Inner Product Spaces and Orthogonality

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Inner Product

Definition

Let V be a vector space on $\mathbb{R}$.
We say that a function $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$ is an inner product on V if it satisfies the following:
For all $u, v, w \in V$, $\alpha \in \mathbb{R}$.

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
4. $\langle u, u \rangle \geq 0$
5. $\langle u, u \rangle = 0 \iff u = 0$

Examples

1. The Euclidean inner product on $\mathbb{R}^n$ defined by:

$$\langle u, v\rangle = \sum_{j=1}^{n} x_j y_j = x_1 y_1 + \ldots + x_n y_n,$$

where $u, v \in \mathbb{R}^n$, $u = (x_1, \ldots, x_n)$ and $v = (y_1, \ldots, y_n)$.

2. If $E = \mathcal{C}([0,1])$ the vector space of continuous functions on $[0,1]$. For all $f, g \in E$, we define the inner product of f and g by:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\underset{\cdot}{t}.$$

Remarks

If $(E, \langle\ ,\ \rangle)$ is an inner product space and $u, v, w, x \in E$, $a, b, c, d \in \mathbb{R}$, we have:

$$\langle u+v, w+x\rangle = \langle u, w\rangle + \langle u, x\rangle + \langle v, w\rangle + \langle v, x\rangle.$$

$$\begin{aligned}\langle au+bv, cw+dx\rangle &= ac\langle u, w\rangle + ad\langle u, x\rangle \\ &\quad +bc\langle v, w\rangle + bd\langle v, x\rangle.\end{aligned}$$

Example

Let $u = (x, y)$ and $v = (a, b)$, we define

$$\langle u, v\rangle = 2ax + by - bx - ay$$

$\langle \; , \; \rangle$ is an inner product on $\mathbb{R}^2$.
It is enough to prove that $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \iff u = 0$.

$$\langle u, u\rangle = 2x^2 + y^2 - 2xy = (x - y)^2 + x^2 \geq 0$$

and $\langle u, u\rangle = 0 \iff u = 0$.

Example

Let $u = (x, y, z)$ and $v = (a, b, c)$, we define

$$\langle u, v\rangle = 2ax + by + 3cz - bx - ay + cy + bz$$

$\langle \ , \ \rangle$ is an inner product on $\mathbb{R}^3$.
It is enough to prove that $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \iff u = 0$.

$$\begin{aligned}\langle u, u\rangle &= (y + z - x)^2 - (z - x)^2 + 2x^2 + 3z^2 \\ &= (y + z - x)^2 + (x + z)^2 + z^2 \geq 0\end{aligned}$$

$$\langle u, u\rangle = 0 \iff z = x = y = 0 \iff u = 0.$$

Example

Let $u = (x, y, z)$ and $v = (a, b, c)$, we define

$$\langle u, v\rangle = 2ax + by + cz - bx - ay + cy + bz$$

$\langle\ ,\ \rangle$ is not an inner product on $\mathbb{R}^3$.

$$\begin{aligned}\langle u, u\rangle &= (y+z-x)^2 - (z-x)^2 + 2x^2 + z^2\\ &= (y+z-x)^2 + x^2 + 2xz\\ &= (y+z-x)^2 + (x+z)^2 - z^2.\end{aligned}$$

Example

If $A = (a_{j,k}) \in \mathscr{M}_n(\mathbb{R})$, we define the trace of the matrix A by:

$$\mathrm{tr}(A) = \sum_{j=1}^{n} a_{j,j}$$

and

$$\langle A, B \rangle = \mathrm{tr}(AB^T)$$

for all $A, B \in \mathscr{M}_n(\mathbb{R})$.
$\langle A, B \rangle$ is an inner product on the vector space $\mathscr{M}_n(\mathbb{R})$.

Exercise

If $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$, we define the following functions: $f, g, h, k \colon \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$.

1. $f(u, v) = x_1y_1 + x_2y_2 + 2x_3y_3 + x_2y_1 + 2x_1y_2 + x_2y_3 + y_2x_3$.
2. $g(u, v) = x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2 + 3x_1y_3 + 3x_3y_1$.
3. $h(u, v) =$ $x_1y_1 + x_2y_2 + x_3y_3 + x_2y_1 + x_1y_2 + x_2y_3 + y_2x_3 + x_3y_1 + x_1y_3$.
4. $k(u, v) = x_1y_1 + x_2y_2 + x_3y_3 - x_2y_3 - x_3y_2 + x_1y_3 + y_1x_3$.
   Select from which the functions f, g, h, k is an inner product on $\mathbb{R}^3$.

Solution

1. $f(u, v) - f(v, u) = x_1 y_2 - x_2 y_1$. Then f is not an inner product on $\mathbb{R}^3$.
2. $g(u, u) = 2x_1x_2 + 2x_2x_3 + 6x_1x_3 = 2(x_1 + x_3)(x_2 + 3x_3) - 6x_3^2 = (x_1 + x_2 + 4x_3)^2 - (x_1 - x_2 - 2x_3)^2 - 6x_3^2$. .
   Then g is not an inner product on $\mathbb{R}^3$.
3.
$$\begin{aligned} h(u, u) &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 \\ &= (x_1 + x_2 + x_3)^2 \end{aligned}$$

   Then h is not an inner product on $\mathbb{R}^3$ because

$$h(u, u) = 0 \not\Rightarrow u = 0.$$

4.

$$\begin{aligned} k(u,u) &= x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 + 2x_1x_3 \\ &= (x_1 + x_3)^2 + x_2^2 - 2x_2x_3 \\ &= (x_1 + x_3)^2 + (x_2 - x_3)^2 - x_3^2 \end{aligned}$$

Then k is not an inner product on $\mathbb{R}^3$ because

$$k(u,u) = 0 \not\Rightarrow u = 0.$$

Example

Find the values of a, b such that

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2 + a x_1 y_2 + b x_2 y_1$$

is an inner product on $\mathbb{R}^2$.

Solution

$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle (y_1, y_2), (x_1, x_2) \rangle$ if $a = b$.

$$\begin{aligned} \langle (x_1, x_2), (x_1, x_2) \rangle &= x_1^2 + x_2^2 + 2a x_1 x_2 \\ &= (x_1 + a x_2)^2 + x_2^2 (1 - a^2). \end{aligned}$$

Then $\langle\ ,\ \rangle$ is an inner product on $\mathbb{R}^2$ if and only if $|a| < 1$.

Definition

Let $(E, \langle\ ,\ \rangle)$ be an inner product space.

1. If $u \in E$, we define the norm of the vector u by:
$$\|u\| = \sqrt{\langle u, u\rangle}.$$
2. If $u, v \in E$, we define distance between u and v by:
$$d(u, v) = \|u - v\|.$$
3. We define the angle $0 \leq \theta \leq \pi$ between the vectors $u, v \in E$ by:
$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|.\|v\|}$$

Let the inner product space $\mathscr{M}_2(\mathbb{R}), \langle\ ,\ \rangle)$ defined by:

$$\langle A, B\rangle = \mathrm{tr}(AB^T).$$

Find $\cos\theta$ If θ is the angle between the matrices
$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
$AB^T = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$, $\|A\|^2 = 15$, $\|B\|^2 = 7$.
Then

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{35}}.$$

Theorem (Cauchy-Schwarz Inequality)

If $(E, \langle\ ,\ \rangle)$ is an inner product space and $u, v \in E$, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|. \tag{1}$$

We have the equality in (1) if the vectors u, v are linearly dependent.

Proof

Let $Q(t)$ be the polynomial

$$Q(t) = \|u + tv\|^2 = \|u\|^2 + 2t\langle u, v\rangle + t^2\|v\|^2.$$

Since $Q(t) \geq 0$ for all $t \in \mathbb{R}$, then the discriminant of $Q(t)$ is non positive. Then

$$\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2.$$

If $|\langle u, v\rangle| = \|u\|\|v\|$, this mean that the discriminant of $Q(t)$ is zero. Then the equation $Q(t) = 0$ has a solution. This means that the vectors u, v are linearly dependent.

Theorem

If $(E, \langle\ ,\ \rangle)$ is an inner product space and $u, v \in E$, then

$$\|u + v\| \leq \|u\| + \|v\|.$$

Proof

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\ \|v\| = (\|u\| + \|v\|)^2. \end{aligned}$$

Definition

If $(E, \langle\ ,\ \rangle)$ is an inner product space. We say that the vectors $u, v \in E$ are orthogonal and we denote $u \perp v$ if $\langle u, v\rangle = 0$.

Theorem (Pythagor's Theorem)

If $u \perp v$ if and only if

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = \|u\|^2 + \|v\|^2.$$

Definition

If $(E, \langle \ , \ \rangle)$ is an inner product space. We say that set $S = \{e_1, \dots, e_n\}$ of non zeros vectors is orthogonal if

$$\langle e_j, e_k \rangle = 0, \quad \forall 1 \leq j \neq k \leq n.$$

and we say that S is normal if

$$\|e_j\| = 1, \quad \forall 1 \leq j \leq n.$$

and we say that it is orthonormal if

$$\langle e_j, e_k \rangle = \delta_{j,k}, \quad \forall 1 \leq j, k \leq n.$$

($\delta_{j,k} = 0$ If $j \neq k$ and $\delta_{j,j} = 1$.)

Theorem

Any set of non zero orthogonal vectors is linearly independent .

Theorem

If $(E, \langle\ ,\ \rangle)$ is an inner product space and if $S = \{e_1, \ldots, e_n\}$ is an orthonormal basis of E, then for all $u \in E$

$$u = \langle u, e_1\rangle e_1 + \ldots + \langle u, e_n\rangle e_n.$$

Proof

If $u = \sum_{j=1}^{n} a_j e_j$, then $\langle u, e_k\rangle = \sum_{j=1}^{n} a_j \langle e_j, e_k\rangle = a_k$.

Theorem

(Gramm-Schmidt Algorithm) If $(E, \langle\ ,\ \rangle)$ is an inner product space and $(v_1, \dots, v_n)$ a set of linearly independent vectors in E, there is a unique orthonormal set $(e_1, \dots, e_n)$ such that

1. for all $k \in \{1, \dots, n\}$,
$$\mathrm{Vect}(e_1, \dots, e_k) = \mathrm{Vect}(v_1, \dots, v_k),$$
2. for all $k \in \{1, \dots, n\}$,
$$\langle e_k, v_k \rangle > 0.$$

Proof

We construct in the first time an orthogonal set $(u_1, \dots, u_n)$ such that:

$$\begin{cases} u_1 &= v_1 \\ u_2 &= v_2 - \dfrac{\langle u_1, v_2 \rangle}{\|u_1\|^2} u_1 \\ &\vdots \\ u_n &= v_n - \displaystyle\sum_{i=1}^{n-1} \dfrac{\langle u_i, v_n \rangle}{\|u_i\|^2} u_i. \end{cases}$$

We construct the set $(e_1, \dots, e_n)$ from $(u_1, \dots, u_n)$ as follows:

$$e_k = \frac{u_k}{\|u_k\|}, \quad k \in \{1, \dots, n\}.$$

Example

Let F be the vector sub-space of $\mathbb{R}^4$ spanned by the vectors $S = \{u = (1,1,0,0),\ v = (1,0,-1,0),\ w = (0,0,1,1)\}$.

1. Prove that S is a basis of the sub-space F.
2. In use of Gramm-Schmidt Algorithm, find an orthonormal basis of F. (with respect to the Euclidean inner product).

Solution

1. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with columns the vectors u, v, w.

   The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a row reduced form of the matrix A. This proves that S is a basis of the sub-space F.

2. $u_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0)$, $u_2 = \frac{1}{\sqrt{6}}(1, -1, -2, 0)$,
   $u_3 = \frac{1}{\sqrt{12}}(1, -1, 1, 3)$.
   $\{u_1, u_2, u_3\}$ is an orthonormal basis of the sub-space F.

Exercise

1. Prove that $\langle (a,b),(x,y)\rangle = ax + ay + bx + 2by$ is an inner product in $\mathbb{R}^2$.
2. Use Gramm-Schmidt algorithm to construct an orthonormal basis of $\mathbb{R}^2$ from the basis $\{u_1 = (1,-1), u_2 = (1,2)\}$.

Solution

1. - $\langle (a,b)+(c,d),(x,y)\rangle = (a+c)x+(a+c)y+(b+d)x+2(b+d)y = \langle (a,b),(x,y)\rangle + \langle (c,d),(x,y)\rangle$
   - $\langle (a,b),(x,y)\rangle = ax+ay+bx+2by = \langle (x,y),(a,b)\rangle$
   - $\langle \lambda(a,b),(x,y)\rangle = \lambda ax+\lambda ay+\lambda bx+2\lambda by = \lambda\langle (a,b),(x,y)\rangle$
   - $\langle (a,b),(a,b)\rangle = a^2+2ab+2b^2 = (a+b)^2+b^2 \geq 0$
   - $\langle (a,b),(a,b)\rangle = 0 \iff a+b=0=b \iff a=b=0$
2. The vector u_1 is unitary and the second vector is $v_2=(1,0)$. Then $\{v_1=(1,-1), v_2=(1,0)\}$ is an orthonormal basis.

Example

Let $S = \{u_1, u_2, u_3, u_4\}$ is a basis of the space $\mathscr{M}_2(\mathbb{R})$ such that
$u_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, u_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
We use the Gramm-Schmidt algorithm to construct an orthonormal basis from the basis S.

$$v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

$$\langle u_2, v_1 \rangle = \frac{2}{\sqrt{3}},$$

$$u_2 - \langle u_2, v_1 \rangle v_1 = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

$$v_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

$$\langle u_3, v_1\rangle = \sqrt{3},\ \langle u_3, v_2\rangle = \frac{3}{\sqrt{15}}$$

$$u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 = \frac{1}{5}\begin{pmatrix} -1 & 3 \\ -3 & 4 \end{pmatrix}.$$

$$v_3 = \frac{1}{\sqrt{35}}\begin{pmatrix} -1 & 3 \\ -3 & 4 \end{pmatrix}.$$

$$\langle u_4, v_1\rangle = 0,\ \langle u_4, v_2\rangle = \frac{6}{\sqrt{15}},\ \langle u_4, v_3\rangle = \frac{4}{\sqrt{35}}$$

$$u_4 - \langle u_4, v_1\rangle v_1 - \langle u_4, v_2\rangle v_2 - \langle u_4, v_3\rangle v_3 = \frac{1}{35}\begin{pmatrix} -10 & -39 \\ -29 & -29 \end{pmatrix}.$$

$$v_4 = \frac{1}{\sqrt{7}}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Exercise

Let F be the vector sub-space of the Euclidean space $\mathbb{R}^4$ spanned by the following vectors
$u_1 = (1, 2, 0, 2),\ u_2 = (-1, 1, 1, 1)$.

1. Use Gramm-Schmidt algorithm to construct an orthonormal basis of the vector sub-space F
2. Prove that the set $F^\perp = \{u \in \mathbb{R}^4 :\ \langle u, v\rangle = 0,\ \forall v \in F\}$ is a vector sub-space of $\mathbb{R}^4$.
3. Find an orthonormal basis of the vector sub-space $F^\perp$.

Solution

1. $v_1 = \frac{1}{3}u_1$, $\langle u_2, v_1\rangle = 1$,
   $u_2 - \langle u_2, v_1\rangle v_1 = (0,3,1,-1) - \frac{1}{3}(-1,1,1,1) = \frac{1}{3}(-4,1,3,1)$.
   Then $v_2 = \frac{1}{3\sqrt{3}}(-4,1,3,1)$.
   (v_1, v_2) is an orthonormal basis of the vector sub-space F.
2. If $v_1, v_2 \in F^\perp$, $\alpha, \beta \in \mathbb{R}$ and $u \in F$, then
   $$\langle \alpha v_1 + \beta v_2, u\rangle = \alpha\langle v_1, u\rangle + \beta\langle v_2, u\rangle = 0.$$
   Then $F^\perp$ is a vector sub-space of $\mathbb{R}^4$.
3. Let $u = (x,y,z,t) \in \mathbb{R}^4$.
   $$u \in F^\perp \iff \begin{cases} \langle u, u_1\rangle = 0 \\ \langle u, u_2\rangle = 0 \end{cases} \iff \begin{cases} x + 2y + 2t = 0 \\ -x + y + z + t = 0 \end{cases}$$

$$\begin{cases} x + 2y + 2t = 0 \\ -x + y + z + t = 0 \end{cases} \iff \begin{cases} x = \frac{2}{3}z \\ y = -\frac{z}{3} - t \end{cases}$$

Then $u \in F^{\perp} \iff u = -\frac{z}{3}(-2, 1, -3, 0) + t(0, -1, 0, 1)$.
The vectors $e_1 = (-2, 1, -3, 0), e_2 = (0, -1, 0, 1)$ is an orthogonal basis of the vector sub-space $F^{\perp}$.
$w_1 = \frac{1}{\sqrt{14}} e_1$, $\langle w_1, e_2 \rangle = -\frac{1}{\sqrt{14}}$,
$e_2 - \langle e_2, w_1 \rangle w_1 = \frac{1}{14}(2, 13, 3, 14)$.
Then $(\frac{1}{\sqrt{14}}(-2, 1, -3, 0), \frac{1}{3\sqrt{42}}(2, 13, 3, 14))$ is an orthonormal basis of the vector sub-space $F^{\perp}$.

Consider the following inner product on $\mathbb{R}^3$

$$\langle (x, y, z), (x', y', z') \rangle = 2xx' + 4yy' + zz' + 2xy' + 2yx'.$$

1. Use Gram-Schmidt process on the standard basis $C = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ to get an orthogonal basis $B = \{v_1, v_2, v_2\}$ of $\mathbb{R}^3$.
2. Let $u = (1, 2, 3)$ be a vector in $\mathbb{R}^3$. Compute $[u]_B$ the coordinates of u with respect to the basis B.

1. The basis $\{\frac{1}{\sqrt{2}}u_1, \frac{1}{\sqrt{2}}(u_2 - u_1), u_3\}$ is an orthonormal basis of $\mathbb{R}^3$
2. Let $u = (1, 2, 3)$ be a vector in $\mathbb{R}^3$.
$$[u]_B = \begin{pmatrix} \langle u, v_1 \rangle \\ \langle u, v_2 \rangle \\ \langle u, v_3 \rangle \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \\ 3 \end{pmatrix}.$$