EXPECTATIONS INVOLVING INDEPENDENT R. V. 'S

Q1) Let X_1, X_2 and X_3 be independent r.v.'s with means 4, 9, 3 and variances 3, 7, 5 respectively. For $Y = 2X_1 - 3\,X_2 + 4\,X_3$ and $Z = X_1 + 2\,X_2 - \ X_3$, find:

a. E(Y) and E(Z).

b. V(Y) and V(Z).

Solution :

$E(X_1) = 4\,, Var(X_1) = 3$

$E(X_2) = 9\,, Var(X_2) = 7$

$E(X_3) = 3\,, Var(X_3) = 5$

a) $E(Y) = E(2X_1 - 3X_2 + 4X_3) = 2E(X_1) - 3E(X_2) + 4E(X_3)$
$= 2(4) - 3(9) + 4(3) = 8 - 27 + 12 = -7$
$E(Z) = E(X_1 + 2X_2 - X_3) = E(X_1) + 2E(X_2) - E(X_3)$
$= 4 + 2(9) - 3 = 4 + 18 - 3 = 19$

b) Since X_1, X_2 *and* X_3 are independent
$V(Y) = V(2X_1 - 3X_2 + 4X_3) = 4\,V(X_1) + 9\,V(X_2) + 16\,V(X_3)$
$= 4(3) + 9(7) + 16(5) = 12 + 63 + 80 = 155$

$V(Z) = V(X_1 + 2X_2 - X_3) = V(X_1) + 4V(X_2) + V(X_3)$

$= 3 + 4(7) + 5 = 3 + 28 + 5 = 36$

Q2) If X and Y are independent r.v.'s with E(X)=3, E(Y)=5, V(X)=2, and V(Y)=5, find:

a. E(XY)

b. $E(X^2Y)$

Solution :

X and Y are independent

a) $E(XY) = E(X)E(Y) = 3(5) = 15$

b) $E(X^2Y) = E(X^2)E(Y) = 11(5) = 55$

Where $V(X) = E(X^2) - [E(X)]^2 \iff 2 = E(X^2) - 3^2 \iff E(X^2) = 2 + 9 = 11$

Q3) Let X and Y are independent r.v's with p.d.f $f(x) = e^{-x}; x > 0,$

$f(y) = e^{-y}; y > 0,$ find :

a. E(X) and V(X).
b. E(Y) and V(Y).
c. E(XY).
d. $E(X^2\, Y^3)$.

Solution :

a) $\boldsymbol{E(X)} = \int_0^\infty x\, e^{-x}\, dx = \frac{\Gamma(2)}{1^2} = 1$ $\left[by\, use \int_0^\infty x^a\, e^{-b\,x}\, dx = \frac{\Gamma(a+1)}{b^{a+1}}, \Gamma(a) = (a-1)!\right]$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

Let $u = x \rightarrow \quad du = dx \quad , \quad dv = e^{-x}\, dx \rightarrow \quad v = -e^{-x}$

$E(X) = \int_0^\infty x\, e^{-x}\, dx = [-x\, e^{-x}]_0^\infty + \int_0^\infty e^{-x}\, dx = [-e^{-x}]_0^\infty = 1$

$\boldsymbol{E(X^2)} = \int_0^\infty x^2\, e^{-x}\, dx = \frac{\Gamma(3)}{1^3} = 2$

or use integration by parts

$\Rightarrow Let\, u = x^2 \rightarrow du = 2x\, dx\,, \quad dv = e^{-x}\, dx \rightarrow \quad v = -e^{-x}$

$E(X^2) = [-x^2 e^{-x}]_0^\infty + 2\int_0^\infty xe^{-x}\, dx = 0 + 2E(X) = 2(1) = 2$

$\boldsymbol{V(X)} = E(X^2) - [E(X)]^2 = 2 - 1 = 1$

b) Same solution in part (a) , $E(Y) = 1 \quad , V(Y) = 2$

c) As X and Y are independent $\Rightarrow E(XY) = E(X)E(Y) = 1$

d) As X and Y are independent $\Rightarrow E(X^2Y^3) = E(X^2)E(Y^3) = 2(6) = 12$

$where\; E(Y^3) = \int_0^\infty Y^3\, e^{-y}\, dy = \frac{\Gamma(4)}{1^4} = 6$

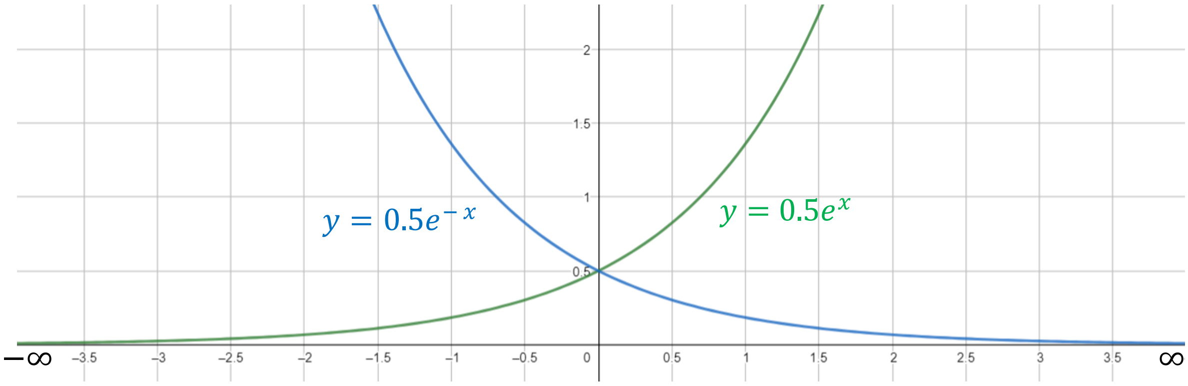
or use integration by parts

$\Rightarrow Let\, u = y^3 \rightarrow du = 3y^2\, dy\,, \quad dv = e^{-y}\, dy \rightarrow \quad v = -e^{-y}$

$E(Y^3) = [-y^3 e^{-y}]_0^\infty + 3\int_0^\infty y^2 e^{-y}\, dy = 0 + 3E(Y^2) = 3(2) = 6$

Q4) A r.v. has $f(x) = \frac{1}{2}\, e^{-|x|}$; $for \; -\infty < x < \infty$, find E(X) and V(X).

Solution :



we know the definition of absolute value is $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{1}{2}e^{-x} \;, & x > 0 \\ \frac{1}{2}e^{x} \;, & x < 0 \end{cases}$$

$$E(X) = \frac{1}{2}\left[\int_{-\infty}^{0} xe^{x}\, dx + \int_{0}^{\infty} xe^{-x}\; dx\right] = \frac{1}{2}[K_1 + K_2] = \frac{1}{2}[-1+1] = 0$$

K_1: $substitution \;\; x = -z \;\Rightarrow\; dx = -dz \;;\;\; -\infty < x < \; 0$

new limits of integration $when \;\; x = 0 \;\Rightarrow\; z = 0$

$when \;\; x = -\infty \;\Rightarrow\; z = \infty$

$$K_1 = \int_{\infty}^{0}(-z)e^{-z}(-dz) = -\int_{0}^{\infty} z\, e^{-z}\, dz = \frac{\Gamma(2)}{1^2} = \mathbf{-1}$$

K_2: $\int_0^{\infty} xe^{-x}\;\, dx = \Gamma(2) = 1$

Or by use Integration by Parts $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

Let $u = x \;\rightarrow\; du = dx \;,\; dv = e^{x}\; dx \;\rightarrow\; v = e^{x}$

$K_1 = [x\, e^{x}]_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\, dx = \mathbf{-1}$

$\boldsymbol{V(X) = E(X^2) - [E(X)]^2}$

$$E(X^2) = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x}\, dx + \int_{0}^{\infty} x^2 e^{-x}\; dx\right] = \frac{1}{2}[K_3 + K_4] = \frac{1}{2}[2+2] = 2$$

K_3: $by \;\; substitution \;\; x = -z \;\rightarrow\; dx = -dz \;;\; -\infty < x < \; 0 \Rightarrow \;\; 0 < z < \infty$

$K_3 = \int_0^{\infty}(z^2)e^{-z}\, dz = \frac{\Gamma(3)}{1^3} = 2$

$K_4 = \int_0^\infty x^2 e^{-x}\ dx = \Gamma(3) = 2$

$$\therefore V(X) = 2 - 0 = 2$$

Q5) Let $X_1, X_2, \dots, X_n$ be independent and identically distributed having mean μ and variance σ^2. Show that $E[\sum_{i=1}^n (X_i - \bar{X})^2] = (n-1)\sigma^2$.

Solution :

$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n (X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \sum_{i=1}^n X_i^2 - 2\bar{X}\sum_{i=1}^n X_i + \sum_{i=1}^n \bar{X}^2$

$= \sum_{i=1}^n X_i^2 - 2\bar{X}\,(n\bar{X}) + n\bar{X}^2 \quad ; \quad \because\ n\bar{X} = \sum_{i=1}^n X_i$

$= \sum_{i=1}^n X_i^2 - n\bar{X}^2$

$\therefore L.H.S = E[\sum_{i=1}^n (X_i - \bar{X})^2] = E[\sum_{i=1}^n X_i^2 - n\bar{X}^2] = E[\sum_{i=1}^n X_i^2] - E[n\bar{X}^2]$

$= \sum_{i=1}^n E(X_i^2) - nE(\bar{X}^2) = n\,\boldsymbol{E}(\boldsymbol{X^2}) - n\,\boldsymbol{E}(\boldsymbol{\bar{X}^2})$

$\because\ V(X) = E(X^2) - [E(X)]^2 \Longrightarrow \sigma^2 = E(X^2) - \mu^2 \Longrightarrow E(X^2) = \sigma^2 + \mu^2$

$\because\ V(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2 \Longrightarrow \frac{\sigma^2}{n} = E(\bar{X}^2) - \mu^2 \Longrightarrow E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu$

$$= n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) = n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2 = \sigma^2(n-1) = R.H.S$$

Or $\sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)\,S^2 \Rightarrow E[\sum_{i=1}^n (X_i - \bar{X})^2] = E[(n-1)\,S^2]$

$= (n-1)E(S^2) = (n-1)\sigma^2$ (Because S^2 is unbiased estimator for σ^2)

Note:

For any distribution has μ *and* σ^2. Then $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ has

$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{n}{n}E(X) = \mu$

$V(\bar{X}) = \frac{1}{n^2}V(\sum_{i=1}^n X_i) = \frac{1}{n^2}\sum_{i=1}^n V(X_i) = \frac{n}{n^2}V(X) = \frac{1}{n}V(X) = \frac{1}{n}\sigma^2$

Q6) If we have

a. $f(x) = \frac{1}{b-a} \quad ; a \le x \le b$

b. $f(x) = \lambda e^{-\lambda x} \quad ; x > 0$

c. $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \quad ; -\infty < x < \infty$

Find E(X) and V(X).

Solution :

a) $f(x) = \frac{1}{b-a} \quad ; a \le x \le b$

$$E(X)=\int_a^b x\,f(x)dx=\frac{1}{b-a}\int_a^b x\,dx=\frac{1}{(b-a)}\left[\frac{x^2}{2}\right]_a^b=\frac{1}{2(b-a)}(b^2-a^2)$$
$$=\frac{1}{2(b-a)}(b+a)(b-a)=\frac{b+a}{2}$$

$$E(X^2)=\int_a^b x^2\,f(x)dx=\frac{1}{b-a}\int_a^b x^2\,dx=\frac{1}{(b-a)}\left[\frac{x^3}{3}\right]_a^b=\frac{(b^3-a^3)}{3(b-a)}$$
$$=\frac{(b-a)(b^2+ab+a^2)}{3(b-a)}=\frac{(b^2+ab+a^2)}{3}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{(b^2+ab+a^2)}{3}-\frac{(b+a)^2}{4}$$
$$=\left(\frac{4}{4}\right)\frac{1}{3}(b^2+ab+a^2)-\left(\frac{3}{3}\right)\frac{1}{4}(b^2+2ab+a^2)$$
$$=\frac{1}{12}(b^2-2ab+a^2)=\frac{1}{12}(b-a)^2$$

b) $f(x)=\lambda e^{-\lambda x}\quad ;\ x>0$

$$E(X)=\int_0^\infty x\,f(x)dx=\lambda\int_0^\infty x\,e^{-\lambda x}\,dx=\lambda\frac{\Gamma(2)}{\lambda^2}=\frac{1}{\lambda}$$

$$\left[\text{by use}\int_0^\infty x^a\,e^{-b\,x}\,dx=\frac{\Gamma(a+1)}{b^{a+1}}\ ,\qquad \Gamma(a)=(a-1)!\right]$$

$$E(X^2)=\int_0^\infty x^2\,f(x)dx=\lambda\int_0^\infty x^2\,e^{-\lambda x}\,dx=\lambda\frac{\Gamma(3)}{\lambda^3}=\frac{2}{\lambda^2}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

c) $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]\quad ;\ -\infty<x<\infty$

$$E(X)=\int_{-\infty}^\infty x\,f(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty x\,\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]dx$$

Let $u=\frac{x-\mu}{\sigma}\ \Rightarrow\ x=\sigma u+\mu\ \to\ dx=\sigma\,du\ \to\ \frac{1}{\sigma}dx=du$

$-\infty<x<\infty\quad\Rightarrow\quad -\infty<u<\infty$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty(\sigma\,u+\mu)e^{-\frac{1}{2}u^2}\,du=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^\infty u\,e^{-\frac{1}{2}u^2}\,du+\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{1}{2}u^2}\,du=\mu$$

Where $\int_{-\infty}^\infty u\,e^{-\frac{1}{2}u^2}\,du=0$ *because it is odd function*, and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}u^2}\ du = 1 \text{ because it is even function}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\ dx$$

Let $u = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma u + \mu \rightarrow dx = \sigma\, du \rightarrow \frac{1}{\sigma}dx = du$

$$-\infty < x < \infty \Rightarrow -\infty < u < \infty$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma u + \mu)^2 e^{-\frac{1}{2}u^2}\ du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\sigma^2 u^2 + 2\sigma\mu\, u + \mu^2) e^{-\frac{1}{2}u^2}\ du$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\ du + \frac{2\sigma\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\ du + \frac{\mu^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du = \sigma^2 + \mu^2$$

$$\text{where } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\ du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du = 1 \text{ because it is even function}$$

$$V(X) = E(X^2) - [E(X)]^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Note:

- $\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{2}u^2}\ du = \frac{2\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} u^2 e^{-\frac{1}{2}u^2}\ du$

  Let $z = u^2 \rightarrow dz = 2u\, du \rightarrow \frac{1}{2u}\, dz = du \rightarrow \frac{1}{2z^{\frac{1}{2}}}\, dz = du$

  $0 < u < \infty \Rightarrow 0 < u < \infty$

  $= \frac{2\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} z\, e^{-\frac{1}{2}z} \frac{1}{2z^{\frac{1}{2}}}\, dz = \frac{\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} Z^{\frac{1}{2}} e^{-\frac{1}{2}Z}\, dz = \frac{\sigma^2}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = \sigma^2$

  $\because \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

- $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u\, e^{-\frac{1}{2}u^2}\ du = \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{1}{2}u^2}\right]_{-\infty}^{\infty} = \frac{-1}{\sqrt{2\pi}}(e^{-\infty} - e^{-\infty}) = 0 \quad \because e^{-\infty} = 0$

- $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}u^2}\ du$

  Let $z = u^2 \rightarrow dz = 2u\, du \rightarrow \frac{1}{2u}\, dz = du \rightarrow \frac{1}{2z^{\frac{1}{2}}}\, dz = du$

  $= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{1}{2} z^{-\frac{1}{2}} e^{-\frac{1}{2}Z}\, dz = \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$

Q7) If $X \sim Exp(2)$ independent of $Y \sim Gamma(3,4)$, find:

a. E(XY).

b. $E(X^2\ Y^3)$.

c. V(X -Y)

d. V(3X + 2Y)

where

	pdf	E(X)	V(X)
$X \sim Exp(\lambda)$	$f(x) = \lambda e^{-\lambda x} \quad ; x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$Y \sim Gamma(\alpha,\beta)$	$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \quad ; y > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$

Solution :

$$X \sim Exp(\lambda = 2); \ \ Y \sim Gamma(\alpha = 3, \beta = 4)$$

a) $E(XY) = E(X)E(Y) = \frac{1}{2}\ \frac{3}{4} = \frac{3}{8}$

b) $E(X^2 Y^3) = E(X^2)E(Y^3) = \frac{1}{2}\ \frac{15}{16} = \frac{15}{32}$

$$V(X) = E(X^2) - [E(X)]^2 \Rightarrow \ \ \boldsymbol{E(X^2) = V(X) + [E(X)]^2} = \frac{\mathbf{1}}{\mathbf{4}} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$E(Y^3) = \int_0^\infty \boldsymbol{y^3\, f(y) dy} \ = \frac{\mathbf{4^3}}{\mathbf{\Gamma(3)}} \int_0^\infty \boldsymbol{y^3\ y^{3-1}\ e^{-4y}\ \ dy}$$

$$= \frac{\mathbf{4^3}}{\mathbf{2}} \int_0^\infty \boldsymbol{y^5\ e^{-4y}\ \ dy} = \frac{\mathbf{4^3}}{\mathbf{2}}\ \frac{\mathbf{\Gamma(6)}}{\mathbf{4^6}} = \frac{\mathbf{5!}}{\mathbf{128}} = \frac{\mathbf{15}}{\mathbf{16}}$$

c) $\boldsymbol{V(X - Y) = V(X) + V(Y)} = \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{3}}{\mathbf{16}} = \frac{\mathbf{7}}{\mathbf{16}}$

d) $V(3X + 2Y) = 9V(X) + 4V(Y) = 9\left(\frac{1}{4}\right) + 4\left(\frac{3}{16}\right) = 3$

Note:

- $\boldsymbol{(b \pm a)^2 = (a^2 \pm 2ab + b^2)}$
- $\boldsymbol{(b^2 - a^2) = (b + a)(b - a)}$
- $\boldsymbol{(b^3 - a^3) = (b - a)(b^2 + ab + a^2)}$

- $f(x)$ even function $\Leftrightarrow f(-x) = f(x) \ \ \Leftrightarrow \ \ \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$
- $f(x)$ odd function $\Leftrightarrow f(-x) = -f(x) \ \ \Leftrightarrow \ \ \int_{-a}^{a} f(x)\, dx = 0$