Topology on the Space Of Holomorphic Functions

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Let Ω be an open subset of $\mathbb{C}$, $\mathscr{C}(\Omega)$ the vector space of continuous functions on Ω and $\mathcal{H}(\Omega)$ the vector subspace of holomorphic functions on Ω.
For all $n \in \mathbb{N}$, we set $K_n = \{z \in \mathbb{C};\ |z| \leq n \text{ and } d(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{n}\}$.
The sequence of compacts $(K_n)_n$ is increasing and $\bigcup_{n\geq 1} K_n = \Omega$.
We define the following sequence of semi-norms and distance the on $\mathscr{C}(\Omega)$

$$||f - g||_n = \sup_{z \in K_n} |f(z) - g(z)|$$

and

$$d(f, g) = \sum_{n=1}^{\infty} \frac{||f - g||_n}{2^n(1 + ||f - g||_n)}.$$

Proposition

The mapping $d: \mathscr{H}(\Omega) \times \mathscr{H}(\Omega) \longrightarrow \mathbb{R}$ *is a distance.*

Proof

$d(f,g) = 0 \iff f = g$ on K_n for all $n \in \mathbb{N}$, then $f = g$ on Ω. d is symmetric. To prove the triangle inequality, we use the following inequality: for $s, t \in [0, +\infty[$, we have

$$\frac{t}{1+t+s} \leq \frac{t}{1+t} \Rightarrow \frac{t+s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s}.$$

Since $||f-g||_n \leq ||f-h||_n + ||h-g||_n$, then $\dfrac{||f-g||_n}{1+||f-g||_n} \leq \dfrac{||f-h||_n + ||h-g||_n}{1+||f-h||_n + ||h-g||_n}$, because the mapping $t \longmapsto \frac{t}{1+t}$ is increasing, which proves the triangle inequality. $\square$

Proposition

The convergence relative to the metric d is equivalent to the uniform convergence on compact subsets of Ω.

we recall the definition of the uniformly convergence on compact subsets:

Definition

A sequence $(f_n)_n$ of continuous functions on an open set Ω is called uniformly convergent on compact subsets of Ω to f, if for all $\varepsilon > 0$ and any compact K subset of Ω, there exists a integer $N = N(K, \varepsilon)$ such that $\sup_{z \in K} |f_n(z) - f(z)| \leq \varepsilon$, for $n \geq N$.

Proof

Let $(f_n)_n$ be a sequence of continuous functions which converges with respect to the metric d to a function f, $(\lim_{n\to+\infty} d(f_n, f) = 0)$, then

$$\forall \varepsilon > 0;\ \exists N \in \mathbb{N};\ d(f_n, f) \le \varepsilon;\ \forall n \ge N.$$

Thus for all $n \ge N$ and for all $m \in \mathbb{N}$, $\dfrac{1}{2^m}\dfrac{||f_n - f||_m}{1 + ||f_n - f||_m} \le \varepsilon$. It results that the sequence $(f_n)_n$ converges uniformly on compact subsets of Ω to f. (The sequence $(K_n)_n$ is exhaustive.)

Conversely, if a sequence $(f_n)_n$ converges uniformly on compact subsets of Ω to f, then $f \in \mathcal{C}(\Omega)$. Furthermore $\forall \varepsilon > 0;\ \exists N \in \mathbb{N};\ \sum_{k=N}^{+\infty} \frac{1}{2^n} \leq \varepsilon$ and there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $||f_n - f||_N \leq \varepsilon$. Since the sequence $(K_n)_n$ is increasing, then $||f_n - f||_k \leq \varepsilon$ for all $n \geq M$ and all $k \leq N$. For $j \geq M$

$$\begin{aligned} d(f_j, f) &= \sum_{n=1}^{N-1} \frac{1}{2^n} \frac{||f_j - f||_n}{1 + ||f_j - f||_n} + \sum_{n=N}^{+\infty} \frac{1}{2^n} \frac{||f_j - f||_n}{1 + ||f_j - f||_n} \\ &\leq \varepsilon \sum_{n=1}^{N-1} \frac{1}{2^n} + \varepsilon \sum_{n=N}^{+\infty} \frac{1}{2^n} \leq 2\varepsilon. \end{aligned}$$

Then $d(f_j, f) \leq 2\varepsilon$ for all $j \geq M$. $\square$

Theorem

The space $(\mathscr{C}(\Omega), d)$ is a complete metric space. The subspace $\mathcal{H}(\Omega)$ is closed subspace, thus it is complete.

Proof

Let $(f_n)_n$ be a Cauchy sequence of $\mathscr{C}(\Omega)$. For all $z \in \Omega$, the sequence $(f_n(z))_n$ is a Cauchy sequence in $\mathbb{C}$, thus it is convergent. We denote $f(z)$ its limit. Let K be a compact subset of Ω. Since $\lim_{j,k\to+\infty} d(f_j, f_k) = 0$, then $\lim_{j,k\to+\infty} ||f_j - f_k||_K = 0$ and $\lim_{j\to+\infty} \sup_{z\in K} \Big| f_j(z) - f(z) \Big| = 0$. Therefore the sequence $(f_n)_n$ converges uniformly on any compact subset to f and f is continuous, which proves that $\mathscr{C}(\Omega)$ is complete.

We know that if a sequence of holomorphic functions $(f_n)_n$ which converges uniformly on any compact subset to f, the function f is holomorphic. Then $\mathcal{H}(\Omega)$ is a closed subspace of $\mathscr{C}(\Omega)$, which is complete, then $\mathcal{H}(\Omega)$ is also complete.

$\square$

Theorem

Let $(f_n)_n$ be a sequence of holomorphic functions on a domain Ω. We assume that for all n, the function f_n never vanishing on Ω and the sequence $(f_n)_n$ converges uniformly on any compact subset to a function $f \not\equiv 0$. Then f never vanishing on Ω.

Proof

Since the sequence $(f_n)_n$ is uniformly convergent on any compact subset of Ω, then f is holomorphic. We assume that f is not identically zero and there exists z_0 which is a zero of multiplicity $k \geq 1$ of f. Let $r > 0$ such that $f(z) \neq 0$ for all $z \in \overline{D(z_0, r)} \setminus \{z_0\}$ and let γ be the closed curve defined by the circle of radius r and centered at z_0 traversed in the counterclockwise direction. Then $\dfrac{1}{2i\pi}\displaystyle\int_\gamma \dfrac{f'(z)}{f(z)}\, dz = k$. Since f never vanishing on γ, the sequence $\left(\dfrac{f_n'}{f_n}\right)_n$ converges uniformly on γ to $\dfrac{f'}{f}$, thus

$$k = \frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \lim_{n\to+\infty} \frac{1}{2i\pi}\int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = 0,$$

which is absurd. $\square$

Corollary

Let $(f_n)_n$ be a sequence of holomorphic functions on a domain Ω which converges uniformly on any compact subset to a function f. We assume that f has some zeros in Ω. Then there exists a rank N such that, f_n has some zeros in Ω, whenever $n \geq N$.

Theorem

Let $(f_n)_n$ be a sequence of holomorphic functions on a domain Ω which converges uniformly on any compact subset to a function f. We assume that there exists a disc $\overline{D(z_0, r)} \subset \Omega$ such that f never vanishing on the circle $\mathscr{C}(z_0, r) = \{z \in \mathbb{C};\ |z - z_0| = r\}$, then there exists an integer N such that for all $n \geq N$, the functions f and f_n have the same number of zeros in the disc $D(z_0, r)$.

Proof

Since f never vanishing on $\mathscr{C}(z_0, r)$, then $\alpha = \inf_{z\in C(z_0,r)} |f(z)| > 0$ and there exists $N \in \mathbb{N}$ such that for $n \geq N$, $\sup\limits_{|z-z_0|=r} |f_n(z) - f(z)| \leq \alpha/2 < \alpha < |f(z)|$. Thus for $|z - z_0| = r$, $|f_n(z) - f(z)| < |f(z)|$. By Rouché's theorem the functions f_n and f have the same number of zeros on $D(z_0, r)$ for $n \geq N$. $\square$

Theorem

Let $(f_n)_n$ be a sequence of injective holomorphic functions on a domain Ω which converges uniformly on any compact subset to a function f, then f is constant or injective on Ω.

Proof

Let $z_1 \neq z_2$ be two points of Ω. There exists U_1 and U_2 two disjoints connected open subsets Ω containing respectively z_1 and z_2. The sequence $(g_n)_n$ defined by $g_n(z) = f_n(z) - f_n(z_1)$ is a sequence of injective functions on U_2 which converges uniformly on any compact subset of U_2 to the function g defined by $g(z) = f(z) - f(z_1)$. The functions g_n never vanishing on U_2 (connected), thus either $f(z) \equiv f(z_1)$ on U_2 and thus $f(z) \equiv f(z_1)$ on Ω, or $f(z) \neq f(z_1)$ on U_2, in particular $f(z_2) \neq f(z_1)$.

$\square$

Definition

1. *Let K be a compact subset of Ω, a family $\mathcal{F}$ of $\mathcal{H}(\Omega)$ is called bounded on K, if there exists $M > 0$ such that* $\sup_{z \in K} |f(z)| \leq M, \forall f \in \mathcal{F}$.
2. *A family $\mathcal{F}$ is called locally bounded if $\mathcal{F}$ is bounded on any compact of Ω.*
3. *A family $\mathcal{F}$ is called equicontinuous at $z_0 \in \Omega$ if* $\forall\ \varepsilon > 0;\ \exists \eta > 0$ *such that if* $|z - z_0| < \eta$, *then* $|f(z) - f(z_0)| < \varepsilon, \forall\ f \in \mathcal{F}$.

Theorem

Every locally bounded family $\mathcal{F}$ of $\mathcal{H}(\Omega)$ on Ω is equicontinuous at any point of Ω.

Proof

Let $z_0 \in \Omega$ and $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$. If $|z - z_0| < \frac{r}{2}$ and $|z' - z_0| < \frac{r}{2}$, we have

$$f(z) - f(z') = \frac{1}{2i\pi} \int_\gamma f(w)(\frac{1}{w-z} - \frac{1}{w-z'})\ dw,$$

with $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$.

$$f(z) - f(z') = \frac{(z-z')}{2i\pi} \int_\gamma \frac{f(w)}{(w-z)(w-z')}\ dw.$$

Let $M = \sup_{f \in \mathcal{F}} \sup_{w \in \mathscr{C}(z_0, r)} |f(w)|$, we have
$|f(z) - f(z')| \leq \frac{4M}{r}|z - z'|$, for all z and z' in $D(z_0, \frac{r}{2})$ and all $f \in \mathcal{F}$, thus $\mathcal{F}$ is equicontinuous on Ω.

$\square$

Theorem

Let $\mathcal{F}$ be a family of continuous functions on an open subset Ω. We assume that $\mathcal{F}$ is equicontinuous on Ω.

1. *Let $(f_n)_n$ be a sequence of $\mathcal{F}$ which is pointwise convergent to f on Ω, then f is continuous. Furthermore the sequence $(f_n)_n$ converges to f uniformly on compact subsets of Ω.*
2. *Let E be a dense subset in Ω, if the sequence $(f_n(z))_n$ has a limit for all z in E, then the sequence $(f_n)_n$ converges uniformly on compact subsets of Ω.*

Proof

1) Let $z_0 \in \Omega$. Since $\mathcal{F}$ is equicontinuous at z_0, we have

$$\forall\, \varepsilon > 0; \exists \eta > 0;\ \forall z \in \Omega;\ |z - z_0| < \eta \Rightarrow |g(z) - g(z_0)| < \varepsilon, \quad \forall\, g \in \mathcal{F}.$$

In particular $|f_j(z) - f_j(z_0)| < \varepsilon$, $\forall j \in \mathbb{N}$. By taking the limit, we deduce that $|f(z) - f(z_0)| < \varepsilon$.

It remains to show the uniform convergence on compact subsets of the sequence $(f_n)_n$.

Let K be a compact subset of Ω. For all $w \in K$, there exists an open disc $D(w) \neq \{w\}$ centered at w such that $|g(z) - g(w)| \leq \varepsilon$, $\forall z \in D(w)$ and $\forall\, g \in \mathcal{F}$. K is covered by a finite number of such discs $D(w_j)$, thus $\forall z \in K$, $\exists j$ such that $z \in D(w_j)$.

$$|f_n(z) - f(z)| \leq |f_n(z) - f_n(w_j)| + |f_n(w_j) - f(w_j)| + |f(w_j) - f(z)|.$$

$|f_n(z) - f_n(w_j)| \leq \varepsilon$ because $z \in D(w_j)$,
By taking the limit, we have $|f(z) - f(w_j)| \leq \varepsilon$. There exists an integer N such that for $n \geq N$, $|f_n(w_j) - f(w_j)| \leq \varepsilon$. Thus for $z \in K$, $|f_n(z) - f(z)| \leq 3\varepsilon$. It results that the sequence $(f_n)_n$ converges uniformly on compact subsets to f.
2) Let $z_0 \in \Omega$, we claim that the sequence $(f_n(z_0))_n$ is convergent. $\forall \varepsilon > 0$, $\exists \alpha > 0$, such that if $|z - z_0| < \alpha$, $|g(z) - g(z_0)| \leq \varepsilon$, $\forall\ g \in \mathcal{F}$.

Let $w \in E$ such that $|w - z_0| < \alpha$, then $|g(w) - g(z_0)| \leq \varepsilon$, $\forall\ g \in \mathcal{F}$.
$f_n(z_0) - f_m(z_0) = f_n(z_0) - f_n(w) + f_n(w) - f_m(w) + f_m(w) - f_m(z_0) \Rightarrow$ $|f_n(z_0) - f_n(w)| \leq \varepsilon$.
Since the sequence $(f_n(w))_n$ is convergent, then for all $\varepsilon > 0$, there exists an integer N such that for $n \geq N$, $|f_n(z_0) - f_m(z_0)| \leq 3\varepsilon$, for all $n, m \geq N$. It results that the sequence $(f_n(z_0))_n$ is a Cauchy sequence, then it is convergent. Thus the sequence $(f_n)_n$ is pointwise convergent on Ω and the result is deduced from the first part of the theorem. □

Definition

A family $\mathcal{F} \subset \mathscr{C}(\Omega)$ is called a normal family if for any sequence $(f_n)_n \in \mathcal{F}$, we can extract a convergent subsequence for the topology of the uniform convergence on compact subsets of Ω. (The limit is not in general in $\mathcal{F}$.)

Theorem (Montel's Theorem)

Let $\mathcal{F}$ in $\mathcal{H}(\Omega)$ be a family of locally bounded holomorphic functions, then $\mathcal{F}$ is normal.

Proof

Let $(f_n)_n$ be a sequence of $\mathcal{F}$, $\mathcal{F} \subset \mathscr{C}(\Omega)$. The family is equicontinuous. Let E be a countable dense subset in Ω. We denote $E = \{(w_n)_{n\in\mathbb{N}}\}$.

For w_1, there exists $M_1 > 0$ such that $|g(w_1)| \leq M_1$, $\forall g \in \mathcal{F}$. In particular the sequence $(f_n(w_1))_n$ is bounded in $\mathbb{C}$. Thus we can extract a convergent subsequence denoted $(f_{1,n}(w_1))_n$.

For w_2, the sequence $(f_{1,n}(w_2))_n$ is bounded thus we can extract a convergent subsequence denoted $(f_{2,n}(w_2))_n$. The sequence $(f_{2,n}(w_1))_n$ is convergent. By iteration, for every w_k, there exists a subsequence of $(f_{k-1,n})_n$ denoted $(f_{k,n})_n$ such that the sequences $(f_{k,n}(w_j))_n$ are convergent for any $1 \leq j \leq k$. Set $g_k = f_{k,k}$, for $k \in \mathbb{N}$. The sequence $(g_n)_n$ is convergent on E. In use the previous theorem (**??**), we derive the theorem. $\square$

Exercise 1 :

1) The family $\mathcal{F}_1 = \{\sin nz;\ n \in \mathbb{N}\}$ is not normal on any open subset. Indeed for $z = x + \mathrm{i}y$, $y \neq 0$, $|\sin z|^2 = \sin^2 x + \sinh^2 y$, which is not bounded. Thus it is not normal.

Exercise 2 :

The family $\mathcal{F}_2 = \{f \in \mathcal{H}(D);\ f(0) = -1, f(D) \subset \mathbb{C}\backslash]-\infty, 0]\}$ is normal. Indeed, we consider the mapping $\varphi(z) = \left(\dfrac{1-z}{1+z}\right)^2$ which is a bijective holomorphic function from D onto $\mathbb{C}\backslash]-\infty, 0]$, with $\varphi(0) = 1$. We denote ψ the inverse function of φ and $\mathcal{F}^* = \{g = \psi \circ f;\ f \in \mathcal{F}\}$.

For all $g \in \mathcal{F}^*$, $g\colon D \longrightarrow D$ and $g(0) = 0$. Thus by Schwarz's lemma, $|g(z)| \leq |z|$. Then for all $0 < r < 1$,

$$\sup_{|w|\leq r} |\varphi(w)| \leq \left(\frac{1+r}{1-r}\right)^2 \text{ and } \sup_{|z|\leq r} |f(z)| \leq \left(\frac{1+r}{1-r}\right)^2.$$

$(f = \varphi \circ \psi \circ f)$. Thus the family $\mathcal{F}_2$ is locally bounded and then it is a normal family.

Theorem (Vitali's Theorem)

Let $(f_n)_n$ be a sequence of locally bounded holomorphic functions on a domain Ω. We assume that the sequence $(f_n(z))_n$ is pointwise convergent on E and E has a cluster point (accumulation point) in Ω, then the sequence $(f_n)_n$ converges uniformly on compact subsets to a holomorphic function.

Proof

The sequence $(f_n)_n$ is locally bounded, then it is normal. Let f and g be two limits of the sequence $(f_n)_n$. The functions f and g coincide on E, thus $f \equiv g$ on Ω. Thus the sequence has only one limit. Let f this limit.

If the sequence $(f_n)_n$ is not convergent to f in $\mathcal{H}(\Omega)$, there exists $\varepsilon > 0$, a compact $K \subset \Omega$ and a sequence $(z_n)_n$ in K such that $|f_{n_k}(z_k) - f(z_k)| \geq \varepsilon$, for all $k \geq 1$. We can extract from the sequence $(f_{n_k})_k$ a convergent subsequence. This subsequence must converges to f, which is absurd.

$\square$

Lemma

Let K be a compact subset of Ω. The mapping $M_K : \mathcal{H}(\Omega) \longrightarrow \mathbb{R}$ defined by $M_K(f) = \sup_{z\in K} |f(z)|$, is continuous.

Proof

Let $f, g \in \mathcal{H}(\Omega)$, $g = f + g - f$, thus $|g(z)| \leq |f(z)| + |g(z) - f(z)|$. Then $|M_K(g) - M_K(f)| \leq M_K(f - g)$.

$\square$

Theorem

Let $\mathcal{F}$ be a family of $\mathcal{H}(\Omega)$. $\mathcal{F}$ is a compact subset of $\mathcal{H}(\Omega)$ if and only if $\mathcal{F}$ is closed and locally bounded.

Proof

CN If $\mathcal{F}$ is a compact subset of $\mathcal{H}(\Omega)$, then $\mathcal{F}$ closed and locally bounded on Ω by lemma **??**.

CS Let $(f_n)_n$ be a sequence of $\mathcal{F}$, by Montel's theorem, $(f_n)_n$ is normal, then we can extract a convergent subsequence. The limit of this subsequence is holomorphic and in $\mathcal{F}$ since $\mathcal{F}$ is closed.

□

Theorem

Let $\mathcal{F}$ be a compact subset of $\mathcal{H}(\Omega)$ and $z_0 \in \Omega$, then there exists $g \in \mathcal{F}$ such that $|g'(z_0)| \geq |f'(z_0)|;\ \forall f \in \mathcal{F}$.

Proof

The mapping $f \longmapsto |f'(z_0)|$ is continuous on $\mathcal{H}(\Omega)$ indeed if $(f_n)_n$ is a convergent sequence and f is its limit in $\mathcal{H}(\Omega)$. The sequence $(f_n')_n$ converges also uniformly on compact subsets to f', thus $\lim\limits_{n\to+\infty} |f_n'(z_0)| = |f'(z_0)|$. $\square$

Theorem

Let Ω be an open subset of $\mathbb{C}$, $z_0 \in \Omega$. The set

$$\mathcal{F} = \{f \in \mathcal{H}(\Omega),\ f \text{ injective},\ f(\Omega) \subset \overline{D} \text{ and } |f'(z_0)| \geq 1\}.$$

is compact in $\mathcal{H}(\Omega)$.

Proof

If $\mathcal{F} = \emptyset$, there is nothing to prove. If not the family $\mathcal{F}$ is bounded. Let $(f_n)_n$ be a convergent sequence of $\mathcal{F}$ and f its limit. $|f(z)| \leq 1,\ \forall z \in \Omega$ and $|f'(z_0)| \geq 1$. Thus f is not constant. By theorem **??**, f is injective, thus $f \in \mathcal{F}$ and $\mathcal{F}$ is closed and compact.

□