Conformal Mappings And Riemann's Theorem

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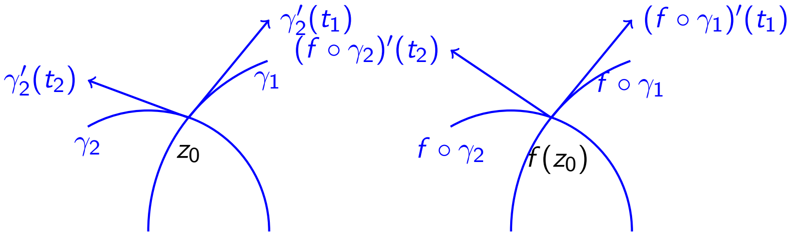
Theorem

Let $f : \Omega \longrightarrow \mathbb{C}$ *be a holomorphic function on an open subset* Ω*.*
Then f *preserves the oriented angles at all* $z \in \Omega$ *where* $f'(z) \neq 0$*.*

Proof


$\gamma_2'(t_1)$
$(f \circ \gamma_2)'(t_2)$
$(f \circ \gamma_1)'(t_1)$
$\gamma_2'(t_2)$
γ_1
$f \circ \gamma_1$
γ_2
z_0
$f \circ \gamma_2$
$f(z_0)$

Let γ_1 and γ_2 be two curves continuously differentiable such that $\gamma_1(t_0) = \gamma_2(t_1) = z_0$, $\gamma_1'(t_0) \neq 0$ and $\gamma_2'(t_1) \neq 0$. The tangent vector to γ_1 (respectively to γ_2) at z_0 is given by $\gamma_1'(t_0)$ (respectively $\gamma_2'(t_1)$). There exists $\lambda > 0$ and $\theta \in \mathbb{R}$ such that $\gamma_2'(t_1) = \lambda e^{i\theta}\gamma_1'(t_0)$. θ is the oriented angle between the tangent vectors to γ_1 and γ_2 at z_0. The tangent vector to $f \circ \gamma_1$ (respectively $f \circ \gamma_2$) at $f(z_0)$ is given by $\gamma_1'(t_0).f'(z_0)$ (respectively $\gamma_2'(t_0).f'(z_0)$) and we have $f'(z_0).\gamma_2'(t_1) = \lambda e^{i\theta}\gamma_1'(t_0)f'(z_0)$. Then θ is again the oriented angle between the tangent vectors to $f \circ \gamma_1$ and $f \circ \gamma_2$. Thus f preserves the oriented angles.

$\square$

Definition

A holomorphic function f on an open set Ω is called a conformal mapping if $f'(z) \neq 0$, $\forall z \in \Omega$.

Recall

1. If $f'(z_0) \neq 0$, then f is injective on a neighborhood of z_0.
2. If $f'(z_0) = 0$, then f is not injective on any neighborhood of z_0.
3. If $f'(z) \neq 0$ for every $z \in \Omega$, then f is locally injective but not necessary injective on Ω. (Example e^z on $\mathbb{C}$).
4. If $f \in \mathcal{H}(\Omega)$ is injective, then $f(\Omega)$ is an open subset and f^{-1} is holomorphic from $f(\Omega)$ onto Ω and $f'(z) \neq 0$, $\forall z \in \Omega$.

Definition

Let Ω_1 and Ω_2 be two open subsets and $f : \Omega_1 \longrightarrow \Omega_2$ a holomorphic function. f is called a conformal mapping from Ω_1 onto Ω_2 if f is an analytic isomorphism from Ω_1 onto Ω_2.

Theorem

For all $a \in D$, the mapping $z \longmapsto h_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ is holomorphic and bijective from D onto itself, $h_a(0) = a$, $h_a(a) = 0$. Furthermore $h_a \circ h_a = \mathrm{id}$ and h_a is bijective from the boundary of the unit disc onto itself. (h_a is a conformal mapping from the unit disc onto itself.)

Proof

$h_a(0) = a$, $h_a(a) = 0$, then $h_a \circ h_a(0) = 0$ and $h_a \circ h_a(a) = a$, then by Schwarz lemma $h_a \circ h_a(z) = z$. Moreover, for $|z| = 1$, $|h_a(z)| = 1$, then by the maximum principle, h_a is a biholomorphism of the unit disc.

$\square$

Theorem

Let f be a conformal mapping from the unit disc D onto itself, then there exists $a \in D$, $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} h_a(z)$.

Proof

Let $a \in D$ such that $f(a) = 0$ and let $g(z) = f \circ h_a(z)$. We have $g(0) = 0$ and $|g(z)| \leq 1$, whenever $z \in D$. By Schwarz's lemma, $|g(z)| \leq |z|$ for $|z| < 1$. But g is bijective from the unit disc onto itself, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $g(z) = \lambda z$. If $\lambda = e^{\mathrm{i}\theta}$, then $f \circ h_a(z) = e^{\mathrm{i}\theta} z$ and $f(z) = e^{\mathrm{i}\theta} h_a(z)$.

□

Theorem

Let $\alpha \in \mathcal{H}^+ = \{z \in \mathbb{C};\ \mathrm{Im} z > 0\}$. *The mapping* $f_\alpha(z) = \dfrac{z-\alpha}{z-\bar{\alpha}}$ *is a conformal mapping from* $\mathcal{H}^+$ *onto* D. $f_\alpha(\mathbb{R}) = \mathscr{C}(0,1) \setminus \{1\}$, f_α *is bijective from* $\mathbb{R}$ *onto* $\mathscr{C}(0,1) \setminus \{1\}$.

Proof

$\alpha = a + \mathrm{i}b$, $z = x + \mathrm{i}y$, with $b > 0$ and $y > 0$.
$f_\alpha(z) = \dfrac{(x-a) + \mathrm{i}(y-b)}{(x-a) + \mathrm{i}(y+b)}$. Since $b > 0$ and $y > 0$,
$(y-b)^2 < (y+b)^2$, thus $|f_\alpha(z)|^2 = \dfrac{(x-a)^2 + (y-b)^2}{(x-a)^2 + (y+b)^2}. < 1$.
$f_\alpha(z_1) = f_\alpha(z_2) \iff z_1(\alpha - \bar{\alpha}) = z_2(\alpha - \bar{\alpha}) \iff z_1 = z_2$, thus f_α is injective. For all $z \in D$, $h_\alpha(w) = z$ with $w = \dfrac{\alpha - \bar{\alpha} z}{1 - z}$. It results that f_α is bijective from $\mathcal{H}^+$ onto D.

If $x \in \mathbb{R}$, $|f_\alpha(x)| = |\dfrac{x-\alpha}{x-\bar{\alpha}}| = 1$, it results that if $z \neq 1$,
$f_\alpha(\dfrac{\alpha - \bar{\alpha} z}{1-z}) = z$.

□

Theorem

Every conformal mapping from $\mathcal{H}^+$ onto D is of the form

$$f(z) = e^{\mathrm{i}\theta} f_\alpha(z) = e^{\mathrm{i}\theta} \frac{z-\alpha}{z-\bar{\alpha}}; \quad \text{whith } \theta \in \mathbb{R},\ \alpha \in \mathcal{H}^+.$$

Proof

Let $\alpha \in \mathcal{H}^+$ be such that $f(\alpha) = 0$, the mapping $g(z) = f \circ f_\alpha^{-1}(z)$ is an automorphism of the unit disc and $g(0) = 0$, then there exists $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta} z$, which yields that $f(z) = e^{i\theta} f_\alpha(z)$, $\forall\ z \in \mathcal{H}^+$. □

Definition

A Möbius transformation or a linear transformation is a mapping of the form $f(z) = \dfrac{az+b}{cz+d}$ *with* $ad - bc \neq 0$.

Remarks 1 :

1) Note that if $ad = bc$ the same expression would yield a constant.

2) The coefficients aren't unique, since we can multiply them all by any nonzero complex constant.

3) The Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$ is defined on $\mathbb{C} \setminus \{\frac{-d}{c}\}$ if $c \neq 0$. We add to $\mathbb{C}$ a new point denoted ∞ and we define $f(\dfrac{-d}{c}) = \infty$, $f(\infty) = \dfrac{a}{c}$ if $c \neq 0$ and $f(\infty) = \infty$ if $c = 0$. The set $\mathbb{C} \cup \{\infty\}$ is called the extended complex plane and denoted by $\mathbb{C}_\infty$. A Möbius transformation is then defined on $\mathbb{C}_\infty$ with values in $\mathbb{C}_\infty$.

4) Any Möbius transformation is a bijective mapping from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$. Indeed if $w \in \mathbb{C}_\infty$,
$w = f(z) = \dfrac{az+b}{cz+d} \Leftrightarrow z = \dfrac{-dw+b}{cw-a} = f^{-1}(w)$ which is a Möbius transformation. Thus f is a bijection from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$.
5) The set $\mathcal{H}$ of Möbius transformations is a group under composite of mappings.

6) Let f be a Möbius transformation, $f(z) = \dfrac{az+b}{cz+d}$. We can suppose that $ad - bc = 1$, and we associate to f the matrix $M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This matrix is in $\mathrm{SL}(2, \mathbb{C})$, the special linear group of $\mathbb{C}^2$.

Inversely if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$, we associate the Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ and the matrix $-M_f$ gives the same Möbius transformation. Thus we can identify the group of Möbius transformations with the the projective special linear group $\mathrm{PSL}(2, \mathbb{C})$, the group of 2×2 matrices with complex coefficients, determinant $= 1$, modulo the equivalence relation $A \sim -A$.

7) Let $A = \begin{matrix} a & b \\ c & d \end{matrix}$, $B = \begin{matrix} a' & b' \\ c' & d' \end{matrix}$ two matrix in $\mathrm{PSL}(2, \mathbb{C})$,
$f(z) = \dfrac{az+b}{cz+d}$ and $g(z) = \dfrac{a'z+b'}{c'z+d'}$ the associate Möbius transformations, then $f \circ g$ is the Möbius transformation associate to the matrix AB.
8) If f is a Möbius transformation associated to the matrix A, then f^{-1} is the Möbius transformation associated to the matrix A^{-1}.

Lemma

The Möbius transformation $z \longmapsto f(z) = \dfrac{1}{z}$ *transforms a general circle to a general circle in* $\mathbb{C}_\infty$. *(cf theorem* **??**, *chapter I)*

Proof

Let $a \in \mathbb{C}$, $r > 0$ and $\mathscr{C}(a, r)$ the circle of radius $r > 0$ and centered at a.

$z \in \mathscr{C}(a, r) \iff |z - a|^2 = r^2 \iff |z|^2 - 2\mathrm{Re} z\bar{a} = r^2 - |a|^2.$

First ca case $r = |a|$, which is equivalent to $0 \in \mathscr{C}(a, r)$. This condition is equivalent that the pole 0 of f is on the circle $\mathscr{C}(a, r)$.
We set $w = \dfrac{1}{z}$, then $z \in \mathscr{C}(a, r) \iff 1 - \mathrm{Re}\bar{w}\bar{a} = 0$. Then the image under f of the circle $\mathscr{C}(a, r)$ in $\mathbb{C}_\infty$ is the straight line of equation $1 - \mathrm{Re}\bar{w}\bar{a} = 0$.
Second case $r \neq |a|$, then the pole 0 of f is not on the circle $\mathscr{C}(a, r)$.

$z \in \mathscr{C}(a, r) \iff |w|^2 - 2\mathrm{Re}\bar{w}(\dfrac{\bar{a}}{|a|^2 - r^2}) + \dfrac{1}{|a|^2 - r^2} = 0$, which is the equation of the circle of radius R, with $R^2 = \dfrac{1}{r^2 - |a|^2} - \dfrac{|a|^2}{(r^2 - |a|^2)^2}$, $R = \dfrac{r}{|r^2 - |a|^2|}$ and centered at $\dfrac{\bar{a}}{|a|^2 - r^2}$.

We deduce that if the pole of f belongs to the circle $\mathscr{C}(a, r)$, the image of this circle is a straight line and passes through the pole, and if the pole is not on the circle, its image under f is a circle.

By topological considerations of connectedness, we deduce that

1. If $0 \in \mathscr{C}(a,r)$, then the image under f of the disc $D(a,r)$ is a half-plane delimited by $f(\mathscr{C}(a,r))$.
2. If $0 \in D(a,r)$, then the image under f of the disc $D(a,r)$ is the complementary of the disc $D(\frac{\bar{a}}{|a|^2 - r^2}, \frac{r}{|r^2 - |a|^2|})$.
3. If 0 belongs to the complementary of the disc $D(a,r)$, then the image under f of this disc is the disc $D(\frac{\bar{a}}{|a|^2 - r^2}, \frac{r}{|r^2 - |a|^2|})$.

Remark 2 :
Since $f \circ f = \mathrm{Id}$, we deduce that the image under f of a straight line passing through the origin 0 is a straight line passing through the origin, the image under f of a circle passing through the origin is a straight line and the image under f of a circle or a straight line which not passing through the origin is a circle.

In what follows, a straight line in $\mathbb{C}_\infty$ is a straight line in $\mathbb{C}$ which we add the point ∞. Moreover we define a **general circle** in $\mathbb{C}_\infty$, any circle or a straight line.

Theorem

A Möbius transformation transforms a general circle to a general circle in $\mathbb{C}_\infty$.

We agree to set that ∞ is the pole of the function $f(z) = az$, when $a \neq 0$.

Proof

Let f be a Möbius transformation, $f(z) = \dfrac{az+b}{cz+d}$.

1. If $c = 0$, $f(z) = |\dfrac{a}{d}|e^{i\theta}z + \dfrac{b}{d}$, then f is the composite of translation, rotation and a dilation. These mappings preserve the set of general circles in $\mathbb{C}_\infty$.
2. If $c \neq 0$, then $f(z) = \dfrac{a}{c} + \dfrac{bc-ad}{c(cz+d)}$. We set $f_1(z) = cz$, $f_2(z) = z + d$, $f_3(z) = \dfrac{1}{z}$, $f_4(z) = \dfrac{bc-ad}{c}z$ and $f_5(z) = \dfrac{a}{c}z$. Then $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, which is the composite of a translation, rotation, dilation and an inversion. Every of these mappings preserves the set of general circles in $\mathbb{C}_\infty$.

□

Remark 3 :
We deduce that if the pole of the Möbius transformation f belongs to the general circle $\mathcal{F}$, then the image under f of this general circle is a straight line and if the pole not belongs to $\mathcal{F}$, then $f(\mathcal{F})$ is a circle.

Examples

1) Let $\mathcal{H}^+ = \{z \in \mathbb{C};\ \mathrm{Im} z > 0\}$, $\mathcal{D} = \{z = x + \mathrm{i}y \in \mathbb{C};\ y = 0\}$ and $f(z) = \dfrac{1}{1-z}$. $f(\mathcal{D}) = \mathcal{D}$ and $f(\mathrm{i}) = \frac{1+\mathrm{i}}{2}$, then $f(\mathcal{H}^+) = \mathcal{H}^+$.
2) Let $Q = \{z = x + \mathrm{i}y \in \mathbb{C};\ x > 0\}$,
$\Delta = \{z = x + \mathrm{i}y \in \mathbb{C};\ x = 0\}$ and $f(z) = \dfrac{1}{1-z}$. Then the pole 1 of f is not on Δ, then $f(\Delta)$ is a circle. To identify this circle, it suffices to determine the image of three points of Δ. We find that the image of Δ is the circle of radius $\frac{1}{2}$ and centered at $\frac{1}{2}$ and as $f(1) = \infty$, then the image under f of Q is the complementary of the closed disc of radius $\frac{1}{2}$ and centered at $\frac{1}{2}$.

3) Let $f(z) = \dfrac{1}{1-z}$, D the unit disc and let $\mathscr{C}$ be the unit circle. Since the pole of f is on $\mathscr{C}$, then $f(\mathscr{C})$ is a straight line. To determine this line, it suffices of determine the image of two points of $\mathscr{C}$. $f(\mathrm{i})$ and $f(-\mathrm{i})$ on the straight line of equation $x = \frac{1}{2}$. Since $f(0) = 1$, then $f(D)$ is the half plane $\{x + \mathrm{i}y \in \mathbb{C};\ x > \frac{1}{2}\}$.

Remarks 4 :

1. A Möbius transformation different to the identity has at most two fixed points. indeed the equation $z = \dfrac{az+b}{cz+d}$ for $z \neq \infty$ is equivalent to $cz^2 + (d-a)z - b = 0$ which has at most two solutions in $\mathbb{C}$ and exactly two solutions in $\mathbb{C}_\infty$.
2. It results that two Möbius transformations which coincide at three different points in $\mathbb{C}_\infty$ are equal.

Definition

Let α, β and γ be three distinct elements of $\mathbb{C}_\infty$. We define the Möbius transformation called the cross ratio by

$$S(z) = (z, \alpha, \beta, \gamma) = \frac{z-\beta}{z-\gamma}\frac{\alpha-\gamma}{\alpha-\beta}, \text{ if } \alpha, \beta, \gamma \in \mathbb{C}.$$

$$(z, \alpha, \beta, \gamma) = S(z) = \frac{z-\beta}{z-\gamma}, \text{ if } \alpha = \infty.$$

$$(z, \alpha, \beta, \gamma) = S(z) = \frac{\alpha-\gamma}{z-\gamma}, \text{ if } \beta = \infty.$$

and $$(z, \alpha, \beta, \gamma) = S(z) = \frac{z-\beta}{\alpha-\beta}, \text{ if } \gamma = \infty.$$

The transformation S is the only Möbius transformation which verifies $S(\alpha) = 1,\ S(\beta) = 0$ and $S(\gamma) = \infty$.

Remarks 5 :

1) The cross ratio is invariant under Möbius transformations. i.e. for any Möbius transformation T we have $(z, \alpha, \beta, \gamma) = (T(z), T(\alpha), T(\beta), T(\gamma))$. Indeed if we denote S_1 the cross ratio defined by $S_1(z) = (z, T(\alpha), T(\beta), T(\gamma))$, then S_1 verifies $S_1(T(\alpha)) = 1,\ S_1(T(\beta)) = 0$ and $S_1(T(\gamma)) = \infty$. Then $S_1 \circ T(\alpha) = 1,\ S_1 \circ T(\beta) = 0$ and $S_1 \circ T(\gamma) = \infty$, thus $S_1 o T = S$.

2) For all $z_1, z_2, z_3 \in \mathbb{C}_\infty$ different and $w_1, w_2, w_3 \in \mathbb{C}_\infty$ different, there exists one and only one Möbius transformation which transforms z_1 to w_1, z_2 to w_2 and z_3 to w_3. Indeed let $S_1(z) = (z, z_1, z_2, z_3)$ and $S_2(z) = (z, w_1, w_2, w_3)$. The Möbius transformation $T = S_2^{-1} \circ S_1$ fulfills the desired property.
3) A Möbius transformation is a conformal mapping. Thus it preserves the angles.

Lemma

The cross ratio $[z_1, z_2, z_3, z_4]$ is real if and only if all z_1, z_2, z_3, z_4 lie in the same general circle. Further, if $[z_1, z_2, z_3, z_4] < 0$, then the points z_1, z_2, z_3, z_4 have to appear in this general circle in the following order: z_1, z_2, z_3, z_4.

Proof

Let $\mathscr{C}$ be the unique general circle passing through z_2, z_3 and z_4 and f the Möbius transformation sending z_2 to 0, z_3 to 1 and z_4 to ∞. Then $f(\mathscr{C})$ is the real axis. Now $z_1 \in \mathscr{C}$ if and only if $f(z_1) \in f(\mathscr{C}) = \mathbb{R}$, which proves the first part of the lemma. If $f(z_1) < 0$, then $f(z_j)$ appear in the line in the following order $f(z_1) < f(z_2) < f(z_3) < f(z_4)$, and hence the same is true about the inverse image. □

Theorem (Ptolemy's Theorem)

Given four points A, B, C and D on the plane. The following holds

$$\overline{AB}\ \overline{CD} + \overline{BC}\ \overline{AD} \geq \overline{AC}\ \overline{BD}.$$

Equality holds if and only if A, B, C, D lie in a circle and appear in alphabetical order (clockwise or counterclockwise).

Proof

Let z_1, z_2, z_3 and z_4 be the complex numbers representing A, B, C and D, respectively. We can easily check the identity

$$(z_1 - z_2)(z_3 - z_4) + (z_1 - z_4)(z_2 - z_3) = (z_1 - z_3)(z_2 - z_4).$$

Hence, using the triangle inequality, we have:

$$|z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3| \geq |z_1 - z_3||z_2 - z_4|,$$

which proves the first part of the theorem. So, the equality holds when both vectors involved have the same direction and orientation, i.e, we have equality if and only if $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in \mathbb{R}^+$. Or equivalently, if the cross ratio $[z_1, z_2, z_3, z_4]$ is real and negative. The result then follows by lemma 3.5.

Definition

We say that two points z_1 and z_2 are symmetric with respect to the circle $\mathscr{C}(a, r)$ if a, z_1 and z_2 are on the same half-straight line outgrowing of a and $|z_1 - a||a - z_2| = r^2$.

Remarks 6 :

1) We can easily prove that in this definition each straight line or circle passing through z_1 and z_2 intersects $\mathscr{C}(a, r)$ orthogonally.

2) If $z \in \mathbb{C}$, then the symmetry of z with respect to the circle $\mathscr{C}(a, r)$ is $z' = a + \dfrac{r^2}{\bar{z} - \bar{a}}$. If we denote $S_{\mathscr{C}}(z) = a + \dfrac{r^2}{\bar{z} - \bar{a}}$ which designs the image of the symmetric of z with respect to the circle $\mathscr{C}(a, r)$, then the mappings $T(z) = \overline{\mathscr{S}_{\mathscr{C}}(z)}$ and $H(z) = S_{\mathscr{C}}(\bar{z})$ are Möbius transformations.

Let $\mathscr{D}$ be a straight line of equation $z = \alpha + xe^{\mathrm{i}\theta}$ ($x \in \mathbb{R}$, α and θ are fixed in $\mathbb{R}$.) An immediate computation shows that the affix of the symmetric of z with respect to the straight line $\mathscr{D}$ is $z' = S_{\mathscr{D}}(z) = \alpha + e^{2i\theta}(\bar{z} - \alpha)$. Then the mappings $T(z) = \overline{S_{\mathscr{D}}(z)}$ and $H(z) = S_{\mathscr{D}}(\bar{z})$ are also Möbius transformations.

Theorem

Every Möbius transformation transforms two symmetric points with respect to a general circle to two points symmetric with respect to the general circle image.

Proof

Let $\mathcal{F}$ be a general circle and f a Möbius transformation. We denote by $S(z)$ the symmetric of z with respect to $\mathcal{F}$, $H = f(\mathcal{F})$ and $T(z)$ the symmetric of z with respect to H.
To prove the theorem, it suffices to prove that $T \circ f(z) = f \circ S(z)$. From the previous remark $\overline{T \circ f}$ and $\overline{f \circ S}$ are Möbius transformations. Then it suffices to prove that $T \circ f$ and $f \circ S$ coincide on three different points. It is obvious that these Möbius transformations coincide on $\mathcal{F}$.

□

1) **Characterization of Möbius transformations which transform the unit disc on itself.**

Let h be such Möbius transformation and $a \in D$ such that $h(a) = 0$, thus $h(\frac{1}{\bar{a}}) = \infty$. ($\frac{1}{\bar{a}}$ is the symmetric of a with respect to the unit circle. If $a = 0$, $\frac{1}{\bar{a}} = \infty$). Then that $h(z) = k\dfrac{a - z}{1 - \bar{a}z}$.
Moreover h transforms D on itself, then $k = e^{i\theta}$, with $\theta \in \mathbb{R}$.

2) **Characterization of Möbius Transformations Which Transform the Upper Half Plane on the Unit Disc**

Let $\mathcal{H}^+$ be the upper half plane and h a Möbius transformation which transforms $\mathcal{H}^+$ on the unit disc D. There exists $\alpha \in \mathcal{H}^+$ such that $h(\alpha) = 0$, then $h(\bar{\alpha}) = \infty$ and $h(z) = e^{i\theta}\dfrac{z-\alpha}{z-\bar{\alpha}}$.

3) **Characterization of Conformal Mappings Which Transform a Crescent on a Half Plane**

We consider the open subset Ω defined by the region of $\mathbb{C}$ between two arc of circles $\mathscr{C}_1$ and $\mathscr{C}_2$. (cf figure **??**). Ω is a simply connected domain. The mapping $f : \Omega \longrightarrow \mathbb{C}$ defined by $f(z) = \dfrac{z-a}{z-b}$ transforms the domain Ω onto the domain Ω' defined by the sector between two half-lines L_1 and L_2 outgrowing of 0 and the angle between L_1 and L_2 is equal to α, where α is the angle between the circles $\mathscr{C}_1$ and $\mathscr{C}_2$ at a. (L_1 is the image of the arc of circle $\mathscr{C}_1$ under f and L_2 is the image of the arc of circle $\mathscr{C}_2$ under f). The mapping $z \longmapsto z^{\frac{\pi}{\alpha}}$ transforms the domain Ω' on a half plane.

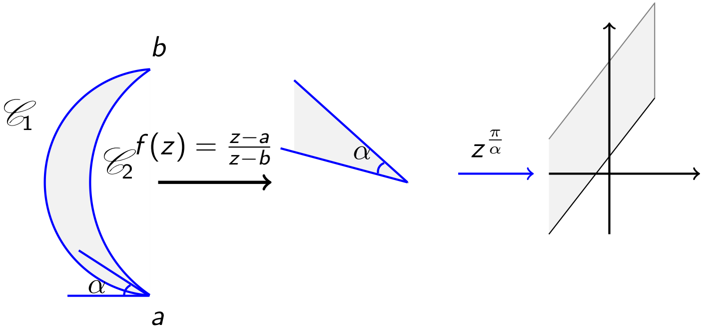

b
$\mathscr{C}_1$
$\mathscr{C}_2$
$f(z) = \frac{z-a}{z-b}$
α
$z^{\frac{\pi}{\alpha}}$
α
a

figure 1:

Exercise 1 :

Let $\Omega = \{z \in \mathbb{C};\ |z - \frac{i}{2}| < 1,\ |z + \frac{i}{2}| < 1\}$.

1. Prove that Ω is simple connected.
2. Let $\mathcal{C}_1 = \{z \in \mathbb{C};\ |z - \frac{i}{2}| = 1\}$, $\mathcal{C}_2 = \{z \in \mathbb{C};\ |z + \frac{i}{2}| = 1\}$, $A = -\sqrt{3}/2$ and $B = \sqrt{3}/2$. We consider the function $f(z) = \dfrac{z + \sqrt{3}/2}{z - \sqrt{3}/2}$.
   a) Give the angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ at A.
   b) Find $f(\Omega)$ and deduce a conformal mapping from Ω onto the upper half plane.

Solution

1. Ω is convex, then it is simple connected.
2. a) Let $\gamma_1(t) = \dfrac{\mathrm{i}}{2} + e^{\mathrm{i}t}$ and $\gamma_2(t) = -\dfrac{\mathrm{i}}{2} + e^{\mathrm{i}t}$ for $t \in [0, 2\pi]$.
   $\gamma_1(t) = -\sqrt{3}/2 \iff t = \dfrac{7\pi}{6}$ and
   $\gamma_2(t) = -\sqrt{3}/2 \iff t = \dfrac{5\pi}{6}$, then since $\gamma_1'(t) = \mathrm{i}e^{\mathrm{i}t}$ and $\gamma_2'(t) = \mathrm{i}e^{\mathrm{i}t}$, the angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ at A is $\frac{\pi}{3}$.
   b) $f(-\dfrac{\mathrm{i}}{2}) = e^{\frac{2\mathrm{i}\pi}{3}}$ and $f(\dfrac{\mathrm{i}}{2}) = e^{\frac{4\mathrm{i}\pi}{3}}$ and $f(0) = -1$, then $f(\Omega) = \{z = re^{\mathrm{i}\theta};\ \frac{2\pi}{3} < \theta < \frac{4\pi}{3},\ r > 0\}$. If $z = re^{\mathrm{i}\theta}$ with $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, $z^{\frac{3}{2}} = r^{\frac{3}{2}}e^{\mathrm{i}\frac{3}{2}\theta}$ with $\pi < \frac{3}{2}\theta < 2\pi$.
   The mapping $h(z) = \frac{z+\mathrm{i}}{z-\mathrm{i}}$ is a conformal mapping from the half plane $\{z = re^{\mathrm{i}\theta};\ \ \pi < \theta < 2\pi\}$ is the unit disc. Then $h \circ g \circ f$ is a conformal mapping from Ω to the unit disc, where $g(z) = z^{\frac{3}{2}}$.

4) **Characterization of Conformal Mappings Which Transform a Domain Delimited by a Semi Circle and a Line onto a Half Plane**

If Ω is the domain of $\mathbb{C}$ delimited by a semi circle of center the origin and radius 1 and contained in the upper half plane. (cf figure 2). The Möbius transformation $f(z) = \dfrac{z-1}{z+1}$ transforms Ω onto the quarter of the plane $\{z \in \mathbb{C};\ x < 0 \text{ and } y > 0\}$ and the mapping $g(z) = -z^2$ transform this quarter of the plane onto the upper half plane.

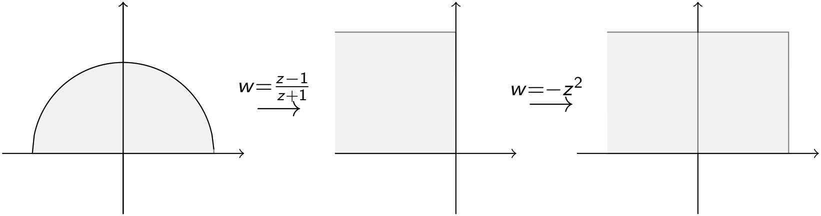


figure 2:

Theorem

Let $\Omega \neq \mathbb{C}$ be a simply connected domain and let $a \in \Omega$. There exists a unique conformal mapping f from Ω onto D such that $f(a) = 0$ and $f'(a) > 0$.

Proof

Uniqueness Let f and g be to such transformations. Bt schwarz lemma, the function $g \circ f^{-1}$ is a conformal mapping from the unit disc onto itself and $g \circ f^{-1}(0) = 0$. Then $g \circ f^{-1}$ is linear. Moreover $(g \circ f^{-1})'(0) > 0$, thus $g \circ f^{-1} = Id \Rightarrow g = f$.

Existence Let $\mathscr{F} = \{f \in \mathcal{H}(\Omega) \text{ injective};\ f(a) = 0, f'(a) > 0, |f(z)| < 1\ \forall\ z \in \Omega\}$. The family $\mathscr{F}$ is normal and let proving that $\mathscr{F}$ is not empty. Let $\alpha \notin \Omega$ and the function $g(z) = (z - \alpha)^{1/2}$. (Since Ω is simple connected and $z - a \neq 0$ for all $z \in \Omega$, then g is well defined on Ω). The function g is holomorphic on Ω, injective and $g(z_1) \neq -g(z_2)$, $\forall\ z_1 \neq z_2$ in Ω. Then by open mapping theorem, there exists $\varepsilon > 0$ such that the disc $\{w \in \mathbb{C};\ |w - g(a)| < \varepsilon\} \subset g(\Omega)$ and $\{w \in \mathbb{C};\ |w + g(a)| < \varepsilon\} \cap g(\Omega) = \emptyset$ (because $g(z_1) \neq -g(z_2)\ \forall\ z_1, z_2 \in \Omega$). Let ψ the Möbius transformation which transforms $\{w \in \mathbb{C};\ |w + g(a)| > \varepsilon\}$ in the unit disc with $\psi(g(a)) = 0$ and $(\psi \circ g)'(a) > 0$.

$$\psi(z) = e^{i\theta} \frac{\varepsilon(g(a) - z)}{2(z + g(a))\overline{g(a)} - \varepsilon^2},$$

θ is such that $(\psi \circ g)'(a) > 0$. Then $\psi \circ g \in \mathscr{F}$, indeed ψ is injective, g is injective, thus $\psi \circ g$ is injective. $\psi \circ g(a) = 0$, $|\psi \circ g(z)| < 1$ by construction.
Let $M = \sup\{f'(a);\ f \in \mathscr{F}\} \leq +\infty$. There exists a sequence $(f_n)_n \in \mathscr{F}$ such that $\lim\limits_{n \to +\infty} f_n'(a) = M$. Since $\mathscr{F}$ is a normal family, we can extract from the sequence $(f_n)_n$ a convergent subsequence, set f its limit for the topology of $\mathcal{H}(\Omega)$. Then f is injective or constant. The function f is not constant because $f'(a) = M > 0$, thus $M < +\infty$ and $f \in \mathscr{F}$.

If f is not surjective, there exists $w \in D$ such that $f(z) \neq w, \ \forall \ z \in \Omega$.
We define the holomorphic functions F and G by:
$F(z) = (\frac{f(z) - w}{1 - \bar{w} f(z)})^{1/2}$ and $G(z) = e^{\mathrm{i}\theta} \frac{F(z) - F(a)}{1 - \overline{F(a)} F(z)}$, with $e^{\mathrm{i}\theta} = \frac{\overline{F'(a)}}{|F'(a)|}$. F is injective, $|F(z)| < 1, \ \forall z \in \Omega, \ G \in \mathscr{F}$ and $G'(a) = \frac{|F'(a)|}{1 - |F(a)|^2} = \frac{1 + |w|}{2\sqrt{w}} f'(a)$. Thus $g'(a) > f'(a)$, which is absurd, then f is surjective and f realizes the conformal mapping from Ω onto the unit disc.

□

Theorem (Caracthéodory's Extension Theorem)

Let Ω be a bounded simply connected domain such that the boundary $\partial\Omega$ is a Jordan curve C and let $f : \Omega \longrightarrow D$ be a conformal mapping from Ω onto D. Then f can be extended to an homeomorphisms from $\overline{\Omega}$ onto $\overline{D}$.