Harmonic Functions of two Variables

BLEL Mongi

Department of Mathematics
King Saud University

February 14, 2023

Definition

A mapping $U\colon \Omega \longrightarrow \mathbb{R}$ *defined on an open subset* Ω *of* $\mathbb{C}$ *twice continuously differentiable (*U *is of class* $\mathcal{C}^2$*) is called harmonic if* $\Delta U = 0$*, known as Laplace equation, with* $\Delta U = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}$*. (*$\Delta$ *is called the Laplace operator).*

Examples

1. $U(x,y) = x^2 - y^2$ is harmonic.
2. If f is holomorphic on Ω, then $\mathrm{Re} f$ and $\mathrm{Im} f$ are harmonic on Ω.

We intend to show that in general any real harmonic function is locally the real part of a holomorphic function.

Theorem

If Ω is a simply connected domain of $\mathbb{C}$ and $U : \Omega \longrightarrow \mathbb{R}$ harmonic on Ω, there exists a holomorphic function f on Ω such that $U = \mathrm{Re} f$ on Ω.

Proof

The mapping $g(z) = \frac{\partial U}{\partial x}(x,y) - \mathrm{i}\frac{\partial U}{\partial y}(x,y)$ is holomorphic on Ω, with $z = x + \mathrm{i}y$. Since Ω is simply connected, g has a primitive in Ω. Let G be any primitive of g. G is holomorphic and

$$\begin{aligned} g(z) &= \frac{\partial U}{\partial x}(x,y) - \mathrm{i}\frac{\partial U}{\partial y}(x,y) = \frac{\partial \mathrm{Re}\, G}{\partial x}(x,y) + \mathrm{i}\frac{\partial \mathrm{Im}\, G}{\partial x}(x,y) \\ &= -\mathrm{i}\frac{\partial \mathrm{Re}\, G}{\partial y}(x,y) + \frac{\partial \mathrm{Im}\, G}{\partial y}(x,y). \end{aligned}$$

Thus

$$\begin{cases} \frac{\partial U}{\partial x} = \frac{\partial \mathrm{Re}\, G}{\partial x} \end{cases}$$

Corollary

Any harmonic function is locally the real part of a holomorphic function.

Corollary

Any harmonic function is infinitely continuously differentiable.

Corollary

If $U\colon D(0,R) \longrightarrow \mathbb{R}$ *is harmonic, then for all* $0 \leq r < R$

$$U(re^{\mathrm{i}\theta}) = \sum_{-\infty}^{+\infty} a_n r^{|n|} e^{\mathrm{i}n\theta},$$

Proof

Let f be a holomorphic function such that $U = \mathrm{Re} f$, $f(z) = \sum_{n=0}^{+\infty} b_n z^n$, then

$$U(re^{\mathrm{i}\theta}) = \mathrm{Re} b_0 + \frac{1}{2}\sum_{n=1}^{+\infty} b_n r^n e^{\mathrm{i}n\theta} + \frac{1}{2}\sum_{n=1}^{+\infty} \overline{b_n} r^n e^{-\mathrm{i}n\theta}.$$

We set $a_0 = \mathrm{Re} b_0$ and for $n \geq 1$, $a_n = \frac{1}{2} b_n$ and for $n \leq -1$, $a_n = \frac{1}{2}\overline{b_{-n}}$. We remark that

$$a_n r^{|n|} = \frac{1}{2\pi}\int_0^{2\pi} U(re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta}\, d\theta.$$

We can prove the same result using Fourier series of functions The mapping $\theta \longmapsto U(re^{i\theta})$ is infinitely continuously differentiable (C^∞) and 2π-periodic, thus $U(re^{i\theta}) = \sum_{n=-\infty}^{+\infty} C_n e^{in\theta}$, for all $r < R$.

The Fourier's coefficients C_n are given by

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} U(re^{i\theta})e^{-in\theta}\, d\theta = a_n r^n.$$

Corollary (Liouville's Theorem)

Any bounded harmonic function on $\mathbb{C}$ is constant.

Proof

Let U be a harmonic function bounded by M on $\mathbb{C}$. For all $r > 0$, we have

$$U(re^{\mathrm{i}\theta}) = \sum_{-\infty}^{+\infty} a_n r^{|n|} e^{\mathrm{i}n\theta}.$$

$$a_n r^{|n|} = \frac{1}{2\pi} \int_0^{2\pi} U(re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta}\, d\theta.$$

Then that $|a_n r^{|n|}| \leq M$ and $a_n = 0$ if $n \neq 0$. □

Corollary

Any harmonic function on $\mathbb{C}$, bounded above or bounded below is constant.

Proof

If we replace U by $-U$, we can suppose that U is bounded above. Since $\mathbb{C}$ is a simply connected domain, there exists a holomorphic function f on $\mathbb{C}$ such that $U = \mathrm{Re} f$. Without loss of generality, we can suppose that U is non positive. Thus $|e^f| = e^{\mathrm{Re} f} = e^U \leq 1$. By Liouville's theorem e^f is constant, then f and U are constant.
$\square$

Theorem

Let U be a harmonic function on a domain Ω. If $\Omega' \neq \emptyset$ is a subdomain of Ω and $U = 0$ on Ω', then $U = 0$ on Ω.

Proof

Suppose first that Ω' is a disc, f analytic on Ω' and $U = \mathrm{Re} f$. In view of the Cauchy-Riemann equations, f is constant on Ω^*, and therefore f is constant on Ω, and hence $U = 0$.

For arbitrary domain, we consider the subset $A = \{z \in \Omega, U = 0 \text{ in a neighborhood of } z\}$. A is open and closed in Ω, then it is equal to Ω. □

Remark 1 :
Let Ω_1, and Ω_2 be two domains such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. If U_1, U_2 are harmonic functions on Ω_1 respectively on U_2 and $U_1 = U_2$ on $\Omega_1 \cap \Omega_2$. These conditions determine a unique harmonic function on $\Omega_1 \cup \Omega_2$ uniquely. Indeed, if V_2 is another harmonic function satisfying the same conditions, then $V_2 - U_2 = 0$ on $\Omega_1 \cap \Omega_2$. In view of the previous theorem, $V_2 = U_2$ on Ω_2.

The function U_2 is called the harmonic continuation (or extension) of U_1, into the domain Ω_2.

Proposition

Let Ω be an open subset of $\mathbb{C}$ and U a harmonic function on $\Omega \setminus \{a\}$, bounded above in a neighborhood of a, $(a \in \Omega)$. Then there exists a constant $c \geq 0$ such that $U - c \ln |z - a|$ can be extended on Ω to a harmonic function.

Proof

We can suppose that $a = 0$ and we consider $R > 0$ such that $D(0, R) \subset \Omega$. We set

$$U_x = \frac{\partial U}{\partial x}, \quad U_y = \frac{\partial U}{\partial y}, \quad U_r = \frac{\partial U}{\partial r} \quad \text{and } U_\theta = \frac{\partial U}{\partial \theta},$$

with $z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta$. We have

$$U_r = U_x\cos\theta + U_y\sin\theta \quad \text{and} \quad U_\theta = -rU_x\sin\theta + rU_y\cos\theta.$$

The mapping $rU_r - \mathrm{i}U_\theta = (x+\mathrm{i}y)(U_x - \mathrm{i}U_y) = zW(z)$ is holomorphic on a neighborhood of 0 except at 0. Let $zW(z) = \sum_{-\infty}^{+\infty} C'_n z^n$ its Laurent expansion. If $C'_n = a'_n + \mathrm{i}b'_n$, we have

$$rU_r = \sum_{-\infty}^{+\infty}(a'_n \cos n\theta - b'_n \sin n\theta)r^n, \quad U_\theta = -\sum_{-\infty}^{+\infty}(b'_n \cos n\theta + a'_n \sin n\theta)r^n.$$

For $0 < r_0 < R$,

$$U(re^{\mathrm{i}\theta}) - U(r_0 e^{\mathrm{i}\theta}) = a'_0 \ln\frac{r}{r_0} + \sum_{n=-\infty, n\neq 0}^{+\infty}(a'_n \cos n\theta - b'_n \sin n\theta)\left(\frac{r^n - r_0^n}{n}\right).$$

$$U(r_0e^{i\theta}) - U(r_0) = -b_0'\theta + \sum_{n=-\infty, n\neq 0}^{+\infty} \left(a_n'(\cos n\theta - 1) - b_n' \sin n\theta\right)\frac{r_0^n}{n}.$$

Since $U(re^{i\theta})$ is 2πperiodic, then $b_0' = 0$. Thus

$$U(re^{i\theta}) - U(r_0) = C + a_0' \ln\frac{r}{r_0} + \sum_{n=-\infty, n\neq 0}^{+\infty} (a_n' \cos n\theta - b_n' \sin n\theta)\frac{r^n}{n},$$

with $C = -\sum_{n=-\infty, n\neq 0}^{+\infty} a_n'\frac{r_0^n}{n}$. Then

$$U(re^{i\theta}) = k\ln r + a_0 + \sum^{+\infty} (a_n \cos n\theta - b_n \sin n\theta)r^n,$$

$k \ln r + a_0 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} U(e^{\mathrm{i}\theta}) d\theta$. Since U is bounded above on a neighborhood of 0, then $k \geq 0$ and $a_n = 0$ and $b_n = 0$, for all $n < 0$. Thus $U - k \ln r$ can be extended to a harmonic function on a neighborhood of 0.

Corollary

Let Ω be an open subset of $\mathbb{C}$ and let $a \in \Omega$. Then any harmonic function U on $\Omega \setminus \{a\}$ bounded on any neighborhood of a can be extended on Ω to a harmonic function.

Proof

If $a = 0$, it results from the previous proposition,

$$U(re^{i\theta}) = k \ln r + a_0 + \sum_{n=-\infty, n\neq 0}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n,$$

Since U is bounded on a neighborhood of 0, then $k = 0$, $a_n = 0$ and $b_n = 0$, for $n < 0$. Then U can be extended to a harmonic

Theorem

Let $U\colon \Omega \longrightarrow \mathbb{R}$ *be a harmonic function. We assume that* $\Omega \supset \overline{D(z_0, R)}$*, then*

$$U(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{\mathrm{i}\theta})\ d\theta, \quad \forall r < R,$$

and

$$U(z_0) = \frac{1}{\pi R^2}\int\int_{D(z_0,R)} U(x,y)\ dxdy.$$

The number $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} U(z_0 + re^{\mathrm{i}\theta})\, d\theta$ is called the mean of f on the circle of center z_0 and radius r and the number $\dfrac{1}{\pi R^2}\displaystyle\int\int_{D(z_0,R)} U(x,y)\, dxdy$ is called the mean of f on the disc of radius R and centered at z_0.

Proof

Let $\varepsilon > 0$ such that $D(z_0, R+\varepsilon) \subset \Omega$. There exists $f \in \mathcal{H}(D(z_0, R+\varepsilon))$ such that $U = \mathrm{Re} f$ on this disc. Since

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta})\ d\theta,$$

then

$$U(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{\mathrm{i}\theta})\ d\theta.$$

Moreover
$$\int_0^R U(z_0) r \, dr = U(z_0)\frac{R^2}{2} = \frac{1}{2\pi}\int_0^R \int_0^{2\pi} U(z_0 + re^{i\theta}) \; rdr \; d\theta.$$
Thus

$$U(z_0) = \frac{1}{\pi R^2} \iint_{D(z_0,R)} U(x,y) \; dx \; dy.$$

□

Corollary (Liouville's Theorem)

Any non negative harmonic function on $\mathbb{C}$ is constant.

This is an other proof of Corollary 1.7. This result is generalized by Picard for harmonic function on $\mathbb{R}^n$, with $n \geq 3$. We yield a proof on $\mathbb{R}^2$, which is the same in $\mathbb{R}^n$, with $n \geq 3$.

Let $a, b \in \mathbb{C}$ and $r = |a - b|$. Then by the mean property

$$\pi R^2 U(a) = \int_{D(a,R)} U(y)dy \leq \int_{D(b,R+r)} U(y)dy = \pi(R+r)^2 U(b).$$

Then $U(a) \leq U(b)$. (It is enough to divide by πR^2 and tends R to $+\infty$.) Thus $U(a) = U(b)$.

Corollary

Any non negative harmonic function on $\mathbb{C}^*$ is constant.

Proof

If U is a non negative harmonic function on $\mathbb{C}^*$, then the function $z \longmapsto U(e^z)$ is a non negative harmonic on $\mathbb{C}$, thus it is constant, which shows that U is constant.

Theorem (Maximum principle)

Let Ω be a bounded domain and U a continuous function on $\overline{\Omega}$ and harmonic on Ω. Then $\sup\limits_{\overline{\Omega}} U = \sup\limits_{\partial\Omega} U$, $\inf\limits_{\overline{\Omega}} U = \inf\limits_{\partial\Omega} U$ and if the maximum or the minimum of U is reached in Ω, then U is constant.

Proof

In considering $-U$ which is harmonic, it suffices to prove the result for the maximum. Let $M = \sup_{\overline{\Omega}} U$ and $A = \{z \in \Omega;\ U(z) = M\}$.

• If $A = \emptyset$ the result is trivial.

• If $A \neq \emptyset$ and $z_0 \in \Omega$ such that $U(z_0) = M$, then there exists $R > 0$ such that $\overline{D(z_0, R)} \subset \Omega$.

$$U(z_0) = \frac{1}{2\pi} \int_0^{2\pi} U(z_0 + re^{i\theta})\ d\theta \quad \forall\ r \leq R.$$

Then $\int_0^{2\pi} \left(U(z_0) - U(z_0 + re^{i\theta})\right)\, d\theta = 0$ and $\left(U(z_0) - U(z_0 + re^{i\theta})\right) \geq 0$. Thus $U(z_0) = U(z_0 + re^{i\theta})$ for all $r \leq R$ and $\theta \in [0, 2\pi]$. Then U is constant on any disc $D(z_0, R)$. It results that $A = \emptyset$ or $A = \Omega$. □

(We remarked in chapter that any function which verifies the Mean Property it fulfills the maximum principle.)

Corollary

Let U and V be two harmonic functions on a bounded domain Ω. We assume that U and V are continuous on $\overline{\Omega}$ and $U_{\restriction \partial\Omega} = V_{\restriction \partial\Omega}$, then $U \equiv V$ on Ω.

Proof

$U - V$ and $V - U$ are harmonic on Ω,
$\sup_{\partial\Omega}(U - V) = \inf_{\partial\Omega}(U - V) = 0$, thus $U \equiv V$.

Corollary (Maximum Principle)

Let $\Omega \neq \mathbb{C}$ be a domain non necessarily bounded of $\mathbb{C}$, and let U be a harmonic function on Ω. We assume that for any sequence $(a_n)_n$ of Ω which converges to a point of $\partial\Omega$ or tends to ∞, $\overline{\lim}_{n\to+\infty} U(a_n) \leq M$, then $U \leq M$ on Ω.

(We say that the sequence $(a_n)_n$ of Ω tends to ∞, if $\lim_{n\to+\infty} |a_n| = +\infty$.)

Proof

Let $M' = \sup_{z\in\Omega} U(z)$. There exists a sequence $(a_n)_n$ of Ω such that $\lim_{n\to+\infty} U(a_n) = M'$. If the sequence $(a_n)_n$ has a limit point b in Ω, then there exists a subsequence $(a_{n_k})_k$ which converges to b and $\lim_{k\to+\infty} U(a_{n_k}) = M'$. By maximum principle, U is constant on Ω.

If the sequence $(a_n)_n$ has no limit point in Ω, then there exists a subsequence $(a_{n_k})_k$ which converges to a point in $\partial\Omega$ or tends to ∞. Then $M' \leq M$. Then $M' \leq M$.

Corollary

Any real harmonic function can not have an isolate zero.

Proof

Let a be a zero of a harmonic function U on a domain Ω. We assume that $U \not\equiv 0$ on Ω. For all $r > 0$ such that $\overline{D(a,r)} \subset \Omega$, by mean value property, the function U has a zero on $\mathcal{C}(a,r)$.

Let Ω be a bounded open subset and ψ a continuous function on $\partial\Omega$. The Dirichlet problem on Ω with the given function ψ on $\partial\Omega$, consists to find a continuous function $U \colon \overline{\Omega} \longrightarrow \mathbb{R}$ and harmonic on Ω such that $U_{\restriction \partial\Omega} = \psi$. If there exists a such function, it is unique.

Poisson Kernel

Let $0 \leq r < 1$. The Poisson kernel is the mapping defined on $\mathbb{R}$ by

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

Properties

1. $P_r(\theta - t) = \dfrac{1-r^2}{1-2r\cos(\theta-t)+r^2} = \mathrm{Re}\,\dfrac{e^{\mathrm{i}t}+re^{\mathrm{i}\theta}}{e^{\mathrm{i}t}-re^{\mathrm{i}\theta}}.$
2. $P_r \geq 0.$
3. $P_r(\theta) = \sum_{-\infty}^{\infty} r^{|n|} e^{\mathrm{i}n\theta}.$
4. $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P_r(\theta)\, d\theta = 1.$
5. For all $0 < \delta < \pi$, $\sup_{\delta \leq \theta \leq 2\pi - \delta} P_r(\theta) \underset{r\to 1}{\longrightarrow} 0.$ ($P_r(\theta) = \dfrac{1-r^2}{|1-re^{\mathrm{i}\theta}|^2} \leq \dfrac{1-r^2}{\sin^2 \delta} \longrightarrow 0.$)

Theorem (Poisson Formula)

Let f be a holomorphic function on a neighborhood of $\overline{D}$, then for all $|z| < 1$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\frac{e^{\mathrm{i}t}+z}{e^{\mathrm{i}t}-z} f(e^{\mathrm{i}t})\, dt. \tag{1}$$

$$U(re^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)U(e^{\mathrm{i}t})\, dt, \quad \text{with } U = \mathrm{Re} f.$$

Proof

The formula (1) for $z = 0$ is the Mean Property.
For $z \neq 0$, we apply the Cauchy's formula to the function f, we find

$$f(z) = \frac{1}{2\mathrm{i}\pi} \int_\gamma \frac{f(w)}{w - z}\, dw,$$

with $\gamma(t) = e^{\mathrm{i}t}$, $t \in [0, 2\pi]$.
If $|z'| > 1$, $\frac{1}{2\mathrm{i}\pi} \int_\gamma \frac{f(w)}{w - z'}\, dw = 0$. In particular for $z' = \frac{1}{\bar{z}}$, with $z = re^{\mathrm{i}\theta}$, we have

$$f(re^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \Big(\frac{1}{e^{\mathrm{i}t} - re^{\mathrm{i}\theta}} - \frac{1}{e^{\mathrm{i}t} - \frac{e^{\mathrm{i}\theta}}{r}}\Big) f(e^{\mathrm{i}t})\, dt.$$

$$\frac{1}{e^{\mathrm{i}t} - re^{\mathrm{i}\theta}} - \frac{1}{e^{\mathrm{i}t} - \frac{e^{\mathrm{i}\theta}}{r}} = P_r(\theta - t).$$

Theorem

Let $\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, 2π-periodic, then there exists a function $U\colon \overline{D(0,R)} \longrightarrow \mathbb{R}$ continuous on $\overline{D(0,R)}$ and harmonic on $D(0,R)$ such that $U(Re^{it}) = \psi(t)$ and for all $0 \leq r < R$

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{\frac{r}{R}}(\theta - t)\psi(t)\ dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\frac{r}{R}}(t)\psi(\theta - t)\ dt.$$

Proof

Let $z_0 = Re^{i\theta_0}$.

$$U(re^{i\theta}) - \psi(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\frac{r}{R}}(t)(\psi(\theta - t) - \psi(\theta_0))\, dt.$$

By the continuity of ψ, for $\varepsilon > 0$, $\exists\ \eta > 0$ be such that $|\alpha - \theta_0| < \eta \Rightarrow |\psi(\alpha) - \psi(\theta_0)| \leq \varepsilon$. For $\theta \in]\theta_0 - \frac{\eta}{2}, \theta_0 + \frac{\eta}{2}[$ and $|t| < \frac{\eta}{2}$, then $|\theta - t - \theta_0| < \eta$.

$$\begin{aligned}\int_{-\pi}^{\pi} P_{\frac{r}{R}}(t)(\psi(\theta - t) - \psi(\theta_0))\, dt &= \int_{|t|<\frac{\eta}{2}} P_{\frac{r}{R}}(t)(\psi(\theta - t) - \psi(\theta_0))\, dt \\ &+ \int_{\frac{\eta}{2}<|t|<\pi} P_{\frac{r}{R}}(t)(\psi(\theta - t) - \psi(\theta_0))\end{aligned}$$

We have

$$\frac{1}{2\pi}\Big|\int_{|t|<\frac{\eta}{2}} P_{\frac{r}{R}}(t)(\psi(\theta-t)-\psi(\theta_0))\ dt\Big| \leq \varepsilon$$

and

$$\frac{1}{2\pi}\Big|\int_{\frac{\eta}{2}<|t|<\pi} P_{\frac{r}{R}}(t)(\psi(\theta-t)-\psi(\theta_0))\ dt\Big| \leq 2M\frac{1-(\frac{r}{R})^2}{\sin^2\frac{\eta}{2}} \leq \varepsilon.$$

For $r \geq r_0$, with r_0 close to R and $M = \sup_{t\in\mathbb{R}}|\psi(t)|$.

Thus $\forall\ \varepsilon > 0,\ \exists\ \eta > 0$ and $0 < r_0 \leq R$ such that if $|\theta - \theta_0| < \frac{\eta}{2}$ and $r \geq r_0$, we have $|U(re^{i\theta}) - \psi(\theta_0)| \leq 2\varepsilon$. Thus U is continuous on $\overline{D(0,R)}$ and $U(Re^{i\theta}) = \psi(\theta)$. U is harmonic on $D(0,R)$ because U is the real part of a holomorphic function.

Remarks 2 :

1. The solution of the Dirichlet problem is unique (by the the maximum principle).
2. If ψ is a locally integrable function on $\mathbb{R}$ and 2π-periodic, then for all $R > 0$, the mapping U defined on $D(0, R)$ by

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{\frac{r}{R}}(\theta - t)\psi(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\frac{r}{R}}(t)\psi(\theta - t)\,dt,$$

for all $r < R$ is harmonic on $D(0, R)$ and for any point of continuity θ_0 of ψ, $\lim_{\theta\to\theta_0, r\to R} U(re^{i\theta}) = \psi(\theta_0)$.

Theorem

Any continuous function on an open subset Ω of $\mathbb{C}$ which verifies the Mean Property is harmonic.

Proof

Let $U\colon \Omega \longrightarrow \mathbb{R}$ be a continuous function which verifies the Mean Property. To show that U is harmonic on Ω, it suffices to show that U is harmonic in a neighborhood of each point. Let D be a disc of center z and of boundary $\mathscr{C}$ contained in Ω. There exists a continuous function V on $\overline{D}$, harmonic on D and equal to U on the circle $\mathscr{C}$. Then $V = U$ on D.

□

Corollary

Let Ω be an open subset of $\mathbb{C}$, the space of harmonic functions equipped with the topology of the uniform convergence on any compact is a complete space.

Proof

It suffices to show that the space of harmonic functions on an open subset Ω is closed in the space of continuous functions on Ω equipped with the topology of the uniform convergence on any compact.

Let $(U_n)_n$ be a sequence of harmonic functions which converges uniformly on compact subsets to a function U on Ω. U is continuous and is the Mean Property, thus U is harmonic.

Theorem

Let U be a locally integrable function on a domain Ω and such that

$$U(a) = \frac{1}{\pi r^2} \int_{D(a,r)} U(x,y)dxdy$$

for all $a \in \Omega$ and all $r > 0$ such that $\overline{D(a,r)} \subset \Omega$, then U is harmonic.

Proof

Il suffices to show that U is continuous.
Let $a \in \Omega$ and $r > 0$ such that $K = \overline{D(a, 2r)} \subset \Omega$. We consider a sequence $(a_n)_n$ which converges to a. We can suppose that $(a_n)_n$ is in the disc $D(a, r)$. Then by dominated convergence theorem

$$\begin{aligned} \lim_{n\to+\infty} U(a_n) &= \frac{1}{\pi r^2}\int_{D(a_n,r)} U(x,y)dxdy = \lim_{n\to+\infty}\frac{1}{\pi r^2}\int_K \chi_{D(a_n,r)} U(x, \\ &= \frac{1}{\pi r^2}\int_{D(a,r)} U(x,y)dxdy = U(a). \end{aligned}$$

An other proof of the Corollary 2.3

Let $r > 0$ such that $\overline{D(a,r)} \subset \Omega$. We consider the harmonic function V solution of the Dirichlet problem on the disc $D(a,r)$ and equal to U on $\partial D(a,r)$. We intend to show that $U = V$ on $D(a,r)$.

Let $\varepsilon > 0$ small enough and the mapping
$U_\varepsilon = U - V - \varepsilon \ln(\frac{x^2+y^2}{r^2})$.
$\lim\limits_{x^2+y^2 \to r^2} U_\varepsilon(x,y) = 0$ and $\lim\limits_{(x,y)\to(0,0)} U_\varepsilon(x,y) = +\infty$. Then by the the Maximum principle $U_\varepsilon \geq 0$ on $D(a,r) \setminus \{0\}$. In making tends ε to 0, we have $U \geq V$. In consider $-U$, we have $U = V$. Thus U can be extended to a harmonic function on Ω.

Theorem (Characterization of Harmonic Functions)

Let $U : \Omega \longrightarrow \mathbb{C}$ be a continuous function. The following properties are equivalent

1. *U is harmonic on Ω.*
2. *U verifies the Mean Property.*
3. *For any disc $\overline{D(a,R)} \subset \Omega$, and any polynomial P,*

$$\sup_{z\in D(a,R)} |(U-P)(z)| = \sup_{z\in \mathscr{C}(a,R)} |(U-P)(z)|.$$

4. *For any disc $\overline{D(a,r)} \subset \Omega$,*

Proof

1) $\Rightarrow$ 2) results from theorem 3.1.
2) $\Rightarrow$ 1) results from theorem 6.1.
1) $\Rightarrow$ 4) results from the Poisson's formula 5.2.
4) $\Rightarrow$ 1) results from theorem 5.3, since the solution of the Dirichlet's problem is harmonic.
1) $\Rightarrow$ 3) results from the fact that any polynomial is a holomorphic function, thus the maximum principle yields the result.
It remains to show that 3 $\Rightarrow$ 4.

Let $a \in \Omega$ and $R > 0$ such that $\overline{D(a,R)} \subset \Omega$ and $\tilde{U}$ the solution of the Dirichlet problem on $D(a,R)$ and equal to U on $\mathscr{C}(a,R)$.
There exist two holomorphic functions g and h on $D(a,R)$ such that $\mathrm{Re} g = \mathrm{Re}\tilde{U}$ and $\mathrm{Re} h = \mathrm{Im}\tilde{U}$.
$\tilde{U}$ is uniformly continuous on the compact $\overline{D(a,R)}$, then for $\varepsilon > 0$, there exists $s \in]0,1[$ such that, whenever $z, w \in \overline{D(a,R)}$ and $|z-w| \leq sR$, $|\tilde{U}(z) - \tilde{U}(w)| \leq \varepsilon$.
The Taylor series of g and h has a radius of convergence at least R, then these series converge uniformly on $\overline{D(a,(1-s)R)}$. If $g(a+z) = \sum_{n=0}^{+\infty} a_n z^n$, for $|z| < R$, there exist $N \in \mathbb{N}$ such that

$$\left| \sum_{n=N+1}^{+\infty} a_n z^n \right| \leq \varepsilon, \quad \forall z \in \overline{D(0,(1-s)R)}.$$

Then whenever $\theta \in [0, 2\pi]$,

$$\left|P(a+Re^{i\theta}) - g(a+(1-s)Re^{i\theta})\right| \leq \varepsilon,$$

$$\left|\mathrm{Re}P(a+Re^{i\theta}) - \mathrm{Re}g(a+(1-s)Re^{i\theta})\right| \leq \varepsilon$$

and

$$\left|\mathrm{Re}P(a+Re^{i\theta}) - \mathrm{Re}g(a+Re^{i\theta})\right| \leq 2\varepsilon.$$

Then

$$\left|\mathrm{Re}(P-g)(a+Re^{i\theta})\right| \leq 2\varepsilon.$$

If $w \in D(a, R)$, the assumption 3) gives that whenever $t \in \mathbb{R}$,

$$\left|(\tilde{U} - P + t)(w)\right| \leq \sup_{\theta \in \mathbb{R}} \left|(\tilde{U} - P + t)(a + Re^{i\theta})\right|$$

Then

$$\left|(\tilde{U} - P + t)(w)\right|^2 \leq \sup_{\theta \in \mathbb{R}} \left|(\tilde{U} - P + t)(a + Re^{i\theta})\right|^2$$

It results that

$$\left|(\tilde{U} - P)(w)\right|^2 + 2t\mathrm{Re}(\tilde{U} - P)(w) \leq \sup_{\theta \in \mathbb{R}} \left|(\tilde{U} - P)(a + Re^{i\theta})\right|^2 + 2t \sup_{\theta \in \mathbb{R}} \left|\mathrm{Re}(\right.$$

If we tend t to $\pm\infty$,

we have

$$\left|\mathrm{Re}(\tilde{U}-P)(w)\right| \le \sup_{\theta\in\mathbb{R}} \left|\mathrm{Re}(\tilde{U}-P)(a+Re^{\mathrm{i}\theta})\right| \le 2\varepsilon.$$

Since $\tilde{U}-P$ is harmonic and $\left|\mathrm{Re}(P-g)(a+Re^{\mathrm{i}\theta})\right| \le 2\varepsilon$, then $\left|\mathrm{Re}(\tilde{U}-g)(w)\right| \le 4\varepsilon$.
We prove in the same way that $\left|\mathrm{Im}(\tilde{U}-g)(w)\right| \le 4\varepsilon$, then $\left|(\tilde{U}-g)(w)\right| \le 4\varepsilon$, which proves that $\tilde{U}=g$.

□

Theorem (The Rado's Theorem)

Let f be a continuous function on an open subset Ω and holomorphic on $\Omega \setminus Z_f$, where $Z_f = \{z \in \Omega;\ f(z) = 0\}$ the zero set of f. Then f is holomorphic on Ω.

Proof

Let P be a polynomial and $R > 0$ such that $\overline{D(a,R)} \subset \Omega$. We claim that $(f - P)$ is harmonic on Ω. By theorem **??**, to prove that f is harmonic on $D(a,R)$, it suffices to prove that the maximum of $|f - P|$ on $D(a,R)$ is reached on $\mathscr{C}(a,R)$.

If $|f-P|$ reaches it maximum at $w \in D(a,R)$ and not on $\mathcal{C}(a,R)$, then $f-P$ is not holomorphic in a neighborhood of w, which proves that w is in the boundary of Z_f. There exists a sequence $(w_n)_n$ of $D(a,R) \setminus Z_f$ which converges to w, where $Z_f = \{z;\ f(z)=0\}$. Then

$$|(f-P)(w_n)| > M = \sup_{z \in \mathscr{C}(a,R)} |(f-P)(z)|, \quad \forall n \in \mathbb{N}.$$

If $m = \sup_{z \in \mathscr{C}(a,R)} |f(z)|$, there exists an integer N such that

$$\left(\frac{|(f-P)(w_n)|}{M}\right)^N > \frac{m}{|f(w_n)|}$$

Let g be the function defined by $g = f(f-P)^N$. Since

Proposition (Harnack's Inequality)

Let Ω be an open subset of $\mathbb{C}$, $a \in \Omega$, $R > 0$ and U a continuous function on $\overline{D(a,R)}$, harmonic on $D(a,R)$ and $U \geq 0$. Then for all $0 \leq r < R$ and all $\theta \in \mathbb{R}$ we have

$$\frac{R-r}{R+r}U(a) \leq U(a+re^{i\theta}) \leq \frac{R+r}{R-r}U(a). \tag{2}$$

Proof

By Poisson Formula, we have

$$U(a+re^{\mathrm{i}t}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} U(a+Re^{\mathrm{i}\theta})\ d\theta.$$

$\dfrac{R-r}{R+r} \leq \dfrac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \leq \dfrac{R+r}{R-r}$. The result is deduced by mean property.

Corollary

Let Ω be a domain of $\mathbb{C}$ and $(U_n)_n$ an increasing sequence of harmonic functions. If the limit of $(U_n(a))_n$ exists and finite at $a \in \Omega$, then the sequence $(U_n)_n$ converges uniformly on compact subsets of Ω to a harmonic function.

Proof

We can assume that $U_n \geq 0$ (if not we take $U_n - U_0$). We set $U(z) = \sup_{n\in\mathbb{N}} U_n(z)$. From the Harnack's inequality.

$$\frac{R - |z-a|}{R + |z-a|} U_n(a) \leq U_n(z) \leq \frac{R + |z-a|}{R - |z-a|} U_n(a).$$

Thus the sequence $(U_n)_n$ converges on any closed disc centered at a in Ω. (Increasing sequence and bounded above). Let $A = \{z \in \Omega;\ (U_n(z))_n \text{ converge}\}$. The set A is non empty because $a \in A$ and A is an open subset from which above.

Let $z_0 \in \overline{A} \cap \Omega$ and $r > 0$ such that $D(z_0, r) \subset \Omega$. There exists $z_1 \in A$ such that $z_1 \in D(z_0, \frac{r}{2})$, thus $z_0 \in D(z_1, \frac{r}{2})$ and in this disc the sequence $(U_n)_n$ converges. Thus $A = \Omega$.

Let prove that U is continuous. Let $z_0 \in \Omega$, by Harnack's inequality, if $z \in D(z_0, R) \subset \Omega$

$$\frac{R - |z - z_0|}{R + |z - z_0|} U(z_0) \leq U(z) \leq \frac{R + |z - z_0|}{R - |z - z_0|} U(z_0).$$

Then

$$\frac{-2|z - z_0|}{R + |z - z_0|} U(z_0) \leq U(z) - U(z_0) \leq \frac{2|z - z_0|}{R - |z - z_0|} U(z_0).$$

Thus U is continuous on Ω. $(U_n)_n$ verifies the mean property, by the monotone convergence theorem, U is harmonic on Ω. By Dini's theorem, the convergence is uniform on any compact of Ω.

For harmonic extension (continuation) we prove the Schwarz reflection principle.

Theorem

Let Ω be a domain in $\mathbb{C}$ symmetric with respect to the real axis. Let $\Omega^+ = \Omega \cap \mathcal{H}^+$, $\Omega^- = \Omega \cap \mathcal{H}^-$ and I a non empty open interval of $\Omega \cap \mathbb{R}$. Suppose that a harmonic function $U(x, y) = U(z)$ on Ω^+ and such that for all $a \in I$, $\lim\limits_{z \in \Omega^+ \longrightarrow a} U(z) = 0$. Then U can be continued (extended) harmonically on the domain Ω. The harmonic continuation is defined by the function $\tilde{U}$ which is equal to U on Ω^+, 0 on the segment I, and $-U(\bar{z})$ on Ω^-.

Proof

We must prove that $\tilde{U}$ is harmonic on the domain Ω. By definition, $\tilde{U}$ is harmonic on the domain $\Omega^+ \cup \Omega^-$. To show that $\tilde{U}$ is also harmonic on the segment I, we consider a disc $D(0,R)$ with $a \in I$ and R is so small that $D(0,R) \subset \Omega$. Let V be the solution of the Dirichlet problem on the disc $D(0,R)$ and equal to $\tilde{U}$ on the boundary of $D(0,R)$.

$$V(z) = V(a+re^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{U}(a+Re^{\mathrm{i}\varphi})\frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-\varphi)}d\theta$$

To prove that $\tilde{U}$ is harmonic on the real axis, we will show that $\tilde{U}(z) = V(z)$ in $D(0,R)$.

The functions V and $\tilde{U}$ are equal on the semi circle $\{z \in \mathbb{C};\ \mathrm{Im} z > 0,\ |z-a| = R\}$. If z lies on the real axis, the integral from the upper and lower semi-circles cancel, hence, $V = 0 = \tilde{U}$ on that part of I which lies in $D(0,R)$. By the Maximum and Minimum Principles, $V(z) = \tilde{U}(z)$ in the upper half of $D(0,R)$.

Suppose first that z is in the upper half of $D(0,R)$. On the boundary arc $\mathrm{Im} z > 0$, $|z-a| = R$, the function V takes the boundary values $\tilde{U}(a + Re^{\mathrm{i}\theta})$. By the same argument $V(z) = \tilde{U}(z)$ in the lower half of $D(0,R)$. Then V is equal to $\tilde{U}$ on $D(0,R)$.

$\square$