Parametric Equations and Polar Coordinates

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1. Parametric Equations of Plane Curves
2. Polar Coordinates
3. Polar Curves

The graph of a function $f\colon I \longrightarrow \mathbb{R}$ (I an interval) is an example of plane curve but it is not general enough to represent all types of plane curves, for example a circle or a vertical line segment are not the graph of functions because two distinct points of a graph have different abscissa. In this section we study the trajectory of a point in the plane whose coordinates $(x(t), y(t))$ depend on a parameter t, these are the parametric curves, or curves verifying a Cartesian equation.

Parametric Equations

Definition

If f and g are continuous functions on an interval I, the set of ordered pairs $(f(t), g(t))$, $t \in I$ is called a plane curve $\mathcal{C}$.
The equations $x = f(t)$ and $y = g(t)$ are called parametric equations of the curve $\mathcal{C}$ and t is called the parameter.
We can also interpret the curve as the vectorial function $\gamma\colon I \longrightarrow \mathbb{R}^2$ defined by $\gamma(t) = (f(t), g(t))$, $t \in I$. In this case $\mathcal{C}$ is called the support of the curve γ.

Definition

1. The curve $\gamma\colon I \longrightarrow \mathbb{R}^2$ is called respectively continuous, differentiable, k-times differentiable, of class $\mathcal{C}^k$, if f and g are continuous, differentiable, k-times differentiable, of class $\mathcal{C}^k$.
2. The Orientation of the curve of the parametric equations $\gamma = (f, g)$ is the direction of movement of the vector γ, for $t \in I$.

Remark

1. If $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ is a curve and the function $f : I \longrightarrow J$ is bijective, then $t = f^{-1}(x)$ and the curve is represented by the equation $y = g(t) = g \circ f^{-1}(x)$ and the curve is the graph of the function $y = g \circ f^{-1}(x)$, for $x \in J$.
2. If $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ is a curve an the function $g : I \longrightarrow J$ is bijective, then $t = g^{-1}(y)$ and the curve is represented by the equation $x = f(t) = f \circ g^{-1}(x)$ and the curve is the graph of the function $x = f \circ g^{-1}(y)$, for $y \in J$.

Examples

1. The graph of a function $y = f(x)$ is a parametric curve of equation
   $\gamma(t) = (x(t), y(t)) = (t, f(t))$.
2. A line of equation $y = ax + b$ is the geometric curve of the mapping
   $\gamma(t) = (t, at + b)$, $t \in \mathbb{R}$, therefore it is parameterizable as in 1).
   The parametrization $(x(t), y(t)) = (a, t)$, $t \in \mathbb{R}$ is a parametrization of the vertical line $x = a$.
3. The circle in $\mathbb{R}^2$ of center (a, b) and of radius $r > 0$ is the curve defined by $\{(x, y) \in \mathbb{R}^2;\ (x - a)^2 + (y - b)^2 = r^2\}$ and it is parameterized by $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$, where
   $\gamma(t) = (a + r\cos(t), b + r\sin(t))$.

Remark

There are infinitely many ways to parametrize a curve.

1. $(x(t), y(t)) = (t, f(t))$ is a parametrization of the graph of the function $y = f(x)$. But $(x(t), y(t)) = (t - a, f(t - a))$ is also a parametrization of this curve.
2. $(a + r\cos(t), b + r\sin(t),\ t \in [0, 2\pi]$ is also a parametrization of the circle of center (a, b) and radius r.

Examples

1. $x(t) = t + 1$, $y(t) = 2t + 3$, $t \in [-1, 2]$. Then $y = 2x + 1$, $x \in [0, 3]$. The parametric equation represents a straight line.
2. $x(t) = t - 1$, $y(t) = t^2$, $t \in [-1, 3]$. Then $y = (x + 1)^2$, $x \in [-2, 1]$. The parametric equation represents a parabola opens upwards with vertex $(-1, 0)$.
3. $x(t) = 2 + 2\cos t$, $y(t) = -1 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $(x - 2)^2 + (y + 1)^2 = 4$. The parametric equation represents a circle with center $(2, -1)$ and radius 2. It is a closed curve and its direction is counter-clockwise.

4. $x(t) = 1 + 3\cos t$, $y(t) = -1 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $\dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{4} = 1$. The parametric equation represents an ellipse with center $(1, -1)$, the endpoints of the major axis are $(4, -1)$, $(-2, -1)$ (its length is 6) and the endpoints of the minor axis are $(1, -3)$, $(1, 1)$ (its length is 4). It is a closed curve and its direction is counter-clockwise.

Tangent to Parametric Curve

Definition

Let $\gamma = (f, g) \colon I \longrightarrow \mathbb{R}^2$ be a parametric curve and let $a \in I$ (I an open interval). We assume that $\gamma(t) \neq \gamma(a)$ for t close to a. We say that this curve has tangent at the point $M_0 = (f(a), g(a))$ if the direction of the vector $M_0M_t = \gamma(t) - \gamma(a)$, ($M_t = \gamma(t)$) has a limit when t tends to a. This means that for $t \in I$ close to a ($t \neq a$), there exists a vector $V(t)$ collinear to the vector M_0M_t such that $\lim\limits_{t \longrightarrow a} V(t) = V \neq 0$. The tangent at $M_0 = \gamma(a)$ to the curve is the line passing through M_0 and parallel to the vector V.

Example

If $\gamma(t) = (t^2, t^3)$ for $t \in \mathbb{R}$. The tangent to the curve $t \longmapsto \gamma(t)$ at $(0,0) = \gamma(0)$ is the real axis. Indeed, $\gamma(t) - \gamma(0) = t^2(1, t)$ which is parallel to the vector $V(t) = (1, t)$ and has the limit $(1, 0)$ when t tends to 0.

Theorem

1. Let $\gamma\colon I \longrightarrow \mathbb{R}^2$ be a plane curve. If γ is differentiable at a and $\gamma'(a) \neq 0$, the curve has a tangent at $M_0 = \gamma(a)$ parallel to the vector $\gamma'(a)$.
2. In general if γ is k-times differentiable at a and $\gamma'(a) = \gamma''(a) = \ldots = \gamma^{(k-1)}(a) = 0$ and $\gamma^{(k)}(a) \neq 0$, then the curve has a tangent at $M_0 = \gamma(a)$ parallel to the vector $\gamma^{(k)}(a)$.

Remark

1. The slope of the tangent line to a parametric curve if it exists is
$$m = \lim_{t \to t_0} \frac{y'(t)}{x'(t)}.$$
2. The tangent line to the parametric curve is horizontal if the slope is equal to zero. In particular if $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.
3. The tangent line to the parametric curve is vertical if the slope is equal to ∞. In particular if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Definition

Let $\gamma = (f(t), g(t))$ be a parametric curve defined on the interval $I = [a, b]$.

1. If γ is injective, the parametric curve is called simple.
2. If $\gamma(a) = \gamma(b)$, the parametric curve is called closed.

Examples

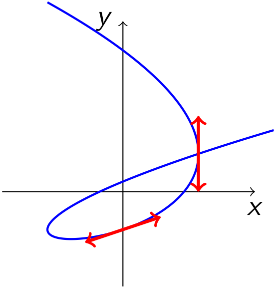
1. $x(t) = 1 + 3\cos t$, $y(t) = -1 + 3\sin(t)$, $t \in [0, 2\pi]$. Then $(x-1)^2 + (y+1)^2 = 9$. Since $x'(t) = -3\sin(t)$ and $y'(t) = 3\cos(t)$, the tangent to the curve is parallel to the $x-$axis at the point $(1, 2)$ for $t = \frac{\pi}{2}$ and at the point $(1, -4)$ for $t = \frac{3\pi}{2}$.
   The tangent to the curve is parallel to the $y-$axis at the point $(4, -1)$ for $t = 0$ and at the point $(-2, -1)$ for $t = \pi$.

2. $x(t) = 3 + 3\cos t$, $y(t) = 2 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{4} = 1$. Since $x'(t) = -3\sin(t)$ and $y'(t) = 2\cos(t)$, the tangent to the curve is parallel to the $x-$axis at the point $(3, 5)$ for $t = \frac{\pi}{2}$ and at the point $(3, 0)$ for $t = \frac{3\pi}{2}$.
The tangent to the curve is parallel to the $y-$axis at the point $(6, 2)$ for $t = 0$ and at the point $(0, 2)$ for $t = \pi$.

3. The slope of the tangent line to the curve $(x(t) = t^3 + 1, y(t) = t^4 - 1)$ at $t = 1$ is $m = \dfrac{y'(1)}{x'(1)} = \dfrac{4}{3}$.
4. Let $x(t) = t^3 - 3t,\ y(t) = t^2 - t - 1,\ t \in \mathbb{R}$.

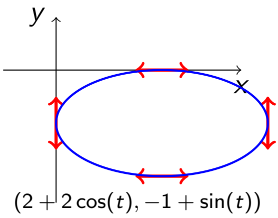
The slope of the curve at $t = -1$ is ∞. The tangent line to the curve at $(2, 1)$ is parallel to the $y-$ axis. The slope of the curve at $(0, -1)$ for $t = 0$ is $\dfrac{1}{3}$. The equation of the tangent line to the curve at $(0, -1)$ is

$y = \dfrac{1}{3}x - 1$.



$(t^3 - 3t, t^2 - t - 1)$

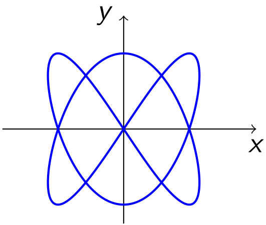
5. Let $x(t) = 2 + 2\cos t,\ y(t) = -1 + \sin(t),\ t \in [0, 2\pi]$.
The slope of the curve is
$m = \dfrac{\cos(t)}{-2\sin(t)}$. The points of the curve at which the tangent line is vertical are $(4, -1)$ and $(0, -1)$. The points of the curve at which the tangent line is horizontal are $(2, 0)$ and $(2, -2)$.



Example

$(x(t), y(t)) = (\sin(2t), \cos(3t))$, for $t \in \mathbb{R}$. The curve is periodic of period 2π. $x(-t) = -x(t)$, $y(-t) = -y(t)$, thus we study the curve on the interval $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the origin.

$x(\pi - t) = x(-t) = -x(t)$,
$y(\pi - t) = y(t)$, thus we study the curve on $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the axis (oy) and a symmetry with respect to the origin. $M_0 = (0,0)$, $f'(0) = (2,3)$,
$f''(0) = (0,0)$ and
$f^{(3)}(0) = (-8,-27)$. $(0,0)$ is an inflection point.



$(x(t), y(t)) = (\sin(2t), \cos(3t))$

Arc Length of Parametric Curve

Definition

Let $\gamma \colon [a, b] \longrightarrow \mathbb{R}^2$ be a smooth curve. The arc length of the curve γ is defined by:

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Remark

The expression of $L(\gamma)$ is invariant by change of parametrization of class $\mathcal{C}^1$ of the curve. Indeed if $\varphi\colon [\alpha, \beta] \longrightarrow [a, b]$ is a strictly increasing function of class $\mathcal{C}^1$. Set $\psi(s) = \gamma(\varphi(s))$, $\psi'(s) = \gamma'(\varphi(s)).\varphi'(s)$, $||\psi'(s)|| = ||\gamma'(\varphi(s))||\varphi'(s)$. $(\varphi'(s) \geq 0)$. Thus from the change of variables formula $(\varphi(\alpha) = a$, $\varphi(\beta) = b)$ we have

$$\int_\alpha^\beta ||\psi'(s)||ds = \int_a^b ||\gamma'(t)||dt.$$

The same result if φ is strictly decreasing.

Examples

1. If the curve is defined in Cartesian coordinates $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$, with $\gamma(t) = (t, y(t))$, $t \in [a,b]$ and y of class $\mathcal{C}^1$.

$$L(\gamma) = \int_a^b \sqrt{1 + (y'(t))^2}dt.$$

   For example, if $y = \tan(t)$, $t \in [0, \frac{\pi}{4}]$, then $L(\gamma) = \ln(1 + \sqrt{2})$.

2. If $\gamma(t) = (\cos(t), \sin(t))$, $t \in [0, 4\pi]$. $L(\gamma) = 4\pi$.

Example

Consider the parametric curve $x(t) = \frac{1}{3}t^3 + 1$, $y(t) = \frac{1}{2}t^2 + 2$, $t \in [0, 2]$. The arc length of this curve is

$$\begin{aligned} L &= \int_0^2 \sqrt{(t^2)^2 + (t)^2} dt = \frac{1}{2} \int_0^2 (t^2 + 1)^{\frac{1}{2}} (2t)\ dt \\ &= \frac{1}{2} \left[\frac{2}{3} (t^2 + 1)^{\frac{3}{2}} \right]_0^2 = \frac{1}{3} \left(5\sqrt{5} - 1 \right). \end{aligned}$$

Surface Area Generated by Revolving a Parametric Curves

Theorem

If $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$ is a smooth parametric curve:

1. The surface area generated by revolving the curve γ around the $x-$axis is

$$S = 2\pi \int_a^b |y(t)|\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

2. The surface area generated by revolving γ around the $y-$axis is

$$S = 2\pi \int_a^b |x(t)|\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Examples

The surface area generated by revolving the following parametric curves:

1. $x(t) = t,\ y(t) = \dfrac{t^3}{3} + \dfrac{1}{4t},\ t \in [1, 2]$, around the $x-$axis.

$$\begin{aligned}
S &= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right) \sqrt{1 + \left(t^2 - \frac{1}{4t^2}\right)^2}\ dt \\
&= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right) \sqrt{t^4 + \frac{1}{2} + \frac{1}{16t^4}}\ dt \\
&= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right) \left(t^2 + \frac{1}{4t^2}\right)\ dt \\
&= 2\pi \int_1^2 \left(\frac{t^5}{3} + \frac{t}{3} + \frac{1}{16t^3}\right)\ dt = \frac{509\pi}{64}
\end{aligned}$$

2. $x(t) = 4\sqrt{t}$, $y(t) = \frac{1}{2}t^2 + \frac{1}{t}$, $t \in [1, 4]$, around the $y-$axis.

$$\begin{aligned} S &= 2\pi \int_1^4 4\sqrt{t}\sqrt{\left(\frac{2}{\sqrt{t}}\right)^2 + \left(t - \frac{1}{t^2}\right)^2}\, dt \\ &= 2\pi \int_1^4 4\sqrt{t}\sqrt{\left(t + \frac{1}{t^2}\right)^2}\, dt \\ &= 2\pi \int_1^4 4\sqrt{t}\left(t + \frac{1}{t^2}\right)\, dt = \frac{288\pi}{5} \end{aligned}$$

Example

Find the surface area generated by revolving the following parametric curves:

1. $x(t) = 3t$, $y = 4t$, $t \in [0, 2]$, around the $x-$axis.
2. $x(t) = t$, $y = 2t$, $t \in [0, 4]$, around the $y-$axis.

In the rectangular coordinates system the ordered pair (a, b) represents a point, where "a" is the $x-$coordinate and "b" is the $y-$coordinate.
The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system, and the **polar axis** is the directed half-line (the non-negative part of the $x-$axis).
If P is any point in the plane different from the origin, then its polar coordinates consists of two components r and θ, where r is the algebraic distance between P and the pole O, and θ is the measure of an angle determined by the polar axis and OP.
Note: The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are:

1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$.
2. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$.

Remark

The polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P are related as follows:

$$x = r\cos\theta, \quad y = r\sin(\theta)$$

Examples

1. If $(r,\theta) = \left(2, \frac{\pi}{2}\right)$, then its other polar coordinates are $\left(2, \frac{\pi}{2} + 2k\pi\right)$ or $\left(-2, \frac{3\pi}{2} + 2n\pi\right)$, $k, n \in \mathbb{Z}$.
2. If $(r,\theta) = \left(-3, \frac{5\pi}{4}\right)$ then its other polar coordinates are $\left(-3, \frac{5\pi}{4} + 2k\pi\right)$ and $\left(3, \frac{\pi}{4} + 2n\pi\right)$, $k, n \in \mathbb{Z}$.
3. The rectangular coordinates (x, y) of the point $(r,\theta) = (-5, \pi)$ are
   $(x, y) = (5, 0)$.
4. The polar coordinates of the point $\left(2\sqrt{3}, -2\right)$ are $(4, -\frac{\pi}{6} + 2k\pi)$, $k \in \mathbb{Z}$ or $(-4, \frac{5\pi}{6} + 2k\pi)$, $k \in \mathbb{Z}$

5. The rectangular coordinates of the point $(r, \theta) = \left(2, \dfrac{\pi}{2}\right)$ are $(x, y) = (0, 2)$.
6. The polar coordinates of the point $\left(\sqrt{2}, \sqrt{2}\right)$ are $\left(2, \dfrac{\pi}{4} + 2k\pi\right)$, $k \in \mathbb{Z}$ or $\left(-2, \dfrac{5\pi}{4} + 2k\pi\right)$, $k \in \mathbb{Z}$.

Definition

A parametric curve $t \longmapsto \gamma(t)$, $(t \in I)$ is called a polar curve if for any $t \in I$, $\gamma(t)$ is determined by a polar coordinates $(r(t), \theta(t))$. In which follows, we study the polar curves with equation $r = f(\theta)$.

A curve in polar coordinates can be studied in Cartesian coordinates by the change of coordinates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$.

Examples

1. The straight lines:

   • Lines passing through the pole:

   Any straight line passing through the pole has the form $\theta = \theta_0$, where θ_0 is the angle between the straight line and the polar axis.

   $\theta = \theta_0 \Rightarrow \tan(\theta) = \tan(\theta_0) \Rightarrow \dfrac{y}{x} = \tan(\theta_0) \Rightarrow y = \tan(\theta_0)\ x.$

   The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan(\theta_0)$.

   For example the equation $\theta = \dfrac{\pi}{4}$ is the equation of a straight line passing through the pole with a slope equals to $\tan\left(\frac{\pi}{4}\right) = 1$. Therefore its equation in $xy-$form is $y = x$.

• Lines perpendicular to the polar axis:
Any straight line perpendicular to the polar axis has the form $r = a\ \sec(\theta)$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$r = a\ \sec(\theta) \Rightarrow r = \frac{a}{\cos(\theta)} \Rightarrow r\cos(\theta) = a \Rightarrow x = a.$$

The straight line $r = a\ \sec(\theta)$ is perpendicular to the polar axis at the point $(r, \theta) = (a, 0)$
For example the equation $r = 3\sec(\theta)$ is a straight line perpendicular to the polar axis and passing through the point $(r, \theta) = (3, 0)$. Therefore its equation in $xy-$form is $x = 3$.
The equation $r = -2\csc(\theta)$ is a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$.
Therefore its equation in the $xy-$form is $y = -2$.

• Lines parallel to the polar axis:
Any straight line parallel to the polar axis has the form $r = a\ \csc(\theta)$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$.
$r = a\ \csc(\theta) \Rightarrow r = \dfrac{a}{\sin(\theta)} \Rightarrow r\sin(\theta) = a \Rightarrow y = a.$
The straight line $r = a\ \sec(\theta)$ is parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \dfrac{\pi}{2}\right)$.

2. Circles:
• Circles of the form $r = a$, where $a \in \mathbb{R}^*$.
The equation $r = a$ represents a circle with center $(0, 0)$ and radius equals $|a|$.
• Circles of the form $r = a\sin(\theta)$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$.
$x = a\sin(\theta)\cos(\theta) = \dfrac{a}{2}\sin(2\theta)$, $y = a\sin^2(\theta) = \dfrac{a}{2} - \dfrac{a}{2}\cos(2\theta)$. Then the equation $r = a\sin(\theta)$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$ represents a circle with center $\left(0, \dfrac{a}{2}\right)$ and radius equals to $\dfrac{|a|}{2}$.

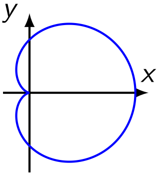
$r = 2\sin(\theta)$ represents a circle with center $(0, 1)$ and radius equals to 1

- Circles of the form $r = a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

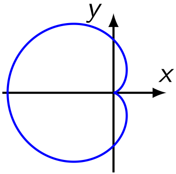
$x = a\cos^2(\theta) = \frac{a}{2} + \frac{a}{2}\cos(2\theta)$, $y = a\sin(\theta)\cos(\theta) = \frac{a}{2}\sin(2\theta)$. Then the equation $r = a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ represents a circle with center $\left(\frac{a}{2}, 0\right)$ and radius equals to $\frac{|a|}{2}$.

$r = 2\cos(\theta)$ represents a circle with center $(1, 0)$ and radius equals to 1

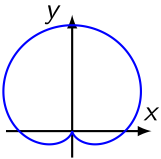
3. The Limaçon curves:
The general form of a Limaçon curve is $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$ and $0 \le \theta \le 2\pi$
• Cardioid (Heart-shaped). It has the form $r(\theta) = a \pm a\sin(\theta)$ or $r(\theta) = a \pm a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $0 \le \theta \le 2\pi$



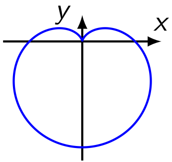
$r = 2 + 2\cos(\theta)$



$r = 2 - 2\cos(\theta)$



$r = 2 + 2\sin(\theta)$

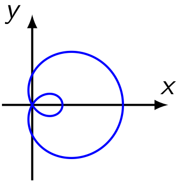


$r = 2 - 2\sin(\theta)$

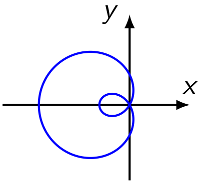
- Limaçon with inner loop:

It has the form $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \leq \theta \leq 2\pi$

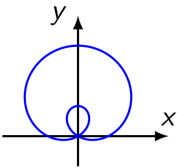
Note: Note that $|a| < |b|$ in this case.



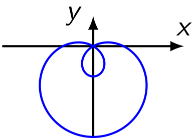
$r = 1 + 2\cos(\theta)$



$r = 1 - 2\cos(\theta)$



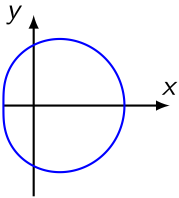
$r = 1 + 2\sin(\theta)$



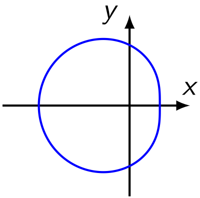
$r = 1 - 2\sin(\theta)$

- Dimpled Limaçon:

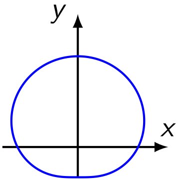
It has the form $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$



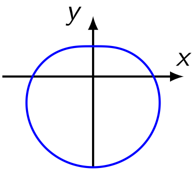
$r = 2 + \cos(\theta)$



$r = 2 - \cos(\theta)$

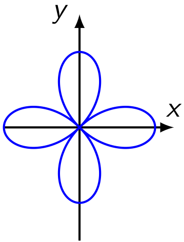


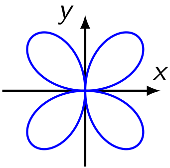
$r = 2 + \cos(\theta)$



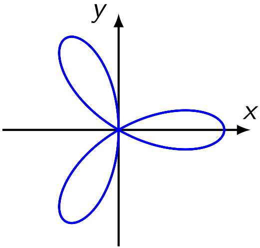
$r = 2 - \sin(\theta)$

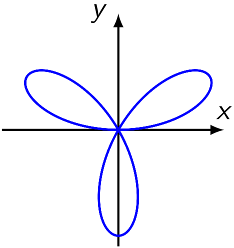
4. Rose curves:
It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \geq 2$
• **n is even**: In this case the number of loops (or leaves) is $2n$.
For example: $r(\theta)=2\cos(2\theta)$ or $r(\theta)=2\sin(2\theta)$, $0\leq\theta\leq 2\pi$.
The number of loops (or leaves) equals 4.





• **n is odd**: In this case the number of loops (or leaves) is n. For example: $r(\theta) = 2\cos(3\theta)$ or $r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$ The number of loops (or leaves) equals 3.





Tests of Symmetry

1. If $r(\theta) = r(-\theta)$, the curve is symmetric with respect to the polar axis (the $x-$axis).
   For example, the circle $r = 4\cos(\theta)$ and the cardioid $r = 2 + 2\cos(\theta)$ are both symmetric with respect to the polar axis.
2. If $r(\theta) = -r(-\theta)$ or $r(\theta) = r(\pi - \theta)$, the curve is symmetric with respect to the $y-$ axis.
   For example the circle $r = 4\sin(\theta)$ and the cardioid $r = 2 + 2\sin(\theta)$ are both symmetric with respect to the $y-$ axis.
3. If $r(\theta) = r(\pi + \theta)$, the curve is symmetric with respect to the pole.
   For example the rose curve $r = \sin(2\theta)$ is symmetric with respect to the pole.

Slope of the Tangent Line to a Polar Curve

Definition

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to the curve $r(\theta)$ at the point $r(\alpha)$ (if it exists) is

$$m = \lim_{\theta \to \alpha} \frac{dy}{dx} = \lim_{\theta \to \alpha} \frac{r(\theta)\cos(\theta) + r'(\theta)\sin(\theta)}{-r(\theta)\sin(\theta) + r'(\theta)\cos(\theta)}.$$

Notes:

1. If $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$, the tangent line to $r = r(\theta)$ is horizontal,
2. If $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$, the tangent line to $r = r(\theta)$ is vertical.

Example

1. Let $r(\theta) = 2\sin(\theta)$, $\theta \in [0, \pi]$.
   $x(\theta) = \sin(2\theta)$ and $\frac{dx}{d\theta} = 2\cos(2\theta)$, $y(\theta) = 2\sin^2(\theta)$ and $\frac{dy}{d\theta} = 2\sin(2\theta)$.
   The tangent line to the curve is vertical if and only if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$. Thus $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$.
   The points of the curve $r(\theta) = 2\sin(\theta)$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical are $(\sqrt{2}, \frac{\pi}{4})$ and $(\sqrt{2}, \frac{3\pi}{4})$.
   The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$. Thus $\theta = 0$, $\theta = \frac{\pi}{2}$ or $\theta = \pi$.

The points of the curve $r(\theta) = 2\sin(\theta)$, $0 \leq \theta \leq \pi$ at which the tangent line to r is horizontal are $(0, 0)$, and $(0, 2)$.

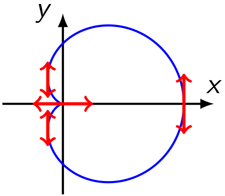
2. Consider the polar curve $r(\theta) = 1 + \cos(\theta)$, $\theta \in [0, 2\pi]$.

$x(\theta) = (1 + \cos\theta)\cos(\theta)$, $\dfrac{dx}{d\theta} = -\sin(\theta)(1 + 2\cos(\theta))$,
$y(\theta) = (1 + \cos(\theta))\sin(\theta)$ and
$\dfrac{dy}{d\theta} = \cos(\theta) + \cos(2\theta) = (2\cos(\theta) - 1)(\cos(\theta) + 1)$.
$\dfrac{dx}{d\theta} = 0 \iff \theta = 0, \pi, 2\pi, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$.
$\dfrac{dy}{d\theta} = 0 \iff \theta = \pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

The slope at the point $r(\pi)$ is

$$\begin{aligned} m &= \lim_{\theta\to\pi} \frac{y'(\theta)}{x'(\theta)} = \lim_{\theta\to\pi} \frac{(2\cos(\theta)-1)(\cos(\theta)+1)}{-\sin(\theta)(1+2\cos(\theta))} \\ &= \lim_{\theta\to\pi} \frac{(2\cos(\theta)-1)(\cos(\theta)+1)}{-\sin(\theta)(1+2\cos(\theta))} = 0. \end{aligned}$$

The tangent line to the curve $r = r(\theta)$ is horizontal at the points $(0,0)$, $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ and $(\frac{3}{4}, -\frac{\sqrt{3}}{4})$
The tangent line to the curve $r = r(\theta)$ is vertical at the points $(2,0)$, $(-\frac{1}{4}, \frac{\sqrt{3}}{4})$ and $(-\frac{1}{4}, -\frac{\sqrt{3}}{4})$

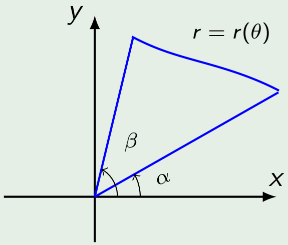


Area Between Polar Curves

Theorem

Let $r\colon [\alpha, \beta] \to \mathbb{R}^+$ be a continuous function, where $0 \leq \alpha < \beta \leq 2\pi$ (generally $0 < \beta - \alpha \leq 2\pi$). Then the area of the region bounded by the curve $r(\theta)$, where $\theta \in [\alpha, \beta]$, is equal to

$$A = \int_\alpha^\beta \frac{1}{2} r^2(\theta) d\theta.$$



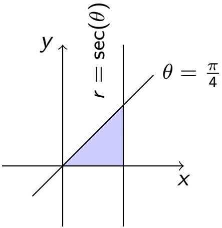
Examples

1. Let $r = \sec(\theta)$. The area of the region bounded by the curve and the straight lines $\theta = 0$ and $\theta = \frac{\pi}{4}$ is

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sec(\theta))^2 d\theta = \frac{1}{2}\left[\tan(\theta)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$

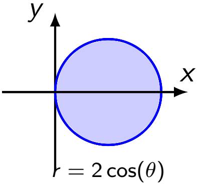
(The area is the area of the triangle of base 1 and height 1).

Note that $r = \sec(\theta)$ is a straight line perpendicular to the polar axis at the point $(r, \theta) = (1, 0)$, $\theta = 0$ is the polar axis and $\theta = \frac{\pi}{4}$ is a straight line passing the pole with a slope equals 1 (in fact it is the line $y = x$).



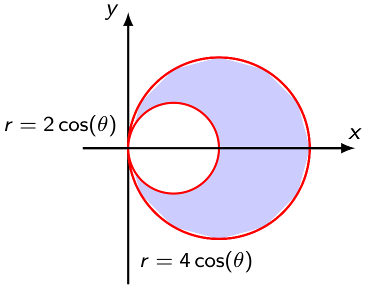
2. Let $r = 2\cos(\theta)$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$:
The polar curve $r = 2\cos(\theta)$ is a circle with center $(1, 0)$ and radius 1. The area inside the curve is:

$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2(\theta)\ d\theta \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1+\cos(2\theta)\right]\ d\theta \\ &= \left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi. \end{aligned}$$



3. Let $r = 4\cos(\theta)$ and $r = 2\cos(\theta)$.

Note that $r = 4\cos(\theta)$ is a circle with center $(2,0)$ and radius 2 and the curve $r = 2\cos(\theta)$ is a the circle with center $(1,0)$ and radius 1. The area inside the curve $r = 4\cos(\theta)$ and outside the curve $r = 2\cos(\theta)$ is:



$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(\theta))^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos(\theta))^2 d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos^2(\theta)\ d\theta = 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1+\cos(2\theta)\right]\ d\theta \\ &= 3\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi \end{aligned}$$

Arc Length of Polar Curve

Definition

The arc length of a smooth polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Examples

1. $r = 1 + \cos(\theta)$, $0 \leq \theta \leq 2\pi$.
   The curve is symmetric with respect to the polar axis then the arc length of the curve is

$$
\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{(1+\cos(\theta))^2 + (-\sin(\theta))^2}\; d\theta \\
&= 2\int_0^{\pi} \sqrt{(1+2\cos(\theta)+\cos^2(\theta)) + \sin^2(\theta)}\; d\theta \\
&= 2\int_0^{\pi} \sqrt{2+2\cos(\theta)}\; d\theta = 2\int_0^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\; d\theta \\
&= 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\; d\theta = 8.
\end{aligned}
$$

2. $r = 2\cos(\theta)$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
   The arc length of the curve is

$$\begin{aligned} L &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(2\cos(\theta))^2 + (-2\sin(\theta))^2}\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta = 2\pi. \end{aligned}$$

3. $r = e^{-\theta}$, $0 \leq \theta \leq \pi$.
   The arc length of the curve is

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \sqrt{2}\int_0^{\pi} e^{-\theta} d\theta = \sqrt{2}\left(1 - e^{-\pi}\right). \end{aligned}$$

Surface Area Generated by Revolving Polar Curve

Definition

The surface area generated by revolving the smooth polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$S = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin(\theta)| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

The surface area generated by revolving the smooth polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \dfrac{\pi}{2}$ is

$$A = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos(\theta)| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Examples

1. Let $r = e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \pi$. The surface area generated by revolving the smooth polar curve around the polar axis is

$$\begin{aligned} S &= 2\pi \int_0^{\pi} \left| e^{\frac{\theta}{2}} \sin(\theta) \right| \sqrt{\left(e^{\frac{\theta}{2}}\right)^2 + \left(\frac{1}{2} e^{\frac{\theta}{2}}\right)^2} \, d\theta \\ &= \pi\sqrt{5} \int_0^{\pi} e^{\theta} \sin(\theta) \, d\theta = \sqrt{5}\pi \left[\frac{1}{2} e^{\theta} (\sin(\theta) - \cos(\theta))\right]_0^{\pi} \\ &= \frac{\sqrt{5}\pi}{2} \left(e^{\pi} + 1\right). \end{aligned}$$

(We use integration by parts).

2. Let $r = 2 + 2\cos(\theta)$, $0 \leq \theta \leq \frac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the polar axis is

$$\begin{aligned}
S &= 2\pi \int_0^{\frac{\pi}{2}} |(2+2\cos(\theta))\sin(\theta)| \sqrt{(2+2\cos(\theta))^2 + (-2\sin(\theta))^2}\ d\theta \\
&= 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos(\theta))\sin(\theta)\sqrt{2+2\cos(\theta)}\ d\theta \\
&= 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos(\theta))^{\frac{3}{2}} \sin(\theta)\ d\theta \\
&= -2\pi \left[\frac{2}{5}(2+2\cos(\theta))^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}} = \frac{16\pi}{5}(8-\sqrt{2}).
\end{aligned}$$

3. Let $r = \cos(\theta)$, $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the line $\theta = \frac{\pi}{2}$ is

$$\begin{aligned} S &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos(\theta)\ \cos(\theta)| \sqrt{(\cos(\theta))^2 + (-\sin(\theta))^2}\ d\theta \\ &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\ d\theta = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(2\theta))\ d\theta \\ &= \pi \left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi^2. \end{aligned}$$

4. Let $r = 2\sin(\theta)$, $0 \leq \theta \leq \frac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the line $\theta = \frac{\pi}{2}$ is

$$\begin{aligned} S &= 2\pi \int_0^{\frac{\pi}{2}} |2\sin(\theta)\ \cos(\theta)| \sqrt{(2\sin(\theta))^2 + (2\cos(\theta))^2}\ d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} \sin(2\theta)\ d\theta = 4\pi. \end{aligned}$$

(The surface area of a sphere of radius 1.)