The Vector Spaces

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Introduction to Vector Spaces

Definition

We say that a non empty set $\mathbb{E}$ is a vector space on $\mathbb{R}$ if:

1. (Closure for the sum operation) $u + v \in \mathbb{E}, \quad \forall u, v \in \mathbb{E}$.
2. (Associativity of the sum operation) $u + (v + w) = (u + v) + w$, for all $u, v, w \in \mathbb{E}$
3. (The identity element) There is $0 \in \mathbb{E}$ called the identity element of the sum operation such that $u + 0 = 0 + u = u, \quad \forall u \in \mathbb{E}$.
4. For all $u \in \mathbb{E}$, there is $v \in \mathbb{E}$ such that $u + v = v + u = 0$. The vector v is called the symmetric of u and written $-u$.
5. (Commutativity) $u + v = v + u, \qquad \forall u, v \in \mathbb{E}$.

1. (The closure of the exterior operation) $\forall a \in \mathbb{R}$ and $u \in \mathbb{E}$, $au \in \mathbb{E}$,
2. If $u, v \in \mathbb{E}$ and $a \in \mathbb{R}$, then $a(u + v) = au + av$.
3. If $u \in \mathbb{E}$ and $a, b \in \mathbb{R}$, then $(a + b)u = au + bu$,
4. If $u \in \mathbb{E}$ and $a, b \in \mathbb{R}$, then $(a.b)u = a(bu)$,
5. If $u \in \mathbb{E}$, then $1.u = u$.

Examples

1. $\mathbb{R}^n$ is a vector space .
2. The set $\{(x, y, 2x + 3y);\ x, y \in \mathbb{R}\}$ is a vector space .
3. The set of polynomials $\mathscr{P} = \mathbb{R}[X]$ is a vector space .
   Also the set of polynomials of degree less then n,
   $\mathscr{P}_n = \mathbb{R}_n[X]$ is a vector space .

The Vector Sub-Spaces

Definition

Let V be a vector space and F a subset of V. We say that F is a sub-space of V if F is vector space with the same operations of the vector space V.

Theorem

Let V be a vector space and F a subset of V.
F is a sub-space of V if and only if

1. $0 \in F$,
2. If $u, v \in F$, then $u + v \in F$,
3. If $u \in F$, $a \in \mathbb{R}$, then $au \in F$.

Examples

1. The set $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix};\ a, b \in \mathbb{R}\}$ is a sub-space of $V = \mathscr{M}_2(\mathbb{R})$.
2. Let $A \in \mathscr{M}_{m,n}(\mathbb{R})$ be a matrix and $F = \{X \in \mathbb{R}^n;\ AX = 0\}$. F is sub-space of $V = \mathbb{R}^n$. (F is the set of solutions of the homogeneous system $AX = 0$).
3. The set $F = \{(x, x+1);\ x \in \mathbb{R}\}$ is not a sub-space of $\mathbb{R}^2$ since $(0, 0) \notin F$.

Example

The set $W = \{A \in \mathscr{M}_n / \ A = -A^T\}$ is a sub-space of $\mathscr{M}_n(\mathbb{R})$.
Indeed: if $A, B \in W$ and $\lambda \in \mathbb{R}$

$$(A + B)^T = A^T + B^T = -A - B$$

and

$$(\lambda A)^T = \lambda A^T = -\lambda A.$$

Then W is a sub-space of $\mathscr{M}_n$.

Example

The set $E = \{(x, y) \in \mathbb{R}^2;\ xy = 0\}$ is not a sub-space since $(1, 0) \in E$ and $(0, 1) \in E$ but $(1, 0) + (0, 1) = (1, 1) \notin E$.

Definition

Let V be a vector space and let $v_1, \ldots, v_n$ be a finite vectors in V. We say that a vector $w \in V$ is a linear combination of the vectors $v_1, \ldots, v_n$ if there is $x_1, \ldots, x_n \in \mathbb{R}$ such that

$$w = x_1 v_1 + \ldots + x_n v_n.$$

Example

The vector $(4, 1, 1)$ is a linear combination of the vectors $(1, 0, 2)$,$(2, -1, 3)$, $(0, -1, 1)$ because

$$(4, 1, 1) = -2(1, 0, 2) + 3(2, -1, 3) - 4(0, -1, 1).$$

Example

The vector $(1, 1, 2)$ is not a linear combination of the vectors $(1, 0, 2)$, $(0, -1, 1)$ because the linear system $(1, 1, 2) = x(1, 0, 2) + y(0, -1, 1)$ don't have a solution.

Example

In $\mathbb{R}^4$ is the vectors $(a, 1, b, 1)$ and $(a, 1, 1, b)$ are linear combination of the vectors $e_1 = (1, 2, 3, 4)$ and $e_2 = (1, -2, 3, -4)$.
The vector $(a, 1, b, 1) \in \mathrm{Vect}(e_1, e_2)$ if and only if the linear system $AX = B$ is consistent with $A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} a \\ 1 \\ b \\ 1 \end{pmatrix}$.
The system is not consistent because the second and the forth equations can not be true in the same time. $((2a - 2b = 1,\ 4a - 4b = 1))$

The vector $(a, 1, 1, b) \in \mathrm{Vect}(e_1, e_2)$ if and only if the linear system $AX = B$ is consistent with $A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 3 & 3 \\ 4 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} a \\ 1 \\ 1 \\ b \end{pmatrix}$.

The system has a unique solution and in this case $a = \dfrac{1}{3}$ and $b = 2$.

Example

Let E be the vector sub-space of $\mathbb{R}^3$ generated by the vectors $(2, 3, -1)$ and $(1, -1, -2)$ and let F be the sub-space of $\mathbb{R}^3$ generated by the vectors $(3, 7, 0)$ and $(5, 0, -7)$.
The sub-spaces E and F are equal.

$$\begin{cases} 2x + y = a \\ 3x - y = b \\ -x - 2y = c \end{cases}$$

This system is equivalent with the following system

$$\begin{cases} x + 2y = -c \\ -3y = a + 2c \\ -7y = b + 3c \end{cases}.$$

This system is consistent if and only if $7a - 3b + 5c = 0$.

We remark that the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are solutions of the system, then $F \subset E$.
With the same method, the vectors $(2, 3, -1)$ and $(1, -1, -2)$ are in the sub-space F. This proves that $E = F$.

Example

Is there $a, b \in \mathbb{R}$ such that the vector $v = (-2, a, b, 5)$ is in the sub-space of $\mathbb{R}^4$ generated by the vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$.

Solution

The vector $v = (-2, a, b, 5)$ is in the sub-space of $\mathbb{R}^4$ generated by the vectors $u = (1, -1, 1, 2)$ and $v = (-1, 2, 3, 1)$ if the following linear system is consistent $AX = B$ with $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 3 \\ 2 & 1 \end{pmatrix}$ and

$$B = \begin{pmatrix} -2 \\ a \\ b \\ 5 \end{pmatrix}.$$

This system is consistent if and only if $3 = a - 2 = \frac{b+2}{4}$.
Then $a = 5$ and $b = 10$.

Theorem

Let A be the matrix of type (m, n) and let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be the matrix of type $(n, 1)$. If $C_1, \ldots, C_n$ are the columns of the matrix A, then

$$AX = x_1 C_1 + \ldots + x_n C_n.$$

Corollary

Let A be a matrix of type (m, n).
The linear system $AX = B$ is consistent if and only if the matrix B is a linear combination of the columns of the matrix A.

Definition

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.
We say that the vector space V is generated (or spanned) by the set S if any vector in V is a linear combination of the vectors $v_1, \ldots, v_n$. (We say also that S is a spanning set of V).

Theorem

Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ and A the matrix of type (m, n) with columns $v_1, \ldots, v_n$.
The set S spans the vector space $\mathbb{R}^m$ if and only if the system $AX = B$ is consistent for all $B \in \mathbb{R}^m$.

Example

Determine whether the vectors $v_1 = (1, -1, 4)$, $v_2 = (-2, 1, 3)$, and $v_3 = (4, -3, 5)$ span $\mathbb{R}^3$.
We solve the following linear system $AX = B$, where

$$A = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{pmatrix},\ B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ for arbitrary $a, b, c \in \mathbb{R}$.

A reduced of the augmented matrix is given by:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -a-2b \\ 0 & 1 & -1 & -a-b \\ 0 & 0 & 0 & 7a+11b+c \end{array}\right].$$

This system has a solution only when $7a + 11b + c = 0$. Thus, the vectors do not span $\mathbb{R}^3$.

Example

Determine whether the vectors $v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, span the vector space $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix};\ a, b \in \mathbb{R}\}$.

$$\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix} = xv_1 + yv_2 \iff \begin{cases} x+2y &= a \\ x+y &= b \\ x+3y &= 2a-b \end{cases}.$$

This system has the unique solution $x = 2b - a$ and $y = a - b$.

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V, then

1. the set W of linear combinations of the vectors of S is a linear sub-space in V.
2. W is the smallest sub-space of V which contains S.
   This sub-space is called the sub-space generated (or spanned) by the set S and denoted by $\langle S \rangle$ or $\mathrm{Vect}(S)$.

Example

Let $F = \{\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix};\ a, b \in \mathbb{R}\}$.

$\begin{pmatrix} a & b \\ 0 & 2a-b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. Then F is the sub-space of $V = \mathscr{M}_2(\mathbb{R})$ spanned by $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right\}$.

Definition

We say that the set of vectors $v_1, \ldots, v_n$ in a vector space V are linearly independent if the equation

$$x_1 v_1 + \ldots, + x_n v_n = 0$$

has 0 as unique solution.

Example

The vectors $u = (1, 1, -2)$, $v = (1, -1, 2)$ and $w = (3, 0, 2)$ are linearly independent in $\mathbb{R}^3$.

$$xu + yv + zw = (0, 0, 0) \iff \begin{cases} x + y + 3z &= 0 \\ x - y &= 0 \\ -2x + 2y + 2z &= 0 \end{cases}.$$

This system has 0 as unique solution.

The matrix of this system is $\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ -2 & 2 & 2 \end{pmatrix}$ and its determinant is -4.

Example

The set of vectors $\{P_1 = 1+x+x^2,\ P_2 = 2-x+3x^2, P_3 = x-x^2\}$ is linearly independent in $\mathscr{P}_2$.

$aP_1+bP_2+cP_3 = 0 \iff (a+2b)+(a-b+c)x+(a+3b-c)x^2 = 0 \iff \begin{cases} a+2b & = & 0 \\ a-b+c & = & 0 \\ a+3b-c & = & 0 \end{cases}.$

Definition

We say that the vectors $v_1, \ldots, v_n$ in a vector space V are linearly dependent if they are not linearly independent.

Example

The vectors $u = (0, 1, -2, 1)$, $v = (1, 0, 2, -1)$ and $w = (3, 2, 2, -1)$ are linearly dependent in $\mathbb{R}^4$.

$$xu + yv + zw = (0,0,0,0) \iff \begin{cases} y + 3z & = & 0 \\ x + 2z & = & 0 \\ -2x + 2y + 2z & = & 0 \\ x - y - z & = & 0 \end{cases}.$$

This system has infinite solutions.

The extended matrix of this system is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ -2 & 2 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$ and the reduced row form of this matrix is : $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V, with $n \geq 2$.
The set S is linearly dependent if and only if there is a vector of S which is a linear combination of the rest of vectors.

Theorem

Let $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ and A the matrix of type (m, n) such that its columns are the vectors of S.
The set S is linearly independent if and only if the homogeneous system $AX = 0$ has 0 as unique solution.

Examples

1. If A is a matrix of type (m, n) with $m < n$. Then the homogeneous system $AX = 0$ has an infinite solutions.
2. If $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$ with $m < n$, then the set S is linearly dependent.

Base and Dimension

Definition

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors in a vector space V.
We say that S is a basis of the vector space V if :

1. The set S generates the vector space V
2. The set S is linearly independent.

Theorem

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V.
Any vector $v \in V$ can be written uniquely as a linear combination of vectors in the basis S.

Remark

Let $S = \{e_1, \dots, e_n\}$ be the set of the vectors in the vector space $\mathbb{R}^n$, where

$$e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1, 0, \dots 0), \dots, e_n = (0, \dots, 0, 1).$$

The set S is a basis of $\mathbb{R}^n$ and is called the natural basis of $\mathbb{R}^n$.

Exercise

Prove that $S = \{1, X, \dots, X^n\}$ is a basis of the vector space $\mathscr{P}_n$.

Example

Let $v_1 = (\lambda, 1, 1)$, $v_2 = (1, \lambda, 1)$ and $v_3 = (1, 1, \lambda)$.
Find the values of $\lambda \in \mathbb{R}$ such that $\{v_1, v_2, v_3\}$ is a basis of the vector space $\mathbb{R}^3$.

Solution

The set $\{v_1, v_2, v_3\}$ is linearly independent if 0 the unique solution of the equation

$$xv_1 + yv_2 + zv_3 = 0.$$

This is equivalent that the following matrix has an inverse :

$$A = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix}.$$

Then $\lambda \notin \{-2, 1\}$.

The set $\{v_1, v_2, v_3\}$ generates the vector space $\mathbb{R}^n$ because the linear system $AX = B$ is consistent for all $B \in \mathbb{R}^n$ since the matrix A has an inverse .

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be a basis of the vector space V and let $T = \{u_1, \ldots, u_m\}$ be a set of vectors.
If $m > n$, then T is linearly dependent .

Corollary

If $S = \{v_1, \ldots, v_n\}$ and $T = \{u_1, \ldots, u_m\}$ are basis of the vector space V, then $m = n$.

Definition

If $S = \{v_1, \ldots, v_n\}$ is a basis of the vector space V then the number of vectors n of S is called the dimension of the vector space V and denoted by: $\dim V = n$.

Theorem

Let V is a vector space of dimension n. If $S = \{v_1, \ldots, v_n\}$ in V. Then
S is linearly independent if and only if S generates the vector space V and this is equivalent also with S is a basis of V.

Theorem

If $S = \{v_1, \ldots, v_n\}$ generates the vector space V, then it contains a basis of the vector space V.

Remark

If $S = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is a set of vectors and F the vector sub-space generated by S. We have the following two algorithms to construct a basis of F.

First Algorithm

1. Construct the matrix A such that its rows are the vectors of S
2. The non zeros rows of any row echelon form of the matrix A are a basis of the vector space $F = \langle S \rangle$.

Second Algorithm

1. Construct the matrix A such that its columns are the vectors of S
2. Take any row echelon form C of the matrix A.
3. Let $C_{k_1}, \dots C_{k_p}$ be the columns which contain a leading number and $k_1 < \dots < k_p$. Then $\{v_{k_1}, \dots, v_{k_p}\}$ is a basis of the vector space $F = \langle S \rangle$.

Theorem

1. If $S = \{v_1, \ldots, v_n\}$ is a set of vectors and generates the vector space V, then S contains a basis of the vector space V.
2. If $S = \{v_1, \ldots, v_n\}$ is a set of linearly independent vectors in the vector space V, then there is a basis T of V which contains the set S.

Example

Let W be the sub-space of $\mathbb{R}^5$ generated by the set of following vectors:
$v_1 = (1, 0, 2, -1, 2)$, $v_2 = (2, 0, 4, -2, 4)$, $v_3 = (1, 2, -1, 2, 0)$, $v_4 = (1, 4, -4, 5, -2)$.

1. Find a basis of the sub-space W in $\{v_1, v_2, v_3, v_4\}$.
2. Find a basis of $\mathbb{R}^5$ and contains $\{v_1, v_3\}$.

Solution

1. Let matrix $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 2 & 4 & -1 & -4 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$ with columns the components of the vectors v_1, v_2, v_3, v_4.

   The reduced row form the matrix A is $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   Then $\{v_1, v_3\}$ is basis of the sub-space W.

2. If $e_1 = (1, 0, 0, 0, 0)$, $e_2 = (0, 1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0, 0)$. Then $\{v_1, v_3, e_1, e_2, e_3\}$ is basis of $\mathbb{R}^5$ and contains $\{v_1, v_3\}$.

Example

Let $W = \{(x, y, z, t) \in \mathbb{R}^4;\ 2x + y + z = 0,\ x - y + z = 0\}$

1. Prove that W is sub-space of $\mathbb{R}^4$
2. Find basis of the sub-space W.

Solution

1. $u = (x, y, z, t) \in W \iff AX = 0$, where
$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$
Since the set of solutions of an homogeneous linear system is a vector sub-space, then W is vector sub-space of $\mathbb{R}^4$.

2. $AX = 0 \iff \begin{cases} 2x + y + z = 0 \\ x - y + z = 0 \end{cases} \iff \begin{cases} x = -2y \\ z = 3y \end{cases}$

$$\iff X = y \begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\{(-2, 1, 3, 0), (0, 0, 0, 1)\}$ is basis of the vector sub-space W.

Example

In the vector space $V = \mathbb{R}^3$, give a set S of vectors in V such that S generates the vector space V and not linearly independent.

Solution

We can take
$S = \{(1,0,0)\}$ and $T = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$.

Coordinate System and Change of Basis

Definition

If $S = \{v_1, \dots, v_n\}$ is a basis of the vector space V and if $v \in V$ such that

$$v = x_1 v_1 + \dots x_n v_n$$

then $(x_1, \dots x_n)$ are called the system of coordinates of the vector v with respect to the basis S. We denote

$$[v]_S = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and called the vector of coordinates of the vector v with respect to the basis S.

Theorem

If $B = \{v_1, \ldots, v_n\}$ and $C = \{u_1, \ldots, u_n\}$ are two basis of the vector space V. We define the matrix ${}_CP_B$ of type n such that its columns are $[v_1]_C, \ldots, [v_n]_C$. This matrix ${}_CP_B$ has an inverse and

$$[v]_C = {}_CP_B[v]_B$$

for all $v \in V$.
The matrix ${}_CP_B$ is called the change of basis matrix from the basis B to the basis C.

Exercise

Let $B = \{v_1 = (0, 1, 1), v_2 = (1, 0, -2), v_3 = (1, 1, 0)\}$ be a basis of the vector space $\mathbb{R}^3$ and let $C = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ be the standard basis of the vector space $\mathbb{R}^3$.

1. Find the following matrix ${}_C P_B$ and ${}_B P_C$.
2. Find $[v]_B$ if $[v]_C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Exercise

1. ${}_CP_B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ ${}_BP_C = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$.
2. $[v]_B = {}_BP_C[v]_C = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Example

Prove that in $\mathbb{R}^3$, the vectors $u = (1, 0, 1)$, $v = (-1, -1, 2)$ and $w = (-2, 1, -2)$ form a basis and find the coordinate system of the vector $X = (x, y, z)$ in this basis.

Solution

The matrix which columns the vectors $u = (1, 0, 1)$, $v = (-1, -1, 2)$ and $w = (-2, 1, -2)$ is $A = \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}$.

Since $|A| = -3$, then $u = (1, 0, 1)$, $v = (-1, -1, 2)$ and $w = (-2, 1, -2)$ is a basis of the vector space $\mathbb{R}^3$.

If $X = au + bv + cw$ then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^{-1}X = \begin{pmatrix} 2y + z \\ \frac{-x+z}{3} \\ \frac{-x+3y+z}{3} \end{pmatrix}$.

Example

Prove that the system of vectors $S = \{(1,1,1),(-1,1,0),(1,0,-1)\}$ is a basis of the vector space $\mathbb{R}^3$.
Find the coordinates of the following vectors $(1,0,0)$, $(1,0,1)$ and $(0,0,1)$ in this basis.
Solution:

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3 \neq 0.$$

Then S is a basis of the vector space $\mathbb{R}^3$.
$(1,0,0) = \frac{1}{3}(1,1,1) - \frac{1}{3}(-1,1,0) + \frac{1}{3}(1,0,-1)$.
Then coordinates in the basis S is $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$.

Solution

$(0,0,1) = \frac{1}{3}(1,1,1) - \frac{1}{3}(-1,1,0) - \frac{2}{3}(1,0,-1)$.
Then coordinates in the basis S is $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$.
$(1,0,1) = (1,0,0) + (0,0,1)$.
Then coordinates in the basis S is $(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$.

Definition

Let A be a matrix of type (m, n).
The vector sub-space of $\mathbb{R}^n$ spanned by the rows of the matrix A is called the row vector space of the matrix A and denoted by: $\mathrm{row}(A)$.
The vector sub-space of $\mathbb{R}^m$ spanned by the columns of the matrix A is called the column vector space of the matrix A and denoted by: $\mathrm{col}(A)$.

Theorem

Let A be a matrix of type (m, n). If B is any matrix which is a result of some row operations on the matrix A, then $\mathrm{row}(A) = \mathrm{row}(B)$.

Theorem

Let A be a matrix of type (m, n) and if B any row echelon form of the matrix A. Then the set of non zero rows of the matrix B are linearly independent.

Definition

Let A be a matrix of type (m, n).
The dimension of the vector space $\mathrm{row}(A)$ is called the rank of the A.
$\mathrm{rank}(A) = \dim(\mathrm{row}(A))$.

Remark

Let A be a matrix of type (m, n).
The rank of the matrix A is the numbers of leading numbers in any row echelon form of the matrix A.

Theorem

Let A be a matrix of type (m, n), then

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)).$$

Corollary

Let A be a matrix of type (m, n), then

$$\mathrm{rank}(A) = \mathrm{rank}(A^T).$$

Corollary

If A is a matrix of type (m, n) and P is any invertible matrix of type m and Q an invertible matrix of type n, then

$$\operatorname{rank}(A) = \operatorname{rank}(PAQ).$$

Proof

There $E_1, \ldots, E_p$ elementary matrix of order m such that $P = E_1 \ldots E_p$.
We know that if E is a elementary matrix which corresponds to an elementary row operation R, then EA is the result of the elementary row operation R on the matrix A. Then

$$\mathrm{rank}(A) = \mathrm{rank}(PA).$$

Also $\mathrm{rank}(PAQ) = \mathrm{rank}(PAQ)^T = \mathrm{rank}(Q^T A^T P^T) = \mathrm{rank}(A^T P^T) = \mathrm{rank}(PA) = \mathrm{rank}(A)$.

Theorem

If A is a matrix of type (m, n). We have the equivalence of the following statements:

1. The homogeneous system $AX = 0$ has 0 as unique solution.
2. The columns of the matrix A are linearly independent .
3. $\mathrm{rank}(A) = n$.
4. The matrix $A^T A$ has an inverse.

Theorem

Let A be a matrix of type (m, n). We have the equivalence of the following statements

1. The system $AX = B$ is consistent for all $B \in \mathbb{R}^m$.
2. The columns of the matrix A generates the vector space $\mathbb{R}^m$.
3. $\mathrm{rank}(A) = m$.
4. The matrix AA^T has an inverse.

Definition

Let A be a matrix of type (m, n). The vector sub-space

$$\{X \in \mathbb{R}^n;\ AX = 0\}$$

is called the nullspace of the matrix A and denoted by: $N(A)$. Its dimension is denoted by $\mathrm{nullity}(A)$.
Also the vector sub-space

$$\{AX;\ X \in \mathbb{R}^n\}$$

is called the image of the matrix A and denoted by: $\mathrm{Im}(A)$.

Theorem

Let A be a matrix of type (m, n). Then $\mathrm{Im}(A) = \mathrm{col}(A)$.

Rank-Nullity Theorem

For any matrix A of type (m, n),

$$\mathrm{nullity}(A) + \mathrm{rank}(A) = n.$$

Example

Let the matrix $A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

1. Find a basis of the vector space $N(A)$.
2. Find a basis of the vector space $\mathrm{Col}(A)$.
3. Find the rank of the matrix A.

Solution

The reduced row form the matrix A is $\begin{pmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

1. $(-3, 2, 1, 0)$, $(-5, 3, 0, 1)$ is basis of the vector space $N(A)$ ..

2. $(0, 1, 2, 1)$, $(-1, 2, 3, 1)$ is a basis of the vector space $\mathrm{Col}(A)$.
3. The rank of the matrix A is 2.

Example

Let $e_1 = (0, 1, -2, 1)$, $e_2 = (1, 0, 2, -1)$, $e_3 = (3, 2, 2, -1)$, $e_4 = (0, 0, 1, 0)$ and $e_5 = (0, 0, 0, 1)$ vectors in $\mathbb{R}^4$.
Is the following statements are true?

1. $\mathrm{Vect}\{e_1, e_2, e_3\} = \mathrm{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}$.
2. $(1, 1, 0, 0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
3. $\mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} = \mathbb{R}^4$.

Solution

1. Let the matrix A which rows are the vectors e_1, e_2, e_3.
   The vector space $\mathrm{Vect}\{e_1, e_2, e_3\}$ is the row vector space of the matrix A.
   The reduced row form of the matrix A is
   $$A_1 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
   Then $\dim \mathrm{Vect}\{e_1, e_2, e_3\} = 2$.
   We have $\mathrm{Vect}\{e_1, e_2, e_3\} = \mathrm{Vect}\{(1,1,0,0), (-1,1,-4,2)\}$ if and only if the rank of the following matrix B is 2

$$B = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ -1 & 1 & -4 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The reduced row form of the matrix B is $A_2 = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then
$\mathrm{Vect}\{e_1, e_2, e_3\} = \mathrm{Vect}\{(1, 1, 0, 0), (-1, 1, -4, 2)\}$.

2. $(1,1,0,0) = e_1 + e_2$, $2(1,1,0,0) = e_3 - e_2$.
   Then $(1,1,0,0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
3. $(1,1,0,0) \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$ and $e_2 \in \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\}$.
   Then $\dim \mathrm{Vect}\{e_1, e_2\} \cap \mathrm{Vect}\{e_2, e_3, e_4\} = 2$ and

$$\dim \mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} \leq 3$$

Then $\mathrm{Vect}\{e_1, e_2\} + \mathrm{Vect}\{e_2, e_3, e_4\} \neq \mathbb{R}^4$.

Example

Let in $\mathbb{R}^3$ the vectors, $u_1 = (1, 2, 1)$, $u_2 = (1, 3, 2)$, $u_3 = (1, 1, 0)$ and $u_4 = (3, 8, 5)$.
Let $F = \mathrm{Vect}(u_1, u_2)$ and $G = \mathrm{Vect}(u_3, u_4)$.
Prove that $F = G$.

Solution

As the vectors u_1, u_2 are linearly independent and also the vectors u_3, u_4 are linearly independent, then
$\dim E = \dim F = 2$.
$F = G$ if and only if the rank of the following matrix is 2, $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \\ 3 & 8 & 5 \end{pmatrix}$.

The reduced row form of this matrix is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $F = G$.

Consider the matrix $A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix}$.

1. Find a basis of $\operatorname{Ker} T$ and a basis of $\operatorname{Im}(T)$.
2. Prove that $\mathbb{R}^4 = \operatorname{Im}(A) \oplus N(A)$.
3. Prove that A and A^2 have the same rank.

The reduced row echelon form of the augmented matrix of the system $AX = 0$ is

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. $N(A) = \{(-z - 2t, -z - 3t, z, t) : z, t \in \mathbb{R}\}$ and $\{(1, 1, -1, 0), (-2, -3, 0, 1)\}$ is a basis of $N(A)$
2. $\{(1, -1, 0, 1), (-1, 1, 1, 0)\}$ is a basis of $\mathrm{Im}(A)$
3. As $\{(1, 1, -1, 0), (-2, -3, 0, 1), (1, -1, 0, 1), (-1, 1, 1, 0)\}$ is a basis of $\mathbb{R}^4$, then $\mathbb{R}^4 = \mathrm{Im}(A) \oplus N(A)$.
4. $$\begin{aligned} X \in N(A^2) &\iff A^2X = 0 \iff AX \in N(A) \\ &\iff A(X) \in N(A) \cap \mathrm{Im}(A) \iff AX = 0. \end{aligned}$$

Then the matrices A and A^2 have the same rank.