Global Expression of Cauchy's Theorem

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February 14, 2023

Definition

Let $\gamma_1, \dots, \gamma_n$ be closed piecewise continuously differentiable paths in an open subset Ω of $\mathbb{C}$. Let $\Gamma = \gamma_1 + \dots + \gamma_n$ be the formal sum of these closed paths defined by

$$\int_\Gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz,$$

for all continuous function f on Ω. Γ will be called a cycle.
By definition the index of the cycle Γ at a point $z \notin \bigcup_{j=1}^{n} (\text{support } \gamma_j)$ is

$$\mathrm{Ind}(\Gamma, z) = \sum_{j=1}^{n} \mathrm{Ind}(\gamma_j, z).$$

The main theorem in this chapter is the following:

Theorem

Let $f \in \mathcal{H}(\Omega)$ and Γ a cycle such that $\mathrm{Ind}(\Gamma, z) = 0,\ \forall\ z \notin \Omega$ then

1. $$f(z).\mathrm{Ind}(\Gamma, z) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma \frac{f(w)}{w - z}\, dw, \qquad \forall\ z \in \Omega \setminus \mathrm{Supp}\Gamma.$$
2. $\displaystyle\int_\Gamma f(w)\, dw = 0.$
3. *If Γ_1 and Γ_2 are two cycles in Ω such that* $\mathrm{Ind}(\Gamma_1, z) = \mathrm{Ind}(\Gamma_2, z);\ \forall\ z \notin \Omega$, *then* $\displaystyle\int_{\Gamma_1} f(w)\, dw = \int_{\Gamma_2} f(w)\, dw.$

Proof

2) and 3) are deduced from 1), indeed to prove 2) with the condition $\mathrm{Ind}(\Gamma, z) = 0, \ \forall \ z \in \mathbb{C} \setminus \Omega$, we consider the function F defined on Ω by

$$F(w) = \begin{cases} (w-z)f(w) & \text{if } w \neq z \\ F(z) = 0 & \end{cases}.$$

$$\frac{1}{2i\pi}\int_\Gamma f(w)\ dw = \frac{1}{2i\pi}\int_\Gamma \frac{F(w)}{w-z}\ dw = F(z)\mathrm{Ind}(\Gamma, z) = 0.$$

To prove 3) it suffices to consider the cycle $\Gamma = \Gamma_1 - \Gamma_2$.

To prove

$$f(z).\mathrm{Ind}(\Gamma, z) = \frac{1}{2\mathrm{i}\pi}\int_\Gamma \frac{f(w)}{w-z}\ dw \tag{1}$$

for $z \in \Omega \setminus \mathrm{Supp}\Gamma$, it suffices to prove

$$\int_\Gamma \frac{f(w)}{w-z}\ dw - \int_\Gamma \frac{f(z)}{w-z}\ dw = 0.$$

For the proof of the theorem 1.2, we need the following lemma:

Lemma

Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and $g : \Omega \longrightarrow \mathbb{C}$ the function defined by

$$g(z,w) = \begin{cases} f'(z) & \text{if} \quad z = w \\ \frac{f(w)-f(z)}{w-z} & \text{if} \quad z \neq w \end{cases}.$$

g is continuous and whenever $w \in \Omega$, the mapping $z \longmapsto g(z,w)$ is holomorphic.

Proof of lemma 1.3

The function g is continuous on $\Omega \setminus \{(a,a);\ a \in \mathbb{C}\}$. For $(a,a) \in \Omega$, there exists $R > 0$ such that $D(a,R) \subset \Omega$. Let $r < R$, $w, z \in \overline{D(a,r)}$ and the path γ defined by $\gamma(t) = tw + (1-t)z$ for $t \in [0,1]$. If $w \neq z$.

$$\begin{aligned}\int_0^1 f'(\gamma(t))\, dt &= \frac{1}{w-z}\int_0^1 f'(\gamma(t))\gamma'(t)\, dt \\ &= \frac{1}{w-z}\int_0^1 (f\circ\gamma)'(t)\, dt \\ &= \frac{f(w)-f(z)}{w-z} = g(w,z).\end{aligned}$$

Thus $g(w,z) - g(a,a) = \int_0^1 (f'(\gamma(t)) - f'(a))\ dt$. Since f' is continuous, g is continuous at (a,a).
We Recall the Fubini's theorem.

Theorem (The Fubini's Theorem)

Let $g\colon [a,b]\times[c,d] \longrightarrow \mathbb{C}$ be a continuous function, then

$$\int_a^b \left(\int_c^d g(t,s)\ ds\right) dt = \int_c^d \left(\int_a^b g(t,s)\ dt\right) ds.$$

Proof of theorem 1.2

The function $h\colon \Omega \longrightarrow \mathbb{C}$ defined by $h(z) = \dfrac{1}{2i\pi}\displaystyle\int_\Gamma g(w,z)\,dw$ is continuous on Ω. Indeed, let $(z_n)_n$ be a convergent sequence in Ω to $z \in \Omega$. The function g is uniformly continuous on any compact. We take $K_1 = \mathrm{Supp}\Gamma$ and K_2 a closed disc centered at z. We deduce that $\lim\limits_{n\to+\infty} g(w,z_n) = g(w,z)$ uniformly with respect to $w \in K_1$. The result follows. (We can use the dominated convergence theorem since for any compact K of Ω, g is bounded on $\mathrm{Supp}(\Gamma) \times K$.)

To prove that h is holomorphic on Ω, we use Morera's theorem and Fubini theorem.
Let Δ be a triangle in Ω.

$$\begin{aligned}\int_{\partial\Delta} h(z)\ dz &= \int_{\partial\Delta}\left(\frac{1}{2i\pi}\int_{\Gamma} g(w,z)\ dw\right)\ dz \\ &= \frac{1}{2i\pi}\int_{\Gamma}\left(\int_{\partial\Delta} g(w,z)\ dz\right)\ dw = 0,\end{aligned}$$

thus h is holomorphic.

We prove now that $h \equiv 0$ on Ω. For this we construct an entire function H, equal to h on Ω and $\lim\limits_{|z| \to +\infty} H(z) = 0$.
Let $V = \{z \in \mathbb{C} \setminus \mathrm{Supp}\Gamma;\ \mathrm{Ind}(\Gamma, z) = 0\}$. V is a non empty open subset, $\Omega^c \subset V$. Let h_1 be the function defined on V by

$$h_1(z) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma \frac{f(w)}{w - z}\, dw.$$

The functions h and h_1 coincide on $\Omega \cap V$, h_1 is holomorphic on V. We define the function H on $\Omega \cup V$ by

$$H(z) = \begin{cases} h(z) & \text{if} \quad z \in \Omega \\ h_1(z) & \text{if} \quad z \in V \end{cases}.$$

H is holomorphic on $\Omega \cup V = \mathbb{C}$ because $\Omega^c \subset V$.
We shall prove that $\lim\limits_{|z| \to +\infty} H(z) = 0$.
Since Γ is a cycle, then for $|z|$ large enough, $\mathrm{Ind}(\Gamma, z) = 0$. Thus the function H is defined by $H(z) = \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_\Gamma \frac{f(w)}{w - z}\, dw$.
$\left| \displaystyle\int_\Gamma \frac{f(w)}{w - z}\, dw \right| \leq \dfrac{1}{|z| - R} \sup\limits_{w \in \mathrm{Supp}\Gamma} |f(w)| L(\Gamma) \underset{|z| \to +\infty}{\longrightarrow} 0$, with $L(\Gamma)$ the length of Γ.

□

Remark 1 :

Let f be a holomorphic function on $D(0,R)$ and $f(z)=\sum_{n=0}^{+\infty} a_n z^n$ the expansion on power series of f. For all $0<r<R$, we denote γ_r the closed curve defined by $\gamma_r(t)=re^{\mathrm{i}t}$, for $t\in[0,2\pi]$. For $0<r_1<r_2<R$, let $\Gamma=\gamma_{r_2}-\gamma_{r_1}$ be the cycle and the function $g(z)=\dfrac{f(z)}{z^{n+1}}$ defined on the punctured disc $\Omega=D(0,R)\setminus\{0\}$ for $n\in\mathbb{N}_0$. Then $\mathrm{Ind}(\Gamma,z)=0$ for all $z\notin\Omega$, thus $\int_\Gamma g(z)dz=0$. We deduce that

$$\frac{1}{2\mathrm{i}\pi}\int_{\gamma_{r_2}}\frac{f(z)}{z^{n+1}}=\frac{1}{2\mathrm{i}\pi}\int_{\gamma_{r_1}}\frac{f(z)}{z^{n+1}},$$

Definition

Let $\gamma_0, \gamma_1 \colon [0,1] \longrightarrow \Omega$ be two closed curves. The curves γ_0 and γ_1 are called homotopically equivalent in Ω if there exists a continuous function $H \colon [0,1] \times [0,1] \longrightarrow \Omega$ such that $H(t,0) = \gamma_0(t)$, $H(0,s) = H(1,s)$ and $H(t,1) = \gamma_1(t)$, $\forall s, t \in [0,1]$.
We say that H is an homotopy between γ_0 and γ_1.
We remark that for all $s \in [0,1]$, the mapping $\gamma_s(t) = H(t,s)$ is a closed curve.

Example

If Ω is a convex open set, all closed curve γ in Ω is homotopically equivalent to a point. It suffices to take the mapping $H(t,s) = (1-s)\gamma_0(t) + s.a$, $a \in \Omega$. The mapping H is continuous, $H(t,0) = \gamma_0(t)$, $H(t,1) = a$, $H(0,s) = H(1,s)$ because $\gamma_0(0) = \gamma_0(1)$.
We have the same result if Ω is starlike with respect to a point.

Lemma

The homotopy's relationship is an equivalence relationship.

• **Reflexivity** Any closed curve γ is homotopically equivalent to itself. It suffices to consider $H(t,s) = \gamma(t)$, $\forall\ s \in [0,1]$.

• **Symmetry** If γ_0 and γ_1 are homotopically equivalent with respect to the mapping H. Let $F\colon [0,1]\times[0,1] \longrightarrow \Omega$ be the mapping defined by $F(t,s) = H(t, 1-s)$. Then $F(t,0) = H(t,1) = \gamma_1(t)$, $F(t,1) = H(t,0) = \gamma_0(t)$. We deduce that γ_1 and γ_0 are homotopically equivalent.

• **Transitivity** If γ_0 and γ_1 are homotopically equivalent with respect to the mapping $H(t,s)$ and γ_1 and γ_2 are homotopically equivalent with respect to the mapping $G(t,s)$. The mapping $F(t,s) = \begin{cases} H(t,2s) & 0 \leq s \leq \frac{1}{2} \\ G(t,2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$. F is continuous and realizes the homotopy between the closed curves γ_0 and γ_2.

Definition

An open subset Ω of $\mathbb{C}$ is called a simply connected domain if

1. *Ω is a domain.*
2. *Any closed curve in Ω is homotopically equivalent to a point.*

Examples

1. Any convex open subset of $\mathbb{C}$ is simply connected and more generally any starlike open subset with respect any point is simply connected. Indeed if Ω is starlike with respect to a point a and $\gamma\colon [0,1] \longrightarrow \Omega$ a closed curve. The mapping $H(t,s) = s\gamma(t) + (1-s)a$ is a homotopy between γ and a.
2. The punctured disc or the annulus are not simply connected.

Theorem

Let Γ_0 and Γ_1 be two closed piecewise continuously differentiable curves homotopically equivalent in Ω, then
$\mathrm{Ind}(\Gamma_0, z) = \mathrm{Ind}(\Gamma_1, z)$, $\forall\ z \notin \Omega$.

Remarks 2 :

1. If Ω is a simply connected domain, then for all closed piecewise continuously differentiable curve in Ω, $\mathrm{Ind}(\gamma, z) = 0$, whenever $z \notin \Omega$. (This remark can be taken also as a definition of a simple connected domain).
2. If Ω is simply connected domain, there is no bounded connected components of Ω^c.

Corollary

If Ω is a simply connected domain, then
a) for all holomorphic function f on Ω and for any closed piecewise continuously differentiable curve γ in Ω, $\int_\gamma f(z)\,dz = 0$,
b) any holomorphic function f on Ω has a primitive in Ω.

Theorem

If Ω is a simply connected domain and f a holomorphic on Ω without zeros, there exists a holomorphic function g on Ω such that $f = \mathrm{e}^g$.

Proof

Let h be a primitive of $\frac{f'}{f}$, then $(\frac{\mathrm{e}^h}{f})' = 0$. There exists $c \in \mathbb{C}^*$ such that $\mathrm{e}^h = cf$, if C is a logarithm of $c \in \mathbb{C}^*$, the function $g = h - C$ answer the theorem.

$\square$

For the proof of theorem 2.4 we need the following lemma:

Lemma

Let γ_0, $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$ *be two closed piecewise continuously differentiable curves in* $\mathbb{C}$ *and let* $z_0 \in \mathbb{C}$ *such that* $|\gamma_1(t) - \gamma_0(t)| < |z_0 - \gamma_0(t)|, \ \forall\ t \in [0,1]$. *Then* $\mathrm{Ind}(\gamma_0, z_0) = \mathrm{Ind}(\gamma_1, z_0)$.

Proof

If $\gamma(t) = \dfrac{\gamma_1(t) - z_0}{\gamma_0(t) - z_0}$, then $1 - \gamma(t) = \dfrac{\gamma_0(t) - \gamma_1(t)}{\gamma_0(t) - z_0}$. The assumption on γ_0 and γ_1 yields that $|1 - \gamma(t)| < 1$, thus $\mathrm{Ind}(\gamma, 0) = 0$. (0 is in the unbounded connected component of $(\mathbb{C} \setminus \mathrm{Supp}\gamma)$). But

$$\begin{aligned} \mathrm{Ind}(\gamma, 0) &= \frac{1}{2i\pi} \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, dt = \frac{1}{2i\pi} \int_0^1 \Big(\frac{\gamma_1'(t)}{\gamma_1(t) - z_0} - \frac{\gamma_0'(t)}{\gamma_0(t) - z_0}\Big)\, dt \\ &= \mathrm{Ind}(\gamma_1, 0) - \mathrm{Ind}(\gamma_0, 0). \end{aligned}$$

Thus $\mathrm{Ind}(\gamma, 0) = \mathrm{Ind}(\gamma_1, z_0) - \mathrm{Ind}(\gamma_0, z_0) = 0$. $\square$

Proof of theorem 2.4

Let $H\colon [0,1]\times[0,1]\longrightarrow\Omega$ be a continuous mapping such that $H(t,0)=\Gamma_0(t)$, $H(t,1)=\Gamma_1(t)$ and $H(0,s)=H(1,s)$ for all $s\in[0,1]$. Let $K=H([0,1]\times[0,1])$ and $\varepsilon>0$ such that $d(K,\Omega^c)\geq 2\varepsilon>0$. Since H is uniformly continuous on the compact set K, there exists $p\in\mathbb{N}$ such that $|H(t,s)-H(t',s')|<\varepsilon$ if $|t-t'|<\frac{1}{p}$ and $|s-s'|<\frac{1}{p}$.
For each $0\leq k\leq p$, we consider the following closed curves
$\gamma_k(t)=H(\frac{j}{p},\frac{k}{p})(pt+1-j)+H(\frac{j-1}{p},\frac{k}{p})(j-pt)$,
for $j-1\leq pt\leq j$ and $1\leq j\leq p$.

We have $|\gamma_k(t) - H(t, \frac{k}{p})| < \varepsilon$ for all $t \in [0,1]$ and $k = 0, \ldots, p$. Indeed for all $j - 1 \leq pt \leq j$,

$$\begin{aligned} |\gamma_k(t) - H(t, \frac{k}{p})| &\leq |H(\frac{j}{p}, \frac{k}{p}) - H(t, \frac{k}{p})|(pt + 1 - j) \\ &\quad +(j - pt)|H(\frac{j-1}{p}, \frac{k}{p}) - H(t, \frac{k}{p})| < \varepsilon. \end{aligned}$$

So is for $|\gamma_k(t) - \gamma_{k-1}(t)| < \varepsilon$. We have then $|\gamma_0(t) - \Gamma_0(t)| < \varepsilon$ for all $t \in [0,1]$.

$|\gamma_p(t) - \Gamma_1(t)| < \varepsilon$ for all $t \in [0,1]$.
Let proving now that $|\gamma_k(t) - z_0| > \varepsilon$ for all $z_0 \notin \Omega$, $k = 0, \dots, p$ and all $t \in [0,1]$.

$$|\gamma_k(t) - z_0| \geq |H(t, \frac{k}{p}) - z_0| - |\gamma_k(t) - H(t, \frac{k}{p})|.$$

$|H(t, \frac{k}{p}) - z_0| \geq 2\varepsilon$ and $|\gamma_k(t) - H(t, \frac{k}{p})| < \varepsilon \Rightarrow |\gamma_k(t) - z_0| > \varepsilon$.

We prove now that $\mathrm{Ind}(\gamma_k, z_0) = \mathrm{Ind}(\gamma_{k-1}, z_0)$.
$\mathrm{Ind}(\gamma_0, z_0) = \mathrm{Ind}(\Gamma_0, z_0)$ and $\mathrm{Ind}(\gamma_p, z_0) = \mathrm{Ind}(\Gamma_1, z_0)$.
We have

$$|\gamma_k(t) - \gamma_{k-1}(t)| < \varepsilon < |\gamma_k(t) - z_0| \Rightarrow \mathrm{Ind}(\gamma_k, z_0) = \mathrm{Ind}(\gamma_{k-1}, z_0).$$

$$|\gamma_0(t) - \Gamma_0(t)| < \varepsilon < |\gamma_0(t) - z_0| \Rightarrow \mathrm{Ind}(\gamma_0, z_0) = \mathrm{Ind}(\Gamma_0, z_0).$$

The same result for the third equality. □

Corollary

If γ_0 and γ_1 are two piecewise continuously differentiable curves and homotopically equivalent in Ω, then for all $f \in \mathcal{H}(\Omega)$

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

If Ω is a domain of $\mathbb{C}$, the following properties are equivalent

1. Ω is simply connected.
2. Two closed curves in Ω are homotopically equivalent in Ω.
3. Any holomorphic function on Ω has a primitive.
4. If $f \in \mathcal{H}(\Omega)$ and γ a closed piecewise continuously differentiable curve in Ω, then $\displaystyle\int_\gamma f(z)\, dz = 0$.
5. For all $z \in \Omega^c$, and for any closed piecewise continuously differentiable curve γ in Ω, $\mathrm{Ind}(\gamma, z) = 0$.
6. For any holomorphic function f on Ω without zeros, there exists a holomorphic function g on Ω such that $f = \mathrm{e}^g$.
7. For any holomorphic function f on Ω without zeros, there exists a holomorphic function g on Ω such that $g^2 = f$.
8. $\Omega = \mathbb{C}$ or Ω is isomorphic to unit disc (Riemann's theorem).

This theorem will be proved later.

Theorem

Let Ω be an open subset containing the annulus $\{z \in \mathbb{C};\ 0 < r_1 \leq |z - z_0| \leq r_2 < +\infty\}$ and let f be a holomorphic function on Ω. Then for all z in the annulus $\{z \in \mathbb{C};\ r_1 < |z - z_0| < r_2\}$,

$$f(z) = \frac{1}{2\mathrm{i}\pi} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\mathrm{i}\pi} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw,$$

with $\gamma_1(t) = z_0 + r_1 \mathrm{e}^{\mathrm{i}t}$ and $\gamma_2(t) = z_0 + r_2 \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$.

Proof

The cycle $\Gamma = \gamma_1 - \gamma_2$ is in Ω and if $|a - z_0| < r_1 < r_2$, $\mathrm{Ind}(\Gamma, a) = 0$.
If $|a - z_0| > r_2 > r_1$, $\mathrm{Ind}(\Gamma, a) = 0$, then $\mathrm{Ind}(\Gamma, a) = 0$ for all $a \notin \Omega$. We derive from theorem 1.2 that

$$f(z)\mathrm{Ind}(\Gamma, z) = \frac{1}{2\mathrm{i}\pi}\int_\Gamma \frac{f(w)}{w - z}\, dw.$$

But if $r_1 < |z - z_0| < r_2$, $\mathrm{Ind}(\Gamma, z) = 1$, thus

$$f(z) = \frac{1}{2\mathrm{i}\pi}\int_{\gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\mathrm{i}\pi}\int_{\gamma_1} \frac{f(w)}{w - z}\, dw.$$

□

Theorem

Let Ω *be the annulus defined by* $\Omega = \{z \in \mathbb{C};\ 0 \leq s_1 < |z - z_0| < s_2 \leq +\infty\}$*. For any holomorphic function* f *on* Ω*, there exist a unique sequence* $(a_n)_{n\in\mathbb{Z}}$ *such that whenever* $z \in \Omega$

$$f(z) = \sum_{-\infty}^{+\infty} a_n(z - z_0)^n, \qquad (2)$$

where, $a_n = \dfrac{1}{2\mathrm{i}\pi} \displaystyle\int_{\gamma_r} \dfrac{f(w)}{(w - z_0)^{n+1}}\, dw$*, for all* $n \in \mathbb{Z}$*.*
$\gamma_r(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$ *with* $s_1 < r < s_2$ *and* $t \in [0, 2\pi]$*.*

The series (2) is absolutely convergent on Ω and uniformly convergent on any compact subset of Ω.

The term $\displaystyle\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is called the singular part of f at z_0 on the annulus.

Proof

Let r_1 and r_2 be two positive numbers such that $s_1 < r_1 < r_2 < s_2$ and let $z \in \Omega$ such that $r_1 < |z - z_0| < r_2$. By theorem 3.1, we have

$$f(z) = \frac{1}{2i\pi}\int_{\gamma_2}\frac{f(w)}{w-z}\,dw - \frac{1}{2i\pi}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw.$$

- Consider the first integral $\frac{1}{2i\pi}\int_{\gamma_2}\frac{f(w)}{w-z}\,dw$.

$\dfrac{1}{w-z} = \dfrac{1}{(w-z_0)-(z-z_0)} = \dfrac{1}{(w-z_0)}\dfrac{1}{1-\frac{z-z_0}{w-z_0}}$. As $|\dfrac{z-z_0}{w-z_0}| < 1$,

$$\frac{1}{w-z} = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(w-z_0)^{k+1}}.$$

If $z \in \overline{D(z_0,r)}$ and $w \in \mathscr{C}(z_0,r_2)$, $|\dfrac{(z-z_0)^k}{(w-z_0)^{k+1}}| \leq \dfrac{1}{r_2}(\dfrac{r}{r_2})^k$. Thus the series $\displaystyle\sum_{n\geq 0} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}$ converges uniformly with respect to w, for $w \in \mathscr{C}(z_0,r_2)$ and with respect to z for $|z-z_0| \leq r$, $r < r_2$. Since the function f is continuous, it is bounded on $\mathscr{C}(z_0,r)$ and

$$\frac{1}{2i\pi}\int_{\gamma_2}\frac{f(w)}{w-z}\,dw=\sum_{k=0}^{\infty}(z-z_0)^k\frac{1}{2i\pi}\int_{\gamma_2}\frac{f(w)}{(w-z)^{k+1}}\,dw.$$

- Consider the second integral $\displaystyle\frac{1}{2i\pi}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw$.

$$\frac{1}{w-z}=\frac{1}{(w-z_0)-(z-z_0)}=\frac{-1}{(z-z_0)}\frac{1}{(1-\frac{w-z_0}{z-z_0})}=$$
$$\frac{-1}{(z-z_0)}\sum_{k=0}^{\infty}(\frac{w-z_0}{z-z_0})^k.$$

If $r > r_1$, $|z - z_0| \geq r$ and $|w - z_0| = r_1$, then the series $\sum_{k\geq 0}(\frac{w - z_0}{z - z_0})^k$ converges uniformly on $\mathscr{C}(z_0, r_1)$ with respect to z such that $|z - z_0| \geq r$. The integral of the previous identity yields

$$\frac{-1}{2i\pi}\int_{\gamma_1}\frac{f(w)}{w - z}\,dw = \sum_{k=0}^{\infty}\frac{1}{(z - z_0)^{k+1}}\frac{1}{2i\pi}\int_{\gamma_1} f(w)(w - z_0)^k\,dw.$$

If $k = -p - 1$, we have $\frac{-1}{2i\pi}\int_{\gamma_1}\frac{f(w)}{w - z}\,dw =$

$$\sum_{k=-\infty}^{-1}(z - z_0)^p\frac{1}{2i\pi}\int_{\gamma_1}\frac{f(w)}{(w - z_0)^{p+1}}\,dw = \sum_{-\infty}^{-1} a_n(z - z_0)^n.$$

The series $\sum_{n\geq 0} a_n(z - z_0)^n$ converges uniformly on

The series $\sum_{n\leq -1} a_n(z-z_0)^n$ converges uniformly on $\{z\in\mathbb{C};\ |z-z_0|\geq r'>r_1\}$. Thus if we take a compact subset K of Ω, there exists r and r' such that $K\subset\{z\in\mathbb{C};\ r'\leq|z-z_0|\leq r\}\subset\{z\in\mathbb{C};\ r_1<|z-z_0|<r_2\}$ and then the series $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ converges uniformly on K.

- **Uniqueness of the coefficients.**

Assume that $f(z) = \sum_{n=-\infty}^{+\infty} b_n(z-z_0)^n$ and the series converges uniformly on any compact subsets of the annulus $\{z \in \mathbb{C};\ s_1 < |z-z_0| < s_2\}$. Let $s_1 < r < s_2$ and $k \in \mathbb{Z}$.
$\frac{f(w)}{(w-z_0)^{k+1}} = \sum_{n=-\infty}^{+\infty} b_n \frac{(w-z_0)^n}{(w-z_0)^{k+1}}$, with $w = z_0 + r\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$, then

$$\frac{1}{2i\pi}\int_{\gamma_r}\frac{f(w)}{(w-z_0)^{k+1}}\,dw=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{(re^{i\theta})^k}\,d\theta=b_k.$$

Thus the coefficients b_k are uniquely determined.

□

Remarks 3 :

Let f be a holomorphic function on the annulus $\{z \in \mathbb{C};\ 0 < |z - z_0| < r\}$.

1. z_0 is an isolated singularity.

$$f(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n + \sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

The series $\sum_{n\geq 0} a_n(z - z_0)^n$ converges for $|z - z_0| < r$ and the series $\sum_{n\leq -1} a_n(z - z_0)^n$ converges for $|z - z_0| > 0$.

Remarks 4 :

1. In the case of a removable singularity (or regular point), the singular part is zero indeed $a_n = \dfrac{1}{2i\pi}\displaystyle\int_{\gamma_s} \dfrac{f(w)}{(w-z_0)^{n+1}}\, dw$, with $0 < s < r$. If $n < 0$, $|a_n| \leq \dfrac{1}{s^n} \displaystyle\sup_{|w-z_0|=s} |f(w)| \underset{s\to 0}{\longrightarrow} 0$, thus $a_n = 0$ if $n < 0$.
2. If z_0 is a pole of order m, the singular part is $\displaystyle\sum_{n=-m}^{-1} a_n(z-z_0)^n$ and $a_{-m} \neq 0$, because $\displaystyle\lim_{z\to z_0} (z-z_0)^m f(z) = \alpha$, with $\alpha \in \mathbb{C}^*$.

Definition

If z_0 is an isolated singularity of a holomorphic function f on $\Omega \setminus \{z_0\}$ and if

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n$$ *on the annulus*

$\{z \in \mathbb{C};\ 0 < |z - z_0| < r\} \subset \Omega$. *The number a_{-1} is called the residue of f at z_0 and denoted by:* $\mathrm{Res}(f, z_0)$.

Remarks 5 :

1. If f is a holomorphic function on $\{z \in \mathbb{C};\ 0 < |z - z_0| < r\}$, for $0 < s < r$,

$$a_{-1} = \frac{1}{2\mathrm{i}\pi} \int_{\gamma_s} f(w)\ dw = \mathrm{Res}(f, z_0).$$

2. (The Bessel's functions)
Let $f(z) = e^{\frac{w}{2}(z - \frac{1}{z})}$.

$$f(z) = e^{\frac{w}{2}(z-\frac{1}{z})} = \sum_{-\infty}^{+\infty} J_n(w) z^n.$$

$$J_n(w) = \frac{1}{2\mathrm{i}\pi} \int_{\mathscr{C}} e^{\frac{w}{2}(z-\frac{1}{z})} \frac{dz}{z^{n+1}}.$$

Theorem (Residue at a simple pole)

If f has a simple pole at z_0, then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

In particular if $f(z) = \dfrac{g(z)}{h(z)}$, *with* $h'(z_0) \neq 0$, $h(z_0) = 0$ *and* $g(z_0) \neq 0$, *then* $\mathrm{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$.

Examples

• If f is a holomorphic function and z_0 is a zero of order k for f, then z_0 is a simple pole for the function $\frac{f'}{f}$ and $\mathrm{Res}(\frac{f'}{f}, z_0) = k$.

Indeed $f(z) = (z - z_0)^k g(z)$, with $g(z_0) \neq 0$, thus

$$\frac{f'(z)}{f(z)} = \frac{k}{(z - z_0)} + \frac{g'(z)}{g(z)}.$$

• If z_0 is a pole of order k for f, then z_0 is a simple pole for the function $\frac{f'}{f}$ and $\mathrm{Res}(\frac{f'}{f}, z_0) = -k$.

Indeed $f(z) = \frac{g(z)}{(z - z_0)^k}$, with $g(z_0) \neq 0$, thus

$$\frac{f'(z)}{f(z)} = \frac{-k}{(z - z_0)} + \frac{g'(z)}{g(z)}.$$

Theorem (Residue at a pole of order m)

If z_0 is a pole of order m for f, then

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right).$$

Theorem (The Residue Theorem)

Let $z_1, \dots, z_p$ *in* Ω *and* γ *a cycle in* $\Omega \setminus \{z_1, \dots, z_p\}$ *such that* $\mathrm{Ind}(\gamma, z) = 0$ *for all* $z \notin \Omega$. *If* $f : \Omega \setminus \{z_1, \dots, z_p\} \longrightarrow \mathbb{C}$ *is a holomorphic, then*

$$\int_\gamma f(z)\, dz = 2i\pi \sum_{j=1}^{p} \mathrm{Res}(f, z_j)\mathrm{Ind}(\gamma, z_j).$$

Proof

Let D_j be a disc centered at z_j and $z_k \notin D_j$, for all $k \neq j$. Then for all $z \in D_j$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_{n,j}(z - z_j)^n,\ z \neq z_j.$$

Define the function f_j by:

$$f_j(z) = \sum_{n=-\infty}^{-1} a_{n,j}(z - z_j)^n.$$

f_j is a holomorphic on $\mathbb{C} \setminus \{z_j\}$ and the function $F = f - \sum_{j=1}^{p} f_j$ is holomorphic on $\Omega \setminus \{z_1, \dots, z_p\}$ and can be extended to a holomorphic function on Ω.

By Cauchy's theorem $\displaystyle\int_\gamma F(z)\ dz = 0$. Then

$$\int_\gamma f(z)\ dz = \sum_{j=1}^{p} \int_\gamma f_j(z)\ dz = 2\mathrm{i}\pi \sum_{j=1}^{p} \mathrm{Res}(f, z_j)\mathrm{Ind}(\gamma, z_j).$$

□

The theorem presented in this section is useful to localize the zeros of a holomorphic function and we derive another proof of the fundamental theorem of Algebra, (D'Alembert's theorem).

Theorem (Rouché's Theorem)

Let f and g be two holomorphic functions on a neighborhood of the disc $\{z \in \mathbb{C};\ |z-a| \leq r\}$ and such that $|f(z)-g(z)| < |f(z)|;\ \forall\, z \in \mathscr{C}(a,r) = \{z \in \mathbb{C};\ |z-a| = r\}$, then f and g have the same number of zeros inside the disc $D(a,r)$. (The zeros are counted according to their order of multiplicity.)

Proof
The function $h = \dfrac{g}{f}$ is holomorphic outside the zeros of f and $|1 - h(z)| < 1$ for all $z \in \mathscr{C}(a, r)$ and $\dfrac{h'}{h} = \dfrac{g'}{g} - \dfrac{f'}{f}$. Let γ be the circle centered at a and of radius r and let $\Gamma(t) = h \circ \gamma(t)$, $\Gamma'(t) = \gamma'(t).h'(\gamma(t))$.

$$\begin{aligned}\int_\gamma \frac{h'(w)}{h(w)}\,dw &= \int_0^{2\pi} \frac{h'(a + r\mathrm{e}^{\mathrm{i}t})}{h(a + r\mathrm{e}^{\mathrm{i}t})} ir\mathrm{e}^{\mathrm{i}t}\,dt = \int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)}\,dt \\ &= \int_\Gamma \frac{dw}{w} = 2\mathrm{i}\pi \mathrm{Ind}(\Gamma, 0) = 0,\end{aligned}$$

because 0 is in the unbounded connected component of the complementary of the support of Γ. Thus

$$\frac{1}{2i\pi}\int_{\gamma}\frac{g'(w)}{g(w)}\,dw = \frac{1}{2i\pi}\int_{\gamma}\frac{f'(w)}{f(w)}\,dw.$$

$\frac{1}{2i\pi}\int_{\gamma}\frac{g'(w)}{g(w)}\,dw$ is the number of zeros of g inside the disc $D(a,r)$, and $\frac{1}{2i\pi}\int_{\gamma}\frac{f'(w)}{f(w)}\,dw$ is the number of zeros of f inside the disc $D(a,r)$.

□

Remark 6 :
The Rouché's theorem remains valid if we replace the circle by a closed curve such that any point inside the curve has an index equal to 1.

Corollary (D'Alembert's Theorem (Fundamental Theorem of Algebra))

Let P be a polynomial of degree $n \geq 1$, then P has n zeros in $\mathbb{C}$ counted according to their order of multiplicities.

Proof

If $P(z) = a_n z^n + \ldots + a_0$, then for $|z|$ large enough, $|P(z) - a_n z^n| < |a_n||z^n|$, because $\lim\limits_{|z|\to+\infty} \left|\dfrac{P(z) - a_n z^n}{a_n z^n}\right| = 0$. It results that P has the same number of zeros that the polynomial $Q(z) = a_n z^n$. □

Example

Let f be a holomorphic function on a neighborhood of the disc $\{z \in \mathbb{C};\ |z| \leq 1\}$ and such that $|f(z)| < 1$ for all $|z| = 1$. The equation $f(z) = z^n$ has exactly n solutions inside the unit disc. In particular f has only one fixed point z_0, $(f(z_0) = z_0)$.

Examples

1. We look for the number of zeros of the polynomial $z^4 + 2z^2 + 3$ inside the disc $D(0,2)$.
   Let $f(z) = z^4$ and $g(z) = z^4 + 2z^2 + 3$.
   $|f(z) - g(z)| \leq 11 < |f(z)| = 16$ for $|z| = 2$. Thus by Rouché's theorem, f and g have the same number of zeros inside the disc $D(0,2)$ which is equal to 4.

2. We consider the polynomial $P(z) = z^7 + 5z^4 + z^3 - z + 1$.
The polynomial P has exactly 4 roots inside the unit disc $\mathbb{D}$, indeed the polynomial $P_1(z) = 5z^4$ has 4 roots inside the unit disc $\mathbb{D}$ and $|P(z) - P_1(z)| < |P_1(z)|$ for all $|z| = 1$.
The polynomial P has exactly 3 roots inside the annulus $\{z \in \mathbb{C};\ 1 < |z| < 2\}$, indeed the polynomial $P_2(z) = z^7$ has 7 roots inside the disc $D(0,2)$ and $|P(z) - P_2(z)| < |P_2(z)|$ for all $|z| = 2$.

3. If $1 < a \in \mathbb{R}$, the equation $z + e^{-z} = a$ has only one solution inside the half plane $\{z \in \mathbb{C};\ \mathrm{Re} z \geq 0\}$. Indeed we consider the closed curve defined by $[-\mathrm{i}R, \mathrm{i}R]$ juxtaposed with the semicircle $|z| = R > 0$ inside the half plane $\{z \in \mathbb{C};\ \mathrm{Re} z \geq 0\}$. Set $f(z) = z - a$ and $g(z) = e^{-z}$. If $a > 1$ then, on the $y-$axis, $|z - a| \geq a > |e^{-\mathrm{i}y}| = 1$. On the semicircle, $|e^{-z}| \leq 1$ and $|z - a| \geq ||z| - a| > 1$ if $R > 1 + a$. Thus, if $R > 1 + a$, we have $|f(z)| > |g(z)|$ on the closed curve. Then, by Rouchś's theorem, $z - a$ and $z - a + e^{-z}$ has the same number of zeros inside the closed curve.

4. We consider the function $f(z) = z^m + \dfrac{1}{z^m}$ defined on $\mathbb{C}^*$. We claim to prove that f takes each non real number exactly m times when z is inside the unit disc. i.e. if $a = a_1 + \mathrm{i}a_2$, $a_2 \neq 0$, then the equation $f(z) - a$ has m zeros inside the unit disc.
If $z = e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$, $f(z) = e^{\mathrm{i}m\theta} + e^{-\mathrm{i}m\theta} = 2\cos m\theta$. Thus $g(z) = f(z) - a = 2\cos m\theta - a_1 - \mathrm{i}a_2$ and the argument of $g(z)$ has a total variation 0 when θ varies between 0 and 2π because the image of the unit circle is an interval. Thus, $\Delta\mathrm{Arg}(g(z)) = 0 = Z - P$, where Z is the number of zeros of g inside the unit disc and P is the number of poles inside. But g has m poles inside the unit disc, then $Z - P = Z - m = 0 \Rightarrow Z = m$.

Theorem

[The open mapping Theorem]
Let f be a non constant holomorphic function on a domain $\Omega \ni z_0$ and let k be the order of multiplicity of the root z_0 for the function $f(z) - f(z_0)$. Then there exists an open neighborhood U of z_0, an open neighborhood $V = f(U)$ of $f(z_0)$ such that for all $w \neq f(z_0)$ in V, there exist k distinct points $z_1, \ldots, z_k$ in U such that $f(z_j) = w$, for all $1 \leq j \leq k$.

Corollary

Any non constant holomorphic function on a domain Ω is open.

Corollary

If $f : \Omega \longrightarrow \mathbb{C}$ is an injective holomorphic function, then $f'(z) \neq 0$ for all $z \in \Omega$.

Proof of theorem 6.1

The zeros of $f'(z)$ and $f(z) - f(z_0)$ are isolated, thus there exists $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$ and $f'(z) \neq 0$, $f(z) - f(z_0) \neq 0,\ \forall\ z \in \overline{D(z_0, r)} \setminus \{z_0\}$. Let γ be the circle of center z_0 and radius r. We have

$$\frac{1}{2i\pi} \int_\gamma \frac{f'(z)}{f(z) - f(z_0)}\ dz = \mathrm{Ind}(f \circ \gamma, f(z_0)) = k. \tag{3}$$

Let V be the connected component of $\mathbb{C} \setminus \mathrm{Im} f \circ \gamma$ which contains $f(z_0)$. V is a open subset. Let $U = D(z_0, r) \cap f^{-1}(V)$, then U is open because f is continuous and $z_0 \in U$. Since the mapping $w \longmapsto \mathrm{Ind}(f \circ \gamma, w)$ is constant on the connected component V of $\mathbb{C} \setminus \mathrm{Im} f \circ \gamma$ which contains $f(z_0)$, then by identity (3) $\mathrm{Ind}(f \circ \gamma, w) = k$, $\forall\ w \in V$. Thus $f(z) - w$ has k solutions in $D(z_0, r)$ for all $w \in V$. The solutions are different because $f'(z) \neq 0$ in $\overline{D(z_0, r)} \setminus \{z_0\}$ and we have $f(U) = V$. $\square$

Theorem

(Local inversion Theorem)
Let f be a holomorphic function on a domain Ω. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$. If $f'(z_0) \neq 0$, then there exist an open neighborhood U of z_0 and an open neighborhood V of w_0 such that f is bijective from U into V. The inverse function f^{-1} is holomorphic.

Proof

The existence of U, V, f^{-1} results by theorem 6.1, the function f^{-1} is continuous because f is open. Furthermore f' never vanishes by Corollary 6.3. Thus f^{-1} is holomorphic. □

Corollary

Let f be an injective holomorphic function on an open subset Ω, then $f(\Omega)$ is an open subset of $\mathbb{C}$ and f is an analytic isomorphism from Ω onto $f(\Omega)$.

Remark 7 :
The function $f(z) = e^z$ is non injective on $\mathbb{C}$ and $f'(z) \neq 0$ for all $z \in \mathbb{C}$. This example shows that we can not replace in the above corollary the assumption f injective by $f'(z) \neq 0;\ \forall\ z \in \Omega$.

Remark 8 :

We consider U and V respectively the neighborhood of z_0 and of $w_0 = f(z_0)$ as in theorem 6.1 and assume that $k = 1$ (i.e. $f'(z_0) \neq 0$). By residue theorem, the unique solution $z = g(w)$ of the equation $w = f(z)$ for $w \in V$ is given by:

$$g(w) = \frac{1}{2i\pi} \int_\gamma \frac{zf'(z)}{f(z) - w}\, dz, \tag{4}$$

where γ is the circle $\mathscr{C}(z_0, r)$ of center z_0 and radius r. More generally for any holomorphic function h on Ω, we have

$$h \circ g(w) = \frac{1}{2i\pi} \int_\gamma \frac{h(z)f'(z)}{f(z) - w}\, dz. \tag{5}$$

It follows from the explicit formula of $g(w)$ that g is holomorphic.

Theorem (Mittag-Leffler's Theorem)

Let $(a_n)_n$ be a sequence of complex numbers such that the sequence $(|a_n|)_n$ is increasing and $|a_1| > 0$. If $f : \mathbb{C} \setminus \{a_n;\ n \in \mathbb{N}\} \longrightarrow \mathbb{C}$ is a holomorphic function such that a_n is a simple poles of f, whenever $n \in \mathbb{N}$, (thus $\lim\limits_{n\to+\infty} |a_n| = +\infty$). We assume that there exists a sequence of circles $(C_N)_N$ centered at the origin such that the sequence $(R_N)_N$ of their radius is increasing and $\lim_{N\to\infty} R_N = +\infty$ and the poles of f are not on C_N for all $N \in \mathbb{N}$. We assume also that there exists M such that $|f| < M < +\infty$ on the circles C_N, whenever $N \in \mathbb{N}$. Then

$$f(z) = f(0) + \sum_{n=1}^{+\infty} \mathrm{Res}(f, a_n)\Big[\frac{1}{z - a_n} + \frac{1}{a_n}\Big]. \tag{6}$$

Proof

For $w \in \mathbb{C}$ which is not a pole of f, the function $g(z) = \dfrac{f(z)}{z-w}$ has w and a_j as poles, whenever $j \in \mathbb{N}$. We have

$$\mathrm{Res}(g, a_n) = \lim_{z \to a_n} (z - a_n)\frac{f(z)}{z-w} = \frac{\mathrm{Res}(f, a_n)}{a_n - w}$$

and

$$\mathrm{Res}(g, w) = \lim_{z \to a_n} (z - w)\frac{f(z)}{z-w} = f(w).$$

Then,

$$\frac{1}{2\mathrm{i}\pi}\int_{C_N} \frac{f(z)}{z-w}dz = f(w) + \sum_{|a_n|<R_N} \frac{\mathrm{Res}(f, a_n)}{a_n - w}.$$

We take this formula at 0, we find

$$\frac{1}{2i\pi}\int_{C_N}\frac{f(z)}{z}dz = f(0) + \sum_{|a_n|<R_N}\frac{\mathrm{Res}(f,a_n)}{a_n}.$$

We deduce from the last formulas that

$$\begin{aligned} f(w)-f(0) &= \sum_{|a_n|<R_N}\Big[\frac{(\mathrm{Res} f,a_n)}{a_n}-\frac{\mathrm{Res}(f,a_n)}{a_n-w}\Big]+\frac{1}{2i\pi}\int_{C_N}f(z)(\frac{1}{z-} \\ &= \sum_{|a_n|<R_N}\Big[\frac{(\mathrm{Res} f,a_n)}{a_n}-\frac{\mathrm{Res}(f,a_n)}{a_n-w}\Big]+\frac{w}{2i\pi}\int_{C_N}\frac{f(z)}{z(z-w)} \end{aligned}$$

If $z \in C_N$, $|z-w| \geq |z| - |w| = R_N - |w|$ and

$$\left|\int_{C_N} \frac{f(z)}{z(z-w)}dz\right| \leq \frac{2\pi M R_N}{R_N(R_N - |w|)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Then

$$f(z) = f(0) + \sum_{n=1}^{+\infty} \mathrm{Res} f(a_n)\left[\frac{1}{z-a_n} + \frac{1}{a_n}\right].$$

□

Remark 9 :
The sequence $(C_N)_N$ of circles can be replaced by a sequence of closed simple curves such that $\lim_{N\to\infty} R_N = +\infty$, with $R_N = \inf_{z\in C_N} |z|$.

Example

In use of Mittag-Leffler's theorem, we prove that

$$\tan z = 2z \sum_{n=0}^{+\infty} \frac{1}{\left(\frac{(2n+1)\pi}{2}\right)^2 - z^2}.$$

Indeed, we consider the function $g(z) = \tan z$. The poles of g are $z_k = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ and the correspondent residue is

$$\mathrm{Res}(g, z_k) = \lim_{z \to \frac{\pi}{2} + k\pi} (z - \frac{\pi}{2} - k\pi) \tan z = \lim_{z \to \frac{\pi}{2} + k\pi} \frac{(z - \frac{\pi}{2} - k\pi) \sin z}{\cos z} = -$$

We show that $|g|$ is bounded on all the circles $C_N = \{z \in \mathbb{C};\ |z| = N\pi\}$. Recall that if $z = x + \mathrm{i}y$, then $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

If $|\mathrm{Im}(z)| > 1$, $|\tan z|^2 = \dfrac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y} \leq \mathrm{cotanh}(1)$.

However if $|\mathrm{Im} z| \leq 1$, $x = \mathrm{Re} z$ is in one of the intervals $[-N\pi, -N\pi + 1]$ and $[N\pi - 1, N\pi]$. We remark that

$|\cos z| > |\cos x| \geq \cos(1)$ and consequently $|\tan z|^2 \leq \dfrac{\cosh^2 1}{\cos^2 1}$.

The function $|g(z)|$ is bounded on C_N by a constant independent of N. Then by Mittag-Leffler's theorem

$$\tan z = -\sum_{n=1}^{+\infty}\Big(\frac{1}{z-(n\pi+\frac{\pi}{2})}+\frac{1}{z+(n\pi+\frac{\pi}{2})}\Big) = \sum_{n=1}^{+\infty}\frac{2z}{(n\pi+\frac{\pi}{2})^2-z^2}.$$

Example

In use of Mittag-Leffler's theorem, we prove that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{+\infty} \frac{2(-1)^n z}{z^2 - n^2\pi^2}.$$

The function $f(z) = \dfrac{1}{\sin z} - \dfrac{1}{z}$ has 0 as a removable singularity. Each point $z = k\pi$, $(k \in \mathbb{Z}^*)$ is a simple pole of f because $\lim_{z\to k\pi}(z - k\pi)f(z) = \lim_{z\to k\pi} \dfrac{(z-k\pi)(z-\sin z)}{z\sin z} = (-1)^k$. (We leave to the reader to show that on the sequence of circles $(C_N)_N$ of center 0 and radius respective $R_N = N\pi + \dfrac{\pi}{2}$, f is uniformly bounded.)

Take the sequence $(a_n = n\pi)_{n\in\mathbb{Z}^*}$. By Mittag-Leffler's theorem, We have

$$\begin{aligned} f(z) &= f(0) + \sum_{k=1}^{+\infty}(-1)^k\Big(\frac{1}{z-k\pi} + \frac{1}{k\pi} + \frac{1}{z+k\pi} - \frac{1}{k\pi}\Big) \\ &= \sum_{k=1}^{+\infty}\frac{2(-1)^k z}{z^2-k^2\pi^2}. \end{aligned}$$

where R is a rational function without poles on the unit circle. We take $z = \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$ and $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$.

$$\begin{aligned} I &= \int_\gamma \frac{1}{\mathrm{i}z} R\left(\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\, dz \\ &= 2\pi \sum \mathrm{Res}\left(\frac{1}{z} R\left(\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right). \end{aligned}$$

The summation is extended to the poles of the function $\left(\frac{1}{z} R\left(\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right)$ in the unit disc.

Example

$$I = \int_0^{2\pi} \frac{dt}{a + \sin t}, \quad a > 1.$$

$I = 2\pi \mathrm{Res}(\frac{2\mathrm{i}}{z^2 + 2\mathrm{i}az - 1}, z_0)$, where z_0 the only pole of the function $(\frac{2\mathrm{i}}{z^2 + 2\mathrm{i}az - 1})$ in the unit disc. $z_0 = -\mathrm{i}a + \mathrm{i}\sqrt{a^2 - 1}$.
The residue is $\frac{\mathrm{i}}{z_0 + \mathrm{i}a}$, and thus

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

where P and Q are two polynomials such that $\deg Q \geq \deg P + 2$ and $Q(x) \neq 0$, $\forall x \in \mathbb{R}$.

We consider the function $f(z) = \dfrac{P(z)}{Q(z)}$ and the closed curve γ_R defined by the semicircle of radius R and centered at 0 situated inside the upper half plane $\mathbb{H}^+ = \{z = x + \mathrm{i}y;\ y > 0\}$. Let Γ_R be the oriented closed curve obtained by the juxtaposition of γ_R and the interval $[-R, R]$. (figure 1). We choose R large enough such that the poles of f are situated inside the disc $D(0, R) = \{z \in \mathbb{C}\ ;\ |z| < R\}$.

$$\int_{\Gamma_R} f(z)\ dz = \int_{\gamma_R} f(z)\ dz + \int_{-R}^{R} f(x)\ dx = 2i\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j).$$

The summation is extended to the poles of the function f situated inside the upper half plane $\mathbb{H}^+ = \{z = x + iy;\ y > 0\}$.

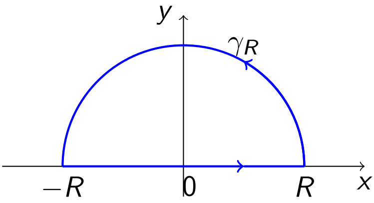


figure 1:

Lemma (First Jordan's Lemma)

Let f be a continuous function defined on a sector $\theta_0 \leq \theta \leq \theta_1$. We assume that

$$\lim_{R \to +\infty} R \sup_{z \in A_R} |f(z)| = 0,$$

then $\lim_{R \to +\infty} \int_{A_R} f(z)\, dz = 0$, *where* A_R *is the curve defined by the arc* $\theta_0 \leq \theta \leq \theta_1$ *and* $|z| = R$.

The lemma results by dominated convergence theorem.

In use of the first Jordan's lemma,

$$\int_{-\infty}^{+\infty} f(x)\; dx = 2\mathrm{i}\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j).$$

Example

$$I = \int_0^{+\infty} \frac{dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{1+x^6}.$$

The poles of f inside the upper half plane $\mathbb{H}^+ = \{z = x + \mathrm{i}y;\ y > 0\}$ are $z_1 = \mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$, $z_2 = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}} = \mathrm{i}$ and $z_3 = \mathrm{e}^{\frac{\mathrm{i}5\pi}{6}}$. Thus $I = \frac{\pi}{3}$.

First case P and Q are two polynomials such that $\deg Q \geq \deg P + 2$, $Q(x) \neq 0$, $\forall x \in \mathbb{R}$ and λ a real number. Let $f(z) = \dfrac{P(z)}{Q(z)}\mathrm{e}^{\mathrm{i}\lambda z}$.

If $\lambda \geq 0$, we integrate the function f on the curve $\gamma_R \cup [-R, R]$, figure 1 and we find

$$\Big|\int_{\gamma_R} f(z)\,dz\Big| \leq \int_0^{\pi} |f(R\mathrm{e}^{\mathrm{i}\theta})|R\,d\theta \underset{R\to+\infty}{\longrightarrow} 0.$$

This yields that $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \mathrm{e}^{\mathrm{i}\lambda x}\, dx = 2\mathrm{i}\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j)$.

If $\lambda \leq 0$, we remark that $I(-\lambda) = \overline{I(\lambda)}$, or we can integrate the function f on the closed curve defined by the juxtaposition of the interval $[-R, R]$ and of the semicircle of radius R and centered at 0, situated inside the lower half plane $\mathcal{H}^- = \{z = x + \mathrm{i}y;\ y < 0\}$, we find, $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \mathrm{e}^{\mathrm{i}\lambda x}\, dx = -2\mathrm{i}\pi \sum_{\mathrm{Im} z < 0} \mathrm{Res}(f, z)$, the summation is extended to the poles of f situated inside the lower half plane $\mathcal{H}^- = \{z = x + \mathrm{i}y;\ y < 0\}$.

Second case $\lambda \in \mathbb{R}^*$, P and Q are two polynomials such that $\deg Q = \deg P + 1$ and $Q(x) \neq 0$, $\forall x \in \mathbb{R}$. We set
$f(z) = \dfrac{P(z)}{Q(z)} \mathrm{e}^{\mathrm{i}\lambda z}$ and $g(z) = \dfrac{P(z)}{Q(z)}$.
The integral is convergent but not absolutely convergent. We can make an integration by parts and we return to the above case. To evaluate the integral, it suffices to evaluate the integral for $\lambda > 0$.

$$\begin{aligned} \left|\int_{\gamma_R} f(z)\, dz\right| &\leq \int_0^{\pi} |g(Re^{i\theta})| Re^{-\lambda R \sin\theta}\, d\theta \leq M \int_0^{\pi} e^{-\lambda R \sin\theta}\, d\theta \\ &\leq 2M \int_0^{\pi/2} e^{-\lambda R \sin\theta}\, d\theta \leq 2M \int_0^{\pi/2} e^{\frac{-2\lambda R\theta}{\pi}}\, d\theta = \frac{2M}{2\lambda R}(\end{aligned}$$

$M = \sup_{R \geq 0} R|g(Re^{i\theta})|$. (We can deduce that $\lim_{R\to+\infty} \int_0^{\pi} e^{-\lambda R \sin\theta}\, d\theta = 0$ by dominated convergence theorem).

Thus for $\lambda > 0$,

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \mathrm{e}^{\mathrm{i}\lambda x}\, dx = 2\mathrm{i}\pi \sum_{\mathrm{Im} z_j > 0} \mathrm{Res}(f, z_j).$$

Example

$a > 0$, $I(\lambda) = \displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{x - \mathrm{i}a}\, dx$.
If $\lambda > 0$, $I(\lambda) = 2\mathrm{i}\pi\mathrm{e}^{-\lambda a}$.
If $\lambda < 0$, $I(\lambda) = 2\mathrm{i}\pi \sum \mathrm{Res}(f, z_j)$, z_j the poles of f inside the lower half plane, but f don't have poles in this half plane, thus $I(\lambda) = 0$.
$I = \displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x}$. We set $f(z) = \dfrac{\mathrm{e}^{\mathrm{i}z}}{z}$. We integrate the function f on the following closed path (figure 2).

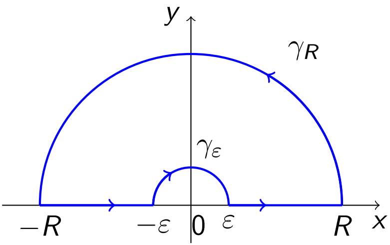


figure 2:

To compute this integral, we need the following lemma

Lemma (Second Jordan's Lemma)

If $f(z) = \dfrac{A}{z} + \sum_{n\geq 0} a_n z^n$, *f defined on a sector* $\theta_0 \leq \theta \leq \theta_1$. *Then*
$$\int_{\gamma_r} f(z)\, dz \underset{r\to 0}{\longrightarrow} i(\theta_1 - \theta_0)A.$$

Proof

$$\int_{\gamma_r} f(z)\,dz = \int_{\theta_0}^{\theta_1} f(re^{i\theta}) ire^{i\theta}\,d\theta = iA\int_{\theta_0}^{\theta_1} d\theta + i\int_{\theta_0}^{\theta_1} g(re^{i\theta}) ire^{i\theta}\,d\theta,$$

g is a holomorphic function, thus $\lim\limits_{r\longrightarrow 0}\int_{\theta_0}^{\theta_1} g(re^{i\theta}) ire^{i\theta}\,d\theta = 0.$

□

We come back to the computation of the following integral $I = \int_{-\infty}^{+\infty} \frac{\sin x}{x} dx$. By residue theorem,

$$\int_{-R}^{-r} f(x)\ dx - \int_{\gamma_r} f(z)\ dz + \int_{r}^{R} f(x)\ dx + \int_{\gamma_R} f(z)\ dz = 0.$$

$$|\int_{\gamma_R} f(z)\ dz| = |\int_0^{\pi} \mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\ d\theta| \leq \int_0^{\pi} \mathrm{e}^{-R\sin\theta}\ d\theta \underset{R\to+\infty}{\longrightarrow} 0.$$

By second Jordan's lemma $\int_{\gamma_r} f(z)\ dz \underset{r\to 0}{\longrightarrow} \mathrm{i}\pi$, thus $I = \pi$.

Example

$I = 2\displaystyle\int_{-\infty}^{+\infty} \frac{x \sin ax \cos bx}{x^2 + c^2} dx$, with $a, b \in \mathbb{R}$ and $c > 0$.
We has the following identity
$2 \sin ax \cos bx = \sin(a + b)x + \sin(a - b)x$. Thus
$I = \mathrm{Im}(I_1) + \mathrm{Im}(I_2)$, with

$$I_1 = \int_{-\infty}^{+\infty} \frac{xe^{\mathrm{i}(a-b)x}}{x^2 + c^2} dx, \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{xe^{\mathrm{i}(a+b)x}}{x^2 + c^2} dx.$$

We remark that if $a = b$ or $a = -b$, the computation of I turns to the computation of I_1 or I_2. We assume that $a \neq b$ and $a \neq -b$.
$I_1 = \mathrm{i}\pi e^{-(a-b)c}$ if $a > b$ and $I_1 = -\mathrm{i}\pi e^{(a-b)c}$ if $a < b$.
Furthermore $I_2 = \mathrm{i}\pi e^{-(a+b)c}$ if $a > -b$ and $I_2 = -\mathrm{i}\pi e^{(a+b)c}$ if $a < -b$.

Thus

$$I = \pi \left(\mathrm{sign}(a-b)e^{-|a-b|c} + \mathrm{sign}(a+b)e^{-|a+b|c} \right).$$

($\mathrm{sign}(x) = 1$, if $x > 0$ and $\mathrm{sign}(x) = -1$, if $x < 0$.)

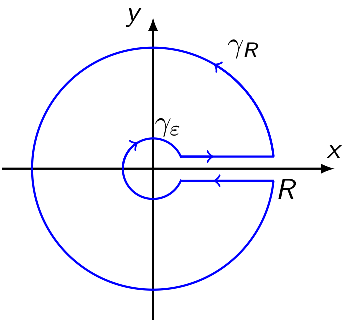
Example

We deduce from the above example that the Fourier Plancherel transform of the function $f(x) = \dfrac{x}{x^2 + c^2}$ is the function

$$g(x) = \int_{-\infty}^{\infty} f(t)e^{-2\mathrm{i}\pi xt}dt = -\mathrm{i}\pi \mathrm{sign}(x)e^{-2\pi|x|c}, \quad \forall\ x \neq 0.$$

The function f is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$. The same for its Fourier Plancherel transform g.

where $Q(x) \neq 0,\ \forall\ x \geq 0$, $\deg Q - \deg P \geq 2$. We consider the closed following curve and $f(z) = \dfrac{P(z)}{Q(z)}(\log z)^2$. ($\log z$ is the determination (branch) of $\log z$ such that $\log z = \ln|z| + i\theta$, $0 < \theta < 2\pi$.)



$$\int_r^R \frac{P(x)}{Q(x)}(\ln x)^2 \, dx \;+\; \int_{\gamma_R} f(z)\, dz + \int_R^r \frac{P(x)}{Q(x)}(\ln x \; + 2\mathrm{i}\pi)^2 \, dx + \int_{\gamma_r} f(z)\, dz = 2\mathrm{i}\pi \sum \mathrm{Res}(f).$$

The summation is extended to the poles of the function f in $\mathbb{C}$.

According to the hypotheses on f, $\displaystyle\int_{\gamma_R} f(z)\, dz \underset{R\to+\infty}{\longrightarrow} 0$ and $\displaystyle\int_{\gamma_r} f(z)\, dz \underset{r\to 0}{\longrightarrow} 0$, thus

$$2\mathrm{i}\pi \sum_{z\in\mathbb{C}} \mathrm{Res}(f, z) = 4\pi^2 \int_0^{+\infty} \frac{P(x)}{Q(x)}\, dx - 4\mathrm{i}\pi \int_0^{+\infty} \ln x \frac{P(x)}{Q(x)}\, dx.$$

Example

$$I = \int_0^{+\infty} \frac{\ln x}{(x+1)(x^2+1)} dx.$$

$\mathrm{Res}(f, \mathrm{i}) = \frac{\pi^2(1+\mathrm{i})}{16}$, $\mathrm{Res}(f, -\mathrm{i}) = \frac{9\pi^2(1-\mathrm{i})}{16}$, $\mathrm{Res}(f, -1) = \frac{-\pi^2}{2}$.
Thus $I = \frac{-\pi^2}{16}$.

Integrals of Type $I(\alpha) = \displaystyle\int_0^{+\infty} \frac{P(x)}{Q(x)} x^{\alpha-1}\, dx$,
with $Q(x) \neq 0 \;\forall\, x \geq 0$, $0 < \alpha < \deg Q - \deg P$. We set
$f(z) = \dfrac{P(z)}{Q(z)} z^{\alpha-1}$, with $z^{\alpha-1} = \mathrm{e}^{(\alpha-1)\log z}$, $\log z$ is the determination (branch) of $\log z$ such that $\log z = \ln|z| + \mathrm{i}\theta$, $0 < \theta < 2\pi$. We take the closed curve defined by the figure (3). For R large enough and r small enough,

$$\int_r^R \frac{P(x)}{Q(x)} x^{\alpha-1}\, dx \quad + \quad \int_{\gamma_R} f(z)\, dz + \int_R^r \frac{P(x)}{Q(x)} \mathrm{e}^{2\mathrm{i}\pi(\alpha-1)} x^{\alpha-1}\, dx$$
$$+ \int_{\gamma_r} f(z)\, dz = 2\mathrm{i}\pi \sum_{z\in\mathbb{C}} \mathrm{Res}(f,z).$$

The summation is extended to the poles of the function f in $\mathbb{C}$.
According to the assumption on f, $\displaystyle\int_{\gamma_R} f(z)\, dz \underset{R\to+\infty}{\longrightarrow} 0$ and
$\displaystyle\int_{\gamma_r} f(z)\, dz \underset{r\to 0}{\longrightarrow} 0.$
Then $(1-\mathrm{e}^{2\mathrm{i}\pi\alpha}) I(\alpha) = 2\mathrm{i}\pi \sum_{z\in\mathbb{C}} \mathrm{Res}(f,z)$.

Example

$$I(\alpha) = \int_0^{+\infty} \frac{x^{\alpha-1}}{x+1}\,dx \quad \text{with } 0 < \alpha < 1.$$

$\mathrm{Res}(f, -1) = -\mathrm{e}^{\mathrm{i}\pi\alpha}$, thus $I(\alpha) = \dfrac{\pi}{\sin \pi\alpha}$.