $\boldsymbol{T}: \mathbb{R}^3 \to P_2$ be the linear transformation: $\boldsymbol{T}(a,b,c) = (a+b) + (b+c)x + (a+c)x^2$, $\forall (a,b,c) \in \mathbb{R}^3$. Then:
   (i) Find the values of q, r, s in the transformation matrix $[\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$ with respect to the bases $\boldsymbol{B}$ and $\boldsymbol{C}$.
   (ii) Find the coordinate vector $[T(1,1,1)]_{\boldsymbol{C}}$.

b) Let $A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$. Then:
   (i) Show that the matrix A is diagonalizable.
   (ii) Find an invertible matrix P and a diagonal matrix D satisfying $P^{-1}AP = D$.
   (iii) Find A^7.

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