Infinite Products

BLEL Mongi

Department of Mathematics
King Saud University

February 14, 2023

Definition

Let $(a_n)_n$ *be a sequence of complex numbers,* $a_n \neq 0$ *for all* $n \in \mathbb{N}$ *and let* $(p_n)_n$ *be the sequence defined by* $p_n = \prod_{k=0}^{n} a_k$.

We say that the infinite product $\prod_{n\geq 0} a_n$ *is convergent if the sequence* $(p_n)_n$ *converges to a non zero complex number and we denote* $\prod_{n=0}^{+\infty} a_n = \lim_{n\to+\infty} \prod_{k=0}^{n} a_k$.

Examples

1. If $a_n = 1 - \dfrac{1}{n+1}$, then $p_n = \dfrac{1}{n+1} \underset{n \to +\infty}{\longrightarrow} 0$ and the infinite product $\prod_{n \geq 1} a_n$ is divergent.
2. If $a_n = 1 + \dfrac{1}{n+1} = \dfrac{n+2}{n+1}$, then $p_n = \dfrac{n+2}{2} \underset{n \to +\infty}{\longrightarrow} +\infty$ and the infinite product $\prod_{n \geq 1} a_n$ is divergent.
3. If $a_n = 1 - \dfrac{1}{n^2}$, $n \geq 2$, then $p_n = \dfrac{n+1}{2n}$ and the infinite product $\prod_{n \geq 1} a_n$ is divergent.

Remark 1 :
If the infinite product $\prod_{n\geq 1} a_n$ is convergent, then $\lim\limits_{n\to+\infty} a_n = 1$.
$\left(\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \dfrac{p_n}{p_{n-1}} = 1.\right)$ The converse is not true. It suffices to take $a_n = 1 - \frac{1}{n+1}$ or $a_n = x$, with $0 < x < 1$.

Proposition

Let $(a_n)_n$ be sequence of non zeros complex numbers. The infinite product $\prod_{n\geq 0} a_n$ is convergent if and only if the series $\sum_{n\geq 0} \log a_n$ is convergent, with $\log a_n = \ln |a_n| + \mathrm{i}\theta_n$, and θ_n is the unique argument of a_n in the interval $]-\pi, \pi]$.

Proof

We set $S_n = \sum_{j=0}^{n} \log a_j$, $p_n = e^{S_n}$. If the series $\sum_{n\geq 0} \log a_n$ is convergent to S, then $\lim_{n\to+\infty} S_n = S$ and $\lim_{n\to+\infty} p_n = e^S \neq 0$. The infinite product is then convergent.

If the infinite product is convergent to $p \neq 0$. Let $\lambda \in \mathbb{C}$ such that $e^\lambda = p$, so $\lim_{n\to+\infty} e^{S_n} = e^\lambda$ and $\lim_{n\to+\infty} e^{S_n-\lambda} = 1$. Then there exists an integer N such that whenever $n \geq N$, $\log(e^{S_n-\lambda})$ is defined. There exists a sequence $(k_n)_n \in \mathbb{Z}$ such that

$$S_n - \lambda = \log(e^{S_n-\lambda}) + 2\mathrm{i}k_n\pi.$$

Since $e^{S_n-\lambda}$ tends to 1, we have

$$\lim_{n\longrightarrow+\infty} S_n - \lambda - 2\mathrm{i}k_n\pi = 0,$$

Furthermore $S_{n+1} - S_n = \log a_n$ tends to 0, then the sequence of integers $(k_{n+1} - k_n)_n$ tends also to 0, then it vanishes from a rank N_1 and $\lim\limits_{n \to +\infty} S_n = \lambda + 2i\pi k_{N_1}$.

Example

$a_n = 1 + \dfrac{1}{n+1}, \ln a_n = \ln(1 + \dfrac{1}{n+1}) \approx \dfrac{1}{n}$. The series $\sum\limits_{n \geq 0} a_n$ is divergent.

$a_n = 1 - \dfrac{1}{n^2}, \ln a_n = \ln(1 - \dfrac{1}{n^2}) \approx \dfrac{-1}{n^2}$. The series $\sum\limits_{n \geq 2} a_n$ is convergent.

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that the infinite product $\prod_{n\geq 0} a_n$ *is convergent if there exists a rank* $n_0 \in \mathbb{N}$ *such that for* $n \geq n_0$, $a_n \neq 0$ *and* $\lim_{n\to+\infty} \prod_{p=n_0}^{n} a_p$ *exists and it is a non zero complex number.*

Definition

We say that the infinite product $\prod_{n\geq 0}(1+u_n)$ *is absolutely convergent if the infinite product* $\prod_{n\geq 0}(1+|u_n|)$ *is convergent.*

Proposition

An infinite product absolutely convergent is convergent.

Lemma

Let $(u_n)_n$ be a sequence of non negative real numbers. The series $\sum_{n\geq 0} u_n$ converges if and only if the infinite product $\prod_{n\geq 0}(1+u_n)$ converges.

Proof

We have, for all $x \geq 0$ $1 + x \leq e^x$. We denote $S_n = \sum_{k=0}^{n} u_k$ and $p_n = \prod_{k=0}^{n}(1 + u_k)$. We have

$$1 + S_n = 1 + \sum_{k=0}^{n} u_k \leq \prod_{k=0}^{n}(1 + u_k) \leq e^{S_n}.$$

(This lemma results also because the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} \ln(1 + u_n)$ have the same nature since $\lim_{x\to 0^+} \frac{\ln(1 + x)}{x} = 1$.) □

Proof of the Proposition 1.5

If the infinite product is absolutely convergent, the series $\sum_{n=0}^{+\infty} |u_n|$ is

To prove that the infinite product $\prod_{n\geq 0}(1+u_n)$ is convergent, it suffices to prove that the series $\sum_{n\geq n_0}|\ln(1+u_n)|$ is convergent.

For $|z|\leq \frac{1}{2}$, $\ln(1+z)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{n+1}z^{n+1}=zh(z)$. For $|z|\leq \frac{1}{2}$, $|h(z)|\leq M$. Then $|\ln(1+u_n)|\leq M|u_n|$, for $n\geq n_0$, thus the series $\sum_{n\geq 0}|\ln(1+u_n)|$ is convergent.

Corollary

If the infinite product $\prod_{n\geq 0} a_n$ *is absolutely convergent, then for all permutation* σ *of* $\mathbb{N}$, *the infinite product* $\prod_{n\geq 0} a_{\sigma(n)}$ *is convergent.*

Proposition

Let $(u_n)_n$ *be a sequence of real numbers such that* $0 \leq u_n < 1$, $\forall n \in \mathbb{N}$.
The infinite product $\prod_{n\geq 0}(1-u_n)$ *is convergent if and only if the series* $\sum_{n\geq 0} u_n$ *is convergent.*

Proof

The sequence $\left(p_n = \prod_{k=0}^{n}(1-u_k)\right)_n$ is decreasing and non negative, then it converges and $0 < p_n \leq e^{-\sum_{k=0}^{n} u_k}$.

If $\sum_{n=0}^{+\infty} u_n = +\infty$, then $\lim_{n\to+\infty} p_n = 0$ and then the infinite product is divergent.

If the series $\sum_{n\geq 0} u_n$ converges. Let $0 < \varepsilon < \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{+\infty} u_n < \varepsilon$.

So for all $N > n_0$,

$$0 < 1-\prod_{n=n_0}^{N}(1-u_n) = |1-\prod_{n=n_0}^{N}(1-u_n)| \leq \prod_{n=n_0}^{N}(1+u_n)-1 \leq e^{\sum_{n=n_0}^{N} u_n}-1 \leq$$

$$0 \leq p_{n_0} - p_N = p_{n_0}(1 - \prod_{n=n_0+1}^{N}(1-u_n)) \leq 2\varepsilon p_{n_0}.$$

It results that $0 < p_{n_0}(1-2\varepsilon)$ and $p_N \geq (1-2\varepsilon)p_{n_0}$. The sequence $(p_n)_n$ is decreasing and bounded above by $p_{n_0}(1-2\varepsilon)$, then it converges to a number $L > 0$, which proves that the infinite product $\prod_{n\geq 0}(1-u_n)$ is convergent.

(This lemma results also from the fact that the series $\sum_{n\geq 0} u_n$ and the series $\sum_{n\geq 0} \ln(1-u_n)$ have the same nature, because

$$\lim_{x\to 0} \frac{-\ln(1-x)}{x} = 1.)$$

□

Theorem

Let $(f_n)_n$ be a sequence of of bounded functions defined on a non empty subset E of $\mathbb{C}$. We assume that the series $\sum_{n\geq 1} |f_n|$ converges uniformly on E, then the infinite product $\prod_{n\geq 0}(1+f_n)$ converges uniformly on E to a function f. Furthermore $f(s_0)=0$ if and only if $1+f_{n_0}(s_0)=0$ for some integer n_0.

Proof

Let $P_n = \prod_{p=1}^{n}(1+f_p)$. For$n < m$,

$|p_n - p_m| = p_n|1 - \prod_{p=n+1}^{m}(1+f_p)|$. For $0 < \varepsilon < \frac{1}{2}$, there exists an

$$|1-\prod_{n+1}^{m}(1+f_j(z))| \leq \prod_{n+1}^{m}(1+|f_j(z)|)-1 \leq e^{\sum_{n+1}^{m}|f_j(z)|}-1 \leq e^{\varepsilon}-1 \leq 2\varepsilon, \quad \forall$$

Then $|p_n(z)| \leq e^{\sum_{j=1}^{n}|f_j(z)|} \leq M < +\infty$, because the series converges uniformly on E.
If $m > n \geq n_0$, $|p_n(z) - p_m(z)| \leq 2\varepsilon e^M$.
The sequence of functions $(p_n)_n$ is then a Cauchy's sequence for the topology of uniform convergence, then it converges uniformly on E.

Let $f(z) = \prod_{n=0}^{+\infty}(1 + f_n(z))$. For $z \in E$ and $m > n \geq n_0$,

$|1 - \prod_{n_0+1}^{m}(1 + f_j(z))| \leq 2\varepsilon.$

Then $\prod_{n_0+1}^{m}(1 + f_j(z))| \geq 1 - 2\varepsilon > 0.$

$p_m(z) = |p_{n_0}(z)| \prod_{n_0+1}^{m}(1 + f_j(z))| \geq |p_{n_0}(z)|(1 - 2\varepsilon).$

If there exists $z \in E$ such that $f(z) = 0$, then $p_{n_0}(z) = 0$ and there exists $j \leq n_0$ such that $1 + f_j(z) = 0$. The converse is false.

Corollary

With the same notations as in theorem 2.1, if $f(z_0) = 0$ and if f is not the zero function ($f \not\equiv 0$), there exists a finite number of index $j \in \mathbb{N}$, such that $1 + f_j(z_0) = 0$.

Theorem

Let $(f_n)_n$ be a sequence of holomorphic functions on a domain Ω. We assume that $f_n \not\equiv 0$ whenever n and the series $\sum_{n\geq 1} |1 - f_n(z)|$ converges uniformly on any compact subset of Ω, then the infinite product $\prod_{n\geq 1} f_n$ converges uniformly on any compact subset of Ω. The limit f is holomorphic on Ω. The function $f \not\equiv 0$ and if $f(z_0) = 0$, then $f_n(z_0) = 0$ for at least one index n and the order of multiplicity of f at z_0 is the sum of the orders of multiplicities of z_0 in the different factors.

Proof

We set $u_j(z) = f_j(z) - 1$ and $p_n(z) = \prod_{j=0}^{n} f_j(z)$. The sequence $(p_n)_n$ converges uniformly on any compact subset of Ω, then the function f defined by $f = \prod_{j=0}^{+\infty} f_j(z)$ is holomorphic on Ω.

For $z \in \Omega$, $|f(z)| \geq |\prod_{j=0}^{n_0} f_j(z)|(1 - 2\varepsilon)$ with n_0 chosen such that $\sum_{n_0+1}^{+\infty} |1 - f_j(z)| < \varepsilon, \quad 0 < \varepsilon < \frac{1}{2}$. The others results are deduced from the previous theorem.

Corollary

With the same conditions as in theorem 2.3, $\dfrac{f'}{f} = \sum_{j=0}^{+\infty} \dfrac{f_j'}{f_j}$ *and the series converges uniformly on any compact subset of* Ω *which not meeting the set of zeros of* f*.*

Proof

The sequence $(p_n(z) = \prod_{j=0}^{n} f_j(z))_n$ converges uniformly on any compact subset of Ω to f. The sequence $(p_n')_n$ converges also uniformly on any compact subset of Ω to f'. Let K be a compact which not intersects the set of zeros of f and $M > 0$ such that on K, $|\dfrac{1}{f}| \leq M$ and $|p_n - f| \leq \dfrac{2}{M}$ for n large enough, then $|\dfrac{1}{p_n}| \leq 2M$ on K for n large enough.

$$|\frac{p_n'}{p_n} - \frac{f'}{f}| = |\frac{fp_n' - p_nf'}{p_nf}| \leq 2M^2|fp_n' - p_nf'|.$$

Then the sequence $(\frac{p_n'}{p_n})_n$ converge uniformly on K to $\frac{f'}{f}$.

Examples

1. Let $a \in \mathbb{C}^*$ be such that $|a| < 1$. We consider the infinite product $\prod_{n\geq 1}(1 + a^n z)$. The series $\sum_{n=1}^{+\infty} |a^n z|$ converges uniformly on any compact subset of $\mathbb{C}$. The set of zeros of $\prod_{n=1}^{+\infty}(1 + a^n z)$ is $\{\frac{-1}{a^n};\ n \in \mathbb{N}\}$.

2. Let $f_n(z) = (1 + \frac{z}{n})$. The infinite product $\prod_{n\geq 1} f_n(z)$ converges only at 0.

3. Let $f_n(z) = (1 + \frac{z}{n})e^{-\frac{z}{n}}$. The infinite product $\prod_{n\geq 1} f_n(z)$ converges uniformly on any compact subset of $\mathbb{C}$ because $|f_n(z) - 1| < \frac{|z|^2}{n^2}M$, for n large enough.

4. Let f be the function $f(z) = z\prod_{n=1}^{+\infty}(1 - \frac{z^2}{n^2})$. f is holomorphic on $\mathbb{C}$ and $f(z) = 0$ if and only if $z \in \mathbb{Z}$.

For $n \notin \mathbb{Z}$, $\frac{f'(z)}{f(z)} = 1 + \sum_{n=1}^{+\infty} \frac{2z}{z^2 - n^2} = \pi\mathrm{cotan}\pi z$, (cf exercise 1, chapter 6). Then for $z \in \mathbb{C} \setminus \mathbb{Z}$, $(\frac{f(z)}{\sin \pi z})' = 0 \Rightarrow f(z) = C \sin \pi z$ on $\mathbb{C}$. But $\frac{f(z)}{z} = \prod_{n=1}^{+\infty}(1 - \frac{z^2}{n^2}) \underset{z\to 0}{\longrightarrow} 1$. It results then $C = \pi$. We deduce the Euler's formula.

5. We consider the geometric series $\sum_{n\geq 1} z^n$, where $z = e^{2\mathrm{i}\pi q}$, $\mathrm{Im} q > 0$. The series $\sum_{n\geq 1} z^n$ is absolutely convergent. We can then define $\prod_{n\geq 1}(1+z^n)$ and $\prod_{n\geq 1}(1-z^n)$. The function $\prod_{n=1}^{+\infty}(1+z^n) = \sum_{n=0}^{+\infty} p(n)z^n$ is holomorphic on the unit disc, $p(n)$ is the number of partitions of the integer n (i.e. the number of $(n_1, \dots, n_s)$ such that $n_1 + \dots n_s = n$).

Definition

We define the following functions

$$E_0(z) = 1 - z; \quad E_1(z) = (1 - z)e^{z}; \quad E_m(z) = (1 - z)e^{\sum_{j=1}^{m} \frac{z^j}{j}}.$$

$E_n(z)$ is an entire function. 1 is a simple zero of E_n. E_n is called the n^{th} elementary Weierstrass's factor.

Lemma

For $|z| < 1$, $|E_n(z) - 1| \leq |z|^{n+1}$.

Proof

$$E_n'(z) = -e^{\sum_{j=1}^n \frac{z^j}{j}} + (1 - z^n)e^{\sum_{j=1}^n \frac{z^j}{j}} = -z^n e^{\sum_{j=1}^n \frac{z^j}{j}}.$$

Since $E_n(z) - 1 = \int_{[0,z]} E_n'(w)dw$, $w = tz$, $t \in [0, 1]$, we have

$$E_n(z) - 1 = z \int_{[0,1]} E_n'(tz)dt = -z^{n+1} \int_0^1 t^m e^{\sum_{j=1}^n \frac{t^j z^j}{j}} dt.$$

$$E_n(1) - 1 = -\int_0^1 t^n e^{\sum_{j=1}^n \frac{t^j}{j}} dt = -1, \text{ because } E_n(1) = 0.$$

For $|z| \leq 1$, $|1 - E_n(z)| \leq |z|^{n+1} \int_0^1 t^n e^{\sum_{j=1}^n \frac{t^j}{j}} dt \leq |z|^{n+1}$

Theorem

Let $(a_n)_n$ be a sequence of complex numbers, $a_n \neq 0$ and $\lim_{n\to+\infty} |a_n| = +\infty$. Let $(k_n)_n$ be a sequence of positive integers chosen such that $\sum_{n=1}^{+\infty} (\frac{r}{r_n})^{1+k_n} < +\infty$, whenever $r > 0$, where $r_n = |a_n|$. Then the infinite product $\prod_{n\geq 1} E_{k_n}(\frac{z}{a_n})$ converges uniformly on any compact of $\mathbb{C}$. The function

$$f(z) = \prod_{n=1}^{+\infty} E_{k_n}(\frac{z}{a_n})$$

is holomorphic on $\mathbb{C}$ and the set of zeros of f, Z_f is the set $\{a_n;\ n \in \mathbb{N}\}$. Furthermore the multiplicity of a zero a of f is equal to the number of integers n such that $a_n = a$.

Remark 2 :
For all $r > 0$, there exists a rank n_0 such that for $n \geq n_0$, $\frac{r}{r_n} < \frac{1}{2}$ and the condition of convergence in the theorem is realized with $k_n = n$.

Proof

By lemma 3.2, we have $|1 - E_{k_n}(\frac{z}{a_n})| \leq |\frac{z}{a_n}|^{1+k_n} \leq (\frac{r}{r_n})^{1+k_n}$. For $|z| \leq r \leq r_n$, the series $\sum_{n\geq 1} |1 - E_{k_n}(z)|$ converges uniformly on any compact subset $\mathbb{C}$. The theorem is deduced since $E_{k_n}(\frac{z}{a_n})$ has only a_n as a simple zero.

Remark 3 :

If $\sum_{n=1}^{+\infty} \frac{1}{r_n} < +\infty$, we can take $k_n = 0$. The canonical product is

$$f(z) = \prod_{n=1}^{+\infty}(1 - \frac{z}{a_n}).$$

If $\prod_{n=1}^{+\infty} \frac{1}{(r_n)^2} < +\infty$, we can take $k_n = 1$ and

$f(z) = \prod_{n=1}^{+\infty}(1 - \frac{z}{a_n})e^{\frac{z}{a_n}}.$

Theorem (Weierstrass's Factorization Theorem)

Let f be an entire function and $A = \{a_1, \ldots, a_n, \ldots\}$ the zeros of f repeated as far as their order of multiplicities. Then there exists an entire function g and a sequence of integers $(k_n)_n$ such that $f(z) = e^{g(z)} \prod_{n=1}^{+\infty} E_{k_n}(\frac{z}{a_n})$. This factorization is not unique because there exist an infinite possible choice of k_n.

Proof

If the sequence $(a_n)_n$ is infinite, then $\lim\limits_{n\to+\infty} |a_n| = +\infty$.
Let h be a Weierstrass's infinite product given in the previous theorem with the sequence $(a_n)_n$. The function $\dfrac{f}{h}$ is holomorphic on $\mathbb{C}$ without zeros. $\mathbb{C}$ is simply connected, then there exists $g \in \mathcal{H}(\mathbb{C})$ such that $\frac{f}{h} = e^g$.

Theorem

let Ω be an open subset of $\mathbb{C}$ and A a discrete closed subset in Ω. For all mapping $a: \stackrel{m}{\longmapsto} m(a)$ from A with values in $\mathbb{N}$, there exists a function $f \in \mathcal{H}(\Omega)$ such that $\forall\ a \in A$, a is a zero of f of order $m(a)$ and $Z_f = A = \{z \in \Omega;\ f(z) = 0\}$.

Proof

We can assume that $\Omega \neq \mathbb{C}$. For the proof it is useful to consider a sequence $(a_n)_n$ such that whenever n, $a_n \in A$ and such that whenever $a \in A$, $\#\{n \in \mathbb{N};\ a = a_n\} = m(a)$.
First case The sequence $(a_n)_n$ is bounded.
Let $b_n \in \Omega^c$ such that $d(a_n, \Omega^c) = |b_n - a_n|$. Then necessary

$$\lim_{n\to+\infty} |a_n - b_n| = 0, \tag{1}$$

if not the sequence $(a_n)_n$ which is bounded has a cluster point (accumulation point) in Ω.
Let's prove that the function defined by the infinite product $\prod_{n\geq 1} E_n(\frac{a_n - b_n}{z - b_n})$ is a solution to the problem. Let K be a compact subset of Ω,

$$|z - b_n| \geq \inf_{w\in K,\delta\notin\Omega} |w - \delta| = d(K, \Omega^c) > 0 \quad \forall\, z \in K. \tag{2}$$

Since $\lim_{n\to+\infty} |a_n - b_n| = 0$, there exists an integer N such that for $n \geq N$

$$|a_n - b_n| \leq \frac{1}{2} d(K, \Omega^c). \tag{3}$$

For $n \geq N$ and $z \in K$ and by equations (2) and (3), $|z - b_n| \geq 2|a_n - b_n|$, let $|\frac{a_n - b_n}{z - b_n}| \leq \frac{1}{2}$. By lemma 3.2, $|E_n(\frac{a_n - b_n}{z - b_n}) - 1| \leq (\frac{1}{2})^{n+1}$ for $n \geq N$ and $z \in K$. Which proves the theorem in the case where the sequence $(a_n)_n$ is bounded.

We prove now that $\lim_{|z|\to+\infty} f(z) = 1$ in the case where the open subset Ω is not bounded.
Since the sequence $(a_n)_n$ is bounded, the sequence $(b_n)_n$ is also bounded. (Because $\lim_{n\to+\infty} a_n - b_n = 0$). Let $M > 0$ be an upper bound of $(|a_n|)_n$ and of $(|b_n|)_n$. For $|z| > 5M$, we have

$$|\frac{a_n - b_n}{z - b_n}| \leq \frac{|a_n| + |b_n|}{|z| - |b_n|} \leq \frac{2M}{5M - M} = \frac{1}{2},$$

in such a way for $|z| > 5M$, we have $|E_n(\frac{a_n - b_n}{z - b_n}) - 1| \leq \frac{1}{2^{n+1}}$, whenever n. The infinite product converges uniformly for $|z| > 5M$ and since $\lim_{|z|\to+\infty} E_n(\frac{a_n - b_n}{z - b_n}) = E_n(0) = 1$, we deduce that $\lim_{|z|\to+\infty} f(z) = 1$.

Second case The sequence $(a_n)_n$ is not bounded.
Let $a \in \Omega$ different of a_n, whenever n. There exists $\varepsilon > 0$ such that $|a_n - a| > \varepsilon$. We consider the function $g\colon \mathbb{C} \setminus \{a\} \longrightarrow \mathbb{C}$ defined by $g(z) = \dfrac{1}{z-a}$. The function g is holomorphic and injective. The sequence $(g(a_n))_n$ is bounded in the open subset $g(\Omega \setminus \{a\}) = \Omega'$. There exists a holomorphic function f on Ω' which vanishes only at the points $(g(a_n))_n$ with multiplicity $m(a_n)$ and $\lim\limits_{|z| \to +\infty} f(z) = 1$. The function $f \circ g$ is holomorphic on Ω and a is a removable singularity because $\lim\limits_{z \to a} f \circ g(z) = 1$. The function $f \circ g$ gives an answer to the problem.

Corollary

Every meromorphic function on Ω is the quotient of two holomorphic functions on Ω.

Proof

Let f be a meromorphic function and let $(a_n)_n$ be the sequence (may be finite) of the poles of f and m_n its multiplicity. By the previous theorem, there exists a holomorphic function g on Ω such a_n is a zero of order m_n of g. The function $h = fg$ is then holomorphic one Ω and $f = \dfrac{h}{g}$.

We consider the function f defined by

$$f(z) = \prod_{j=1}^{+\infty}(1+\frac{z}{j})e^{\frac{-z}{j}}.$$

The infinite product $\prod_{j\geq 1}(1+\frac{z}{j})e^{\frac{-z}{j}}$ defines an entire function f such that $-n$ is a simple zero, whenever $n \in \mathbb{N}$.

Lemma

There exists a constant γ called the Euler's constant such that $f(z-1) = ze^{\gamma} f(z)$.

$$\left(\gamma = \lim_{n\to+\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \ln(n+1)\right) \approx 0,5772156649\right).$$

Proof

The function $g(z) = f(z-1)$ is holomorphic on $\mathbb{C}$ and every non positive integer $(-n), n \in \mathbb{N}$ is a simple zero, then the functions $zf(z)$ and $g(z)$ have the same zeros with the same multiplicity. There exists then an entire function h such that

$$zf(z)e^{h(z)} = g(z) = \lim_{n\to+\infty}\prod_{k=1}^{n}(1+\frac{z-1}{k})e^{\frac{-z+1}{k}}.$$

For $k \neq 1$,

$$(1+\frac{z-1}{k})e^{-\frac{z-1}{k}} = (1+\frac{z}{k-1})\frac{k-1}{k}e^{\frac{-z}{k}+\frac{1}{k}} = (1+\frac{z}{k-1})e^{-\frac{z}{k}}e^{\frac{1}{k}+\ln\frac{k-1}{k}}.$$

$$\begin{aligned}\prod_{k=1}^{n}(1+\frac{z-1}{k})e^{\frac{-z+1}{k}} &= ze^{-(z-1)}\prod_{k=1}^{n-1}(1+\frac{z}{k})e^{-\frac{z}{k+1}}e^{\frac{1}{k+1}+\ln\frac{k}{k+1}} \\ &= ze^{-z}(\prod_{k=1}^{n-1}(1+\frac{z}{k})e^{-\frac{z}{k}})(\prod_{k=1}^{n-1}e^{\frac{z}{k}})(\prod_{k=1}^{n-1}e^{\frac{-z}{k+1}})(\prod_{k=1}^{n-1} e \\ &= ze^{-\frac{z}{n}}e^{\sum_{j=1}^{n}\frac{1}{j}-\ln(n+1)}(\prod_{k=1}^{n-1}(1+\frac{z}{k})e^{-\frac{z}{k}}) \underset{n\to+\infty}{\longrightarrow} ze\end{aligned}$$

We prove that the sequence $\left(\sum_{j=1}^{n}\frac{1}{j}-\ln(n+1)\right)_n$ has a limit.

$\sum_{j=1}^{n}\frac{1}{j}-\ln(n+1)=\sum_{j=1}^{n}\left(\frac{1}{j}-\ln\left(\frac{j+1}{j}\right)\right)$. But for $x>0$,

$0<\frac{x}{1+x}<x$, then $\int_0^x\frac{t}{1+t}\,dt<\frac{x^2}{2}$.

Furthermore $0<x-\ln(1+x)\leq\frac{x^2}{2}$. Then $0<\frac{1}{j}-\ln\frac{j+1}{j}<\frac{1}{2j^2}$ which is the general term of a convergent series.

Definition

The function Γ called the Gamma Euler's function is the meromorphic function defined by

$$\Gamma(z) = \frac{1}{ze^{\gamma z}f(z)}, \quad \frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{j=1}^{+\infty}(1+\frac{z}{j})e^{\frac{-z}{j}}.$$

Any non positive integer is a simple pole of Γ, then the function $\frac{1}{\Gamma}$ is an entire function such that any non positive integer is a simple zero.

Theorem

$$\Gamma(z+1) = z\Gamma(z); \quad \forall z \notin -\mathbb{N}.$$

$$\Gamma(1) = 1, \quad \Gamma(n+1) = n!, \ \forall \ n \in \mathbb{N}.$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{Complement formula} .$$

$$\Gamma(z) = \lim_{n\to+\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \quad n^z = e^{z \ln n}.$$

$$\text{If } \text{Re}z > 0, \quad \Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} \, dt.$$

Proof

1.
$$\begin{aligned}\frac{1}{\Gamma(z+1)} &= (z+1)e^{\gamma(z+1)}f(z+1) = (z+1)e^{\gamma z}e^{\gamma}f(z+1)\\ &= e^{\gamma z}(z+1)e^{\gamma}f(z+1) = e^{\gamma z}f(z) = \frac{1}{z\Gamma(z)}.\end{aligned}$$

2. $\Gamma(1) = \dfrac{1}{e^{\gamma}f(1)}$ and $f(1)e^{\gamma} = f(0) = 1$, then $f(1) = e^{-\gamma}$, $\Gamma(1) = 1$, $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n+1) = n!\Gamma(1) = n!$.

3.
$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(1-z)} &= \frac{1}{-z\Gamma(z)\Gamma(-z)}\\ &= \frac{-1}{z}e^{\gamma z}f(z)(-z)e^{-\gamma z}f(-z)\\ &= zf(z)f(-z) = \frac{\sin \pi z}{\pi},\end{aligned}$$

then $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

An other method

We compare $\sin \pi z$ and $\frac{1}{\Gamma(z)\Gamma(1-z)}$. Since the set of zeros of $\sin \pi z$ is $\mathbb{Z}$ and are simple zeros. The same for the function $\frac{1}{\Gamma(z)\Gamma(1-z)}$, there exists an entire function h such that

$$\frac{1}{\Gamma(z)\Gamma(1-z)} e^{h(z)} = \frac{\sin \pi z}{\pi} \tag{4}$$

But we have

$$\frac{\sin \pi z}{\pi} = z \prod_{\substack{-\infty \\ k \neq 0}}^{+\infty} (1+\frac{z}{k}) e^{-\frac{z}{k}} \qquad (5)$$

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = z e^{\gamma z} \prod_{k=1}^{+\infty} (1+\frac{z}{k}) e^{-\frac{z}{k}} (1-z) e^{\gamma(1-z)} \prod_{k=1}^{+\infty} (1+\frac{1-z}{k}) e^{-\frac{1-z}{k}})$$

In taking the logarithmic derivative of (4) and (5) we find

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + \sum_{\substack{-\infty \\ k \neq 0}}^{+\infty} (\frac{1}{z+k} - \frac{1}{k})$$

$$\frac{(\frac{1}{\Gamma(z)})'}{\frac{1}{\Gamma(z)}} = \frac{1}{z} + \gamma - \sum_{k=1}^{+\infty} (\frac{1}{z+k} - \frac{1}{k})$$

$$\frac{(\frac{1}{\Gamma(1-z)})'}{\cdots} = \frac{-1}{\cdots} \cdots \sum^{+\infty} (\frac{1}{\cdots} - \frac{1}{\cdots})$$

$$\begin{aligned}
h'(z) &= \frac{1}{z} + \sum_{\substack{-\infty \\ k\neq 0}}^{+\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right) - \Big(\frac{1}{z} + \gamma + \sum_{k=1}^{+\infty}\left(\frac{1}{z+k} - \frac{1}{k}\right)\Big) \\
&\quad - \Big(\frac{-1}{1-z} - \gamma - \sum_{k=1}^{+\infty}\left(\frac{1}{1-z+k} - \frac{1}{k}\right)\Big) \\
&= \sum_{-\infty}^{-1}\left(\frac{1}{z+k} - \frac{1}{k}\right) + \Big(\frac{1}{1-z} + \sum_{k=1}^{+\infty}\left(\frac{1}{1-z+k} - \frac{1}{k}\right)\Big) \\
&= \sum_{-\infty}^{-1}\left(\frac{1}{z+k} - \frac{1}{k}\right) - \frac{1}{z-1} - \sum_{k=1}^{+\infty}\left(\frac{1}{z-1-k} + \frac{1}{k}\right).
\end{aligned}$$

$\frac{1}{z-1-k}+\frac{1}{k}=\frac{1}{z-1-k}-\frac{1}{k+1}+\frac{1}{k}-\frac{1}{k+1}$. Then

$$h'(z)=\sum_{k=1}^{+\infty}(\frac{1}{z-k}+\frac{1}{k})-\frac{1}{z-1}-\sum_{k=2}^{+\infty}(\frac{1}{z-k}+\frac{1}{k})-\sum_{k=1}^{+\infty}(\frac{1}{k}-\frac{1}{k+1})$$

and $h'=0$. Since $\lim_{z\to 0}h(z)=0$, then $h=0$ and

$$\frac{1}{\Gamma(z)\Gamma(1-z)}=\frac{\sin\pi z}{\pi}.$$

4. $\dfrac{1}{\Gamma(z)} = ze^{\gamma z}f(z)$ and

$\displaystyle\lim_{n\to+\infty} ze^{z(\sum_{k=1}^{n}\frac{1}{k}-\ln n)}\prod_{k=1}^{n}(1+\frac{z}{k})e^{\frac{-z}{k}} = \frac{1}{\Gamma(z)}$. The convergence is uniform one any compact of $\mathbb{C}\setminus(-\mathbb{N})$.

$\displaystyle\frac{n}{n^z}\prod_{k=1}^{n}\frac{z+k}{k} = \frac{z(z+1)\dots(z+n)}{n^z n!}$, then

$$\frac{1}{\Gamma(z)} = \lim_{n\to+\infty}\frac{z}{n^z n!}(z+1)\dots(z+n).$$

5. If $\mathrm{Re} z > 0$, the integral $\displaystyle\int_0^{+\infty} t^{z-1}e^{-t}\, dt$ is convergent and defines a holomorphic function on $\{z \in \mathbb{C};\ \mathrm{Re} z > 0\}$.

Lemma

The sequence $(f_n)_n$ *defined by* $f_n(z) = \int_0^n (1 - \frac{t}{n})^n t^{z-1}\, dt$ *converges to* $\int_0^{+\infty} e^{-t} t^{z-1}\, dt$*, whenever* $\mathrm{Re} z > 0$*.*

Lemma

$\lim_{n \to +\infty} f_n(z) = \Gamma(z)$*, whenever* $\mathrm{Re} z > 0$*.*

Proof of lemma 4.4

For $0 < t < n$, $0 < e^{-t} - (1 - \frac{t}{n})^n < \frac{t^2}{2n}$. According to the convergence of the integral $\int_0^{+\infty} e^{-t} t^{z-1}\, dt$, for $\mathrm{Re} z > 0$, then $\forall \varepsilon > 0,\ \exists\ n_0$ such that if $n \geq n_0$

$$\left| \int_n^{+\infty} e^{-t} t^{z-1}\, dt \right| \leq \int_n^{+\infty} e^{-t} t^{x-1}\, dt < \frac{\varepsilon}{3},$$

with $z = x + \mathrm{i} y$.

If $n \geq n_0$,

$$\begin{aligned}\int_0^{+\infty} e^{-t} t^{z-1}\, dt - f_n(z) &= \int_0^{n_0} (e^{-t} - (1-\frac{t}{n})^n) t^{z-1}\, dt + \int_{n_0}^{n} (e^{-t} - \\ &+ \int_n^{+\infty} e^{-t} t^{z-1}\, dt,\end{aligned}$$

$\Big|\displaystyle\int_0^{n_0}(e^{-t}-(1-\frac{t}{n})^n)t^{z-1}\ dt\Big|\leq\frac{1}{2n}\int_0^{n_0}t^{x+1}\ dt$, then there exists n_1 such that, whenever $n\geq n_1\geq n_0$,

$\Big|\displaystyle\int_0^{n_0}(e^{-t}-(1-\frac{t}{n})^n)t^{z-1}\ dt\Big|\leq\frac{\varepsilon}{3}$.

$\Big|\displaystyle\int_{n_0}^{n}(e^{-t}-(1-\frac{t}{n})^n)t^{z-1}\ dt\Big|\leq\int_{n_0}^{+\infty}e^{-t}t^{x-1}\ dt\leq\frac{\varepsilon}{3}$.

$\Big|\displaystyle\int_n^{+\infty}e^{-t}t^{z-1}\ dt\Big|\leq\frac{\varepsilon}{3}$, then the sequence $(f_n(z))_n$ converges to $\displaystyle\int_0^{+\infty}e^{-t}t^{z-1}\ dt$.

Proof of lemma 4.5

We introduce a new variable $\tau = \frac{t}{n}$. An integration by part of the integral yields

$$f_n(z) = n^z \int_0^1 (1-\tau)^n \tau^{z-1}\, d\tau = \frac{n^z}{z} n \int_0^1 (1-\tau)^{n-1} \tau^z\, d\tau.$$

We repeat the same operation, we find

$$f_n(z) = \frac{n^z n}{z(z+1)\dots(z+n-1)} \int_0^1 \tau^{z+n-1}\, d\tau = \frac{n^z n}{z(z+1)\dots(z+n)}.$$

Theorem

$$\Gamma(\frac{1}{2})\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}).$$

Proof

$$\begin{aligned}
\frac{d}{dz}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) + \frac{d}{dz}\Big(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\Big) &= \sum_{n=0}^{+\infty}\frac{1}{(n+z)^2} + \sum_{n=0}^{+\infty}\frac{1}{(n+z+\frac{1}{2})^2} \\
&= 4(\sum_{n=0}^{+\infty}\frac{1}{(2n+2z)^2} + \sum_{n=0}^{+\infty}\frac{1}{(2n+2z+1} \\
&= 4\sum_{n=0}^{+\infty}\frac{1}{(n+2z)^2} = 2\frac{d}{dz}\Big(\frac{\Gamma'(2z)}{\Gamma(2z)}\Big).
\end{aligned}$$

This yields that $\Gamma(z)\Gamma(z+\frac{1}{2}) = e^{az+b}\Gamma(2z)$.
For $z=\frac{1}{2}$, $\Gamma(\dfrac{1}{2}) = \sqrt{\pi}$, $\Gamma(1)=\Gamma(2)=1$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \sqrt{\frac{\pi}{2}}$,
then $\frac{a}{2}+b = \frac{1}{2}\ln\pi,\ a+b = \frac{1}{2}\ln\pi - \ln 2 \Rightarrow a = -2\ln 2$,
$b = \frac{1}{2}\ln\pi + \ln 2$. Then we find the Stirling formula

$$\Gamma(\frac{1}{2})\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}). \qquad (6)$$

Proposition

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^{+\infty} \frac{t^{y-1}}{(1+t)^{x+y}}\, dt,$$ *whenever* $y > 0$ *and* $x > 0$.

Proof

$\Gamma(x)\Gamma(y) = \int_0^{+\infty} t^{x-1}e^{-t}\, dt \int_0^{+\infty} s^{y-1}e^{-s}\, ds$, for all $x > 0$ and $y > 0$. The change of variable $s = tv$ yields $\Gamma(x)\Gamma(y) = \int_0^{+\infty} v^{y-1}(\int_0^{+\infty} t^{x+y-1}e^{-t(1+v)}\, dt)dv$. If we set $u = t(1+v)$, we have

$$\Gamma(x)\Gamma(y) = \int_0^{+\infty} v^{y-1}(\int_0^{+\infty} u^{x+y-1}e^{-u}(1+v)^{-x-y}\, du)dv = \Gamma(x+y)\int_0$$

Remarks 4 :

1. For $x = y = \frac{1}{2}$ and the change of variable $v = \tan^2\theta$, we deduce

$$\left[\Gamma(\frac{1}{2})\right]^2 = 2\int_0^{\frac{\pi}{2}} d\theta = \pi \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

2. If $y = 1 - x$,

$$\Gamma(x)\Gamma(1-x) = \int_0^{+\infty} \frac{u^{-x}}{1+u}\, du = \frac{\pi}{\sin(1-x)\pi} = \frac{\pi}{\sin(\pi x)}, \quad \text{for } 0 < x$$

because $\int_0^{+\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}$, for $0 < a < 1$, then
$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, for $0 < x < 1$.
3.

$$\ln(\Gamma(n)) = (n - \frac{1}{2}) \ln n - n + c + o(1), \tag{7}$$

where c is a constant.

Indeed, $\Gamma(n+1) = n!$, for $n \in \mathbb{N}$, $\ln n! = \sum_{j=1}^{n} \ln j$,

$$\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \ln t \, dt = \int_0^{\frac{1}{2}} (\ln(j+t)+\ln(j-t)) \, dt = \int_0^{\frac{1}{2}} (\ln j^2+\ln(1-\frac{t^2}{j^2})) \, dt = \ln j$$

where $c_j = O(\frac{1}{j^2})$. Then

$$\ln(\Gamma(n)) = \ln(n-1)! = \int_{\frac{1}{2}}^{n-\frac{1}{2}} \ln t \, dt - \sum_{j=1}^{n-1} c_j - \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2} = (n-\frac{1}{2})\ln n - n +$$

Lemma

For n large enough

$$\frac{\Gamma(n)}{\Gamma(n+a)} \approx n^{-a}, \quad \text{for } a > 0.$$

Proof

$$\begin{aligned}\lim_{n\to+\infty} \frac{n^a\Gamma(a)\Gamma(n)}{\Gamma(a+n)} &= \lim_{n\to+\infty} n^a \int_0^1 t^{a-1}(1-t)^{n-1}\, dt \\ &= \lim_{n\to+\infty} \int_0^n u^{a-1}(1-\frac{u}{n})^{n-1}\, du = \int_0^{+\infty} u^{a-1}e^{-u}\end{aligned}$$

Then $\lim_{n\to+\infty} \dfrac{n^a\Gamma(n)}{\Gamma(a+n)} = 1.$

Remark 5 :

If x is not an integer, we set $x = n + a$, with $0 < a < 1$. We find for n large enough

$$\Gamma(n+a) = \Gamma(x) \approx \Gamma(n)n^a.$$

In use of the identity (7), we have

$$\begin{aligned}\ln\Gamma(x)=\ln\Gamma(n+a) &= \ln\Gamma(n)+a\ln n+o(1)\\ &= (n-\frac{1}{2})\ln n-n+c_1+a\ln n+o(1)\\ &= (x-\frac{1}{2})\ln x-x+c_2+o(1).\end{aligned}$$

We intend to compute the constant c_2. By (6), we have

$$\Gamma(2x)\Gamma(\frac{1}{2})=2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2}).$$

Furthermore

$$\ln\Big(\Gamma(2x)\Gamma(\frac{1}{2})\Big)=(2x-\frac{1}{2})\ln 2x-2x+c+o(1)+\ln\sqrt{\pi}$$

and

Therefore $\ln\sqrt{\pi} + \frac{1}{2}\ln 2 + o(1) = c - \frac{1}{2} + x\frac{1}{2x} + o(1) = c + o(1)$.
Then $c = \ln\sqrt{2\pi}$ and $\ln\Gamma(x) = (x - \frac{1}{2})\ln x - x + \ln\sqrt{2\pi} + o(1)$.
We deduce

$$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x}\sqrt{2\pi}(1 + o(1)). \tag{8}$$

$$n! = (n+1)^{n+\frac{1}{2}} e^{-n-1}\sqrt{2\pi}(1 + o(1)). \tag{9}$$

Remark 6 :

$\forall\ z \in \mathbb{C} \setminus (-\mathbb{N}), (\dfrac{\Gamma'(z)}{\Gamma(z)})' = \displaystyle\sum_{k=0}^{+\infty} \dfrac{1}{(k+z)^2}$. Then if $z = x > 0$, the previous formula shows that the function $\ln \Gamma$ is convex on $]0, +\infty[$ (i.e. Γ is logarithmic convex on $]0, +\infty[$).

Theorem

Let $f :]0, +\infty[\longrightarrow \mathbb{R}$ *be a convex function such that*

1. $f(1) = 0$.
2. $e^{f(x+1)} = xe^{f(x)}\ \forall\ x > 0$, *then* e^f *is equal to the restriction of the function* Γ *on* $]0, +\infty[$.

Proof

Let $1 < x < 2$ and $n \in \mathbb{N}$. From the second property,

$$f(x+n+1) = f(x) + \sum_{k=0}^{n} \ln(x+k)$$

and

$$f(x+n+1) = f(n+1) + f(x) + \ln x + \sum_{k=0}^{n-1} \ln(1 + \frac{x}{k+1}),$$

because $f(n+1) = \ln n!$. We use the convexity of f, we find

$$f(n+2)-f(n+1) \leq \frac{f(x+n+1)-f(1+n)}{x} \leq \frac{f(n+3)-f(n+1)}{2} \leq f($$

Then

$$f(x)+\Big(\gamma x+\ln x+\sum_{k=0}^{n-1}(\ln(1+\frac{x}{k+1})-(\frac{x}{k+1}))\Big)$$

is between the two following values $(\gamma+\ln(n+1)-\sum_{k=0}^{n-1}\frac{1}{k+1})x$

and $(\gamma+\ln(n+2)-\sum_{k=0}^{n-1}\frac{1}{k+1})x$, this which yields that

$f(x)=-\gamma x-\ln x-\sum_{k=0}^{\infty}(\ln(1+\frac{x}{k+1})-\frac{x}{k+1})$ and then

$f(x)=\ln\Gamma(x)$ for $1<x<2$ and the result is deduced by the