Problem 1: Let X be a random variable with distribution

$$f(x;\theta) = \frac{1}{\theta}\ ;\ \ 0 < x < \theta$$

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be n copies of X

Find $(1-\alpha)100\%$ Confidence interval for θ. (Hint: use $Q = \frac{S}{\theta}$ where $S = \max(\underline{X})$ and $h(q) = nq^{n-1} \quad ; \quad 0 < q < 1$).

Solution

Step1:

$$P\left(q_1 < \frac{S}{\theta} < q_2\right) = P\left(\frac{S}{q_2} < \theta < \frac{S}{q_1}\right) = 1-\alpha$$

$\int_{q_1}^{q_2} nq^{n-1}\,dq = q_2^n - q_1^n = 1-\alpha$ (i)

Step2:

$L = S\left(\frac{1}{q_1} - \frac{1}{q_2}\right)$ (ii)

Must be minimum.

Now, Differentiate (i) with respect to q_2, we get

$$0 = nq_2^{n-1} - nq_1^{n-1}\frac{dq_1}{dq_2} \rightarrow \frac{dq_1}{dq_2} = \left(\frac{q_2}{q_1}\right)^{n-1}$$

Now,let us Differentiate (ii) with respect to q_2

$$\frac{dL}{dq_2} = -\frac{S}{q_1^2}\frac{dq_1}{dq_2} + \frac{S}{q_2^2} = -\frac{S}{q_1^2}\left(\frac{q_2}{q_1}\right)^{n-1} + \frac{S}{q_2^2} = -S\frac{\left(q_2^{n+1} - q_1^{n+1)}\right)}{q_2^2 q_1^{n+1}} < 0$$

i.e L is decresing function of q_2 and we get the minimum value when $q_2 = 1$ (maximum)

then $q_1 = \sqrt[n]{\alpha}$

and the confidence interval is $\left(S, \frac{S}{\sqrt[n]{\alpha}}\right)$

Problem 2: Let X be a random variable with distribution

$$f(x;\theta) = e^{-(x-\theta)}\ ;\ \ x > \theta$$

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be n copies of X

Find $(1-\alpha)100\%$ Confidence interval for θ. (Hint: use $Q = n(S-\theta)$ where $S = \min(\underline{X})$ and $h(q) = e^{-q};\quad q > 0$).

Solution

Step1:

$$P(q_1 < n(S-\theta) < q_2) = P\left(\frac{q_1}{n} < (S-\theta) < \frac{q_2}{n}\right) = P\left(S - \frac{q_2}{n} < \theta < S - \frac{q_1}{n}\right) = 1-\alpha$$

$\int_{q_1}^{q_2} e^{-q}\, dq = e^{-q_1} - e^{-q_2} = 1-\alpha$ (i)

Step2:

$L = \frac{1}{n}(q_2 - q_1)$ (ii)

Must be minimum.

Now, Differentiate (i) with respect to q_1, we get

$$0 = -e^{-q_1} + e^{-q_2}\frac{dq_2}{dq_1} \rightarrow \frac{dq_2}{dq_1} = e^{q_2 - q_1}$$

Now,let us Differentiate (ii) with respect to q_1

$$\frac{dL}{dq_1} = \frac{1}{n}\left(\frac{dq_2}{dq_1} - 1\right) = \frac{1}{n}(e^{q_2-q_1} - 1) = \frac{1-\alpha}{n}e^{q_2} > 0$$

i.e L is increasing function of q_1 and we get the minimum value when $q_1 = 0$ (minimum value)

then $q_2 = -\ln\alpha$

and the confidence interval is $\left(S + \frac{\ln\alpha}{n}, S\right)$