Diagonalization of Matrices

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Eigenvalue and Eigenvector

Definition

If $A \in \mathscr{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$.
We say that λ is an eigenvalue of the matrix A if there is $X \in \mathbb{R}^n \setminus \{0\}$ such that

$$AX = \lambda X.$$

In this case, we say that X is an eigenvector of the matrix A with respect to the eigenvalue λ.

Theorem

If $A \in \mathscr{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$.
λ is an eigenvalue the matrix A if and only if $|\lambda I - A| = 0$.

Definition

If $A \in \mathscr{M}_n(\mathbb{R})$, the polynomial

$$q_A(\lambda) = |\lambda I - A|$$

is called the characteristic equation of the matrix A.

Example

Find the eigenvalues of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}, A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

Theorem

If $A \in \mathscr{M}_n(\mathbb{R})$ and $v_1, \ldots, v_m$ are eigenvectors for different eigenvalues $\lambda_1, \ldots, \lambda_m$, then $v_1, \ldots, v_m$ are linearly independent.

Proof

We do the proof by induction.

The result is true for $m = 1$. We assume the result for m and let $v_1, \ldots, v_{m+1}$ eigenvectors for different eigenvalues $\lambda_1, \ldots, \lambda_{m+1}$.

If

$$a_1 v_1 + \dots a_m v_m + a_{m+1} v_{m+1} = 0$$

then

$$a_1 \lambda_1 v_1 + \dots a_m \lambda_m v_m + a_{m+1} \lambda_{m+1} v_{m+1} = 0$$

Also we have

$$a_1 \lambda_{m+1} v_1 + \dots a_m \lambda_{m+1} v_m + a_{m+1} \lambda_{m+1} v_{m+1} = 0$$

Then

$$a_1(\lambda_1 - \lambda_{m+1}) v_1 + \dots + a_m(\lambda_m - \lambda_{m+1}) v_m = 0.$$

Since $(\lambda_j - \lambda_{m+1}) \neq 0$ for all $j = 1, \dots m$, then $a_1 = \dots = a_m = 0$ and so $a_{m+1} = 0$.

Definition

We say that a matrix $A \in \mathscr{M}_n(\mathbb{R})$ is diagonalizable if there exists an invertible matrix $P \in \mathscr{M}_n(\mathbb{R})$ such that the matrix $P^{-1}AP$ is diagonal .

Remark

If $X_1, \dots, X_n$ are the columns of the matrix P, then the columns of the matrix AP are: $AX_1, \dots, AX_n$.
Moreover if

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{pmatrix}$$

then the columns of the matrix PD are: $\lambda_1 X_1, \dots, \lambda_n X_n$.
Then $P^{-1}AP = D \iff PD = AP$ and the columns of the matrix P form a basis of $\mathbb{R}^n$ and eigenvectors of the matrix A.

Theorem

The matrix $A \in \mathscr{M}_n(\mathbb{R})$ is diagonalizable if and only if it has n eigenvectors linearly independent, then these vectors form a basis of the vector space $\mathbb{R}^n$.

Examples

Prove that the following matrices are diagonalizable and find an invertible matrix $P \in \mathscr{M}_n(\mathbb{R})$ such that the matrix $P^{-1}AP$ is diagonal and find A^{15}.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\ A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix},\ A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

Definition

Let $A \in \mathscr{M}_n(\mathbb{R})$ and λ an eigenvalue of the matrix A. We define

$$E_\lambda = \{X \in \mathbb{R}^n;\ AX = \lambda X\}$$

This space is called called the eigenspace associated to the eigenvalue λ.

Remark

If λ is an eigenvalue of the matrix $A \in \mathscr{M}_n(\mathbb{R})$, then $E_\lambda = \{X \in \mathbb{R}^n;\ AX = \lambda X\}$ is vector sub-space of $\mathbb{R}^n$. Its dimension is called the the geometric multiplicity of λ.

Definition

If $A \in \mathscr{M}_n(\mathbb{R})$ and the characteristic function

$$q_A(\lambda) = (\lambda - \lambda_1)^m Q(\lambda)$$

such that $Q(\lambda_1) \neq 0$ we say that m is the algebraic multiplicity of the eigenvalue λ_1.

Theorem

If $A \in \mathscr{M}_n(\mathbb{R})$ and the characteristic function

$$q_A(\lambda) = C(\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_p)^{m_p}$$

then A is diagonalizable if and only if the algebraic and geometric multiplicities are the same.

Remark

Special case

If $A \in \mathscr{M}_n(\mathbb{R})$ and has n different eigenvalues, then A is diagonalizable.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$$

Solution

The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & 4 \\ -4 & -3-\lambda \end{vmatrix} = (1-\lambda)^2.$$

Then the matrix is not diagonalizable.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} -10-\lambda & -6 \\ 18 & 11-\lambda \end{vmatrix} = (\lambda - 2)(1 + \lambda).$$

Then the matrix is diagonalizable.

$E_{-1} = \langle(-2, 3)\rangle$ and $E_2 = \langle(1, -2)\rangle$.
The diagonal matrix is $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$
and the matrix P is $P = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 5 & 0 & 4 \\ 2 & 1 & 5 \\ -4 & 0 & -3 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & 0 & 4 \\ 2 & 1-\lambda & 5 \\ -4 & 0 & -3-\lambda \end{vmatrix} = (1-\lambda)^3.$$

Then the matrix is not diagonalizable.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

$E_1 = \langle(0,1,0),(1,0,-1)\rangle$ and $E_2 = \langle(0,1,-1)\rangle$.
Then the matrix is diagonalizable.

the diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

and the matrix P is $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(2-\lambda)^2.$$

The matrix is diagonalizable if and only if the dimension of the vector space E_2 is 2.

$E_2 = \langle (1,1,-1,0), (-1,2,0,1) \rangle$.
Then the matrix A is diagonalizable.
$E_5 = \langle (1,0,0,0) \rangle$ and $E_3 = \langle (3,2,0,0) \rangle$.

The diagonal matrix is $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

and the matrix P is $P = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda-5).$$

$E_1 = \langle (1,0,1), (-2,1,0) \rangle$, $E_5 = \langle (1,1,-1) \rangle$.
Then the matrix A is diagonalizable.

The diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ and the matrix P is $P =$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 7-\lambda & 4 & 16 \\ 2 & 5-\lambda & 8 \\ -2 & -2 & -5-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-1).$$

$E_3 = \langle (1,-1,0),(4,0,-1) \rangle$, $E_1 = \langle (2,1,-1) \rangle$.
Then the matrix A is diagonalizable.

The diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and the matrix P is $P = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

Exercise

Show if the following matrix is diagonalizable and find the matrix P such that the matrix $P^{-1}AP$ is diagonal.

$$A = \begin{pmatrix} 2 & -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 3 \end{pmatrix}$$

Solution The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & -1 & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 0 & \frac{1}{2} \\ -1 & 1 & 1-\lambda & -1 \\ 1 & -1 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^3.$$

The matrix is diagonalizable if and only if the dimension the vector space E_2 is 3.

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$.

1. Find the characteristic equation of the matrix A and deduce the eigenvalues of A.
2. Find a matrix P such that $P^{-1}AP$ is diagonal.
3. Without computing A^2, prove that $A^2 = I$.

1. $q_A(\lambda) = (1-\lambda)^2(1+\lambda)$. The 1 and -1 are the eigenvalues of A.
2. If $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$, we have $P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
3. $A^2 = PD^2P^{-1} = PIP^{-1} = I$.