Applications of Definite Integrals

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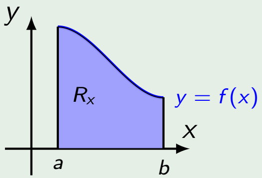
Definition

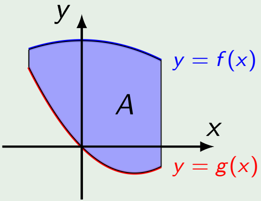
Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a non negative continuous function, the integral $\int_a^b f(x)dx$ represents the area of the region R_x delimited by the graphs of f, the axis of equations: $x = a$, $x = b$ and $y = 0$ (the $x-$axis).



Theorem

If f and g are two continuous functions on $[a, b]$ and $f(x) \geq g(x)$, $\forall x \in [a, b]$. Then the area A of the region bounded by the graphs of f,g, $x = a$ and $x = b$ is

$$A = \int_a^b f(x) - g(x) dx.$$

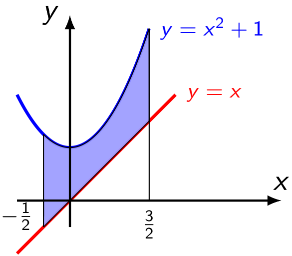


Example

Let $f(x) = x^2 + 1$ and $g(x) = x$.

The area of the shaded region is

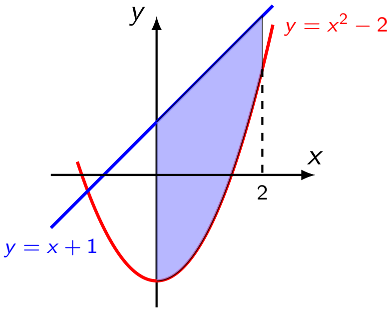
$$A = \int_{-\frac{1}{2}}^{\frac{3}{2}} (x^2 + 1 - x) dx = \frac{13}{6}.$$



Example

Let $f(x) = x^2 - 2$ and $g(x) = x + 1$ on the interval $[0, 2]$.
The area of the region between the graphs of the functions f and g on the interval $[0, 2]$ is

$$\begin{aligned} A &= \int_0^2 (x+1) - (x^2 - 2)dx \\ &= \int_0^2 (x + 3 - x^2)dx \\ &= \frac{16}{3}. \end{aligned}$$



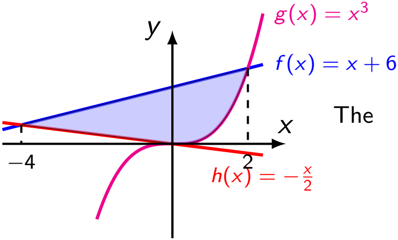
Remark

If f and g are two continuous functions on $[a, b]$. Then the area A of the region bounded by the graphs of f and g is
$A = \int_a^b |f(x) - g(x)|dx.$
For example if there is $c \in]a, b[$ such that $f(x) \geq g(x), \forall x \in [a, c]$ and
$f(x) \leq g(x), \forall x \in [c, b]$, then
$A = \int_a^c f(x) - g(x)dx + \int_c^b g(x) - f(x)dx.$

Example

Consider the functions $f(x) = x + 6$, $g(x) = x^3$ and $h(x) = -\frac{1}{2}x$.
The area A of the region R bounded by the graphs of the functions f, g and h

$f(x) = h(x) \iff x = -4$,
$g(x) = h(x) \iff x = 0$,
$f(x) = g(x) \iff x^3 - x - 6 = 0$. $x = 2$ is the unique solution of this equation.
We have $f(-4) = h(-4) = 2$,
$g(0) = h(0) = 0$ and
$f(2) = g(2) = 8$.



The area of the region is equal to:

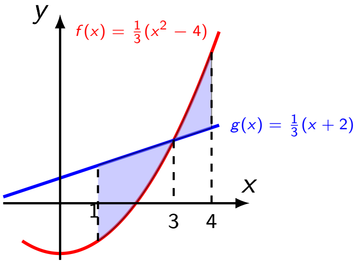
$$A = \int_{-4}^{0} (f(x) - h(x))\,dx + \int_{0}^{2} (f(x) - g(x))\,dx.$$

$$A = \int_{-4}^{0} \left((x+6) + \frac{1}{2}x\right) dx + \int_{0}^{2} \left((x+6) - x^3\right) dx = 22.$$

Example

The area of the region between the graphs of the functions: $f(x) = \frac{1}{3}(x^2 - 4)$ and $g(x) = \frac{1}{3}(x + 2)$ if x is restricted to the interval $[1, 4]$.

$f(x) = g(x) \iff x^2 - x - 6 = 0$. The only solution of this equation on the interval $[1, 4]$ is $x = 3$ and we have $f(3) = g(3) = \frac{5}{3}$.
We have $f \leq g$ on the interval $[1, 3]$ and $g \leq f$ on the interval $[3, 4]$. Then



$$A = \int_1^3 (g(x) - f(x))\,dx + \int_3^4 (f(x) - g(x))\,dx$$
$$= \frac{1}{3}\int_1^3 \Big((x+2) - (x^2 - 4)\Big)\,dx + \frac{1}{3}\int_3^4 \Big((x^2 - 4) - (x+2)\Big)\,dx = \frac{61}{18}.$$

The Disk Method

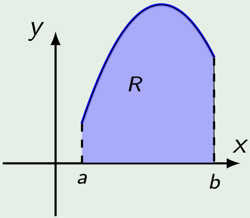
Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a non negative continuous function and R_x the region delimited by the graph of f and the axis: $x = a$, $x = b$ and the $x-$axis. If the region R_x is revolved around the x-axis, the resulting solid is called: the solid of revolution generated by the region R_x.

Examples

1. If $f : [a, b] \longrightarrow \mathbb{R}$ is a constant $c > 0$, then the region under the graph of f on the interval $[a, b]$ is a rectangle. The solid generated by revolving this region around the x-axis is a circular right cylinder.
2. Consider the region under the graph of the function $f(x) = \sqrt{4 - x^2}$ for $x \in [-2, 2]$. If we revolve the region R_x around the x-axis, the solid generated is a ball of radius $r = 2$.

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of f, $y = 0$ $x = a$ and $x = b$ is given by

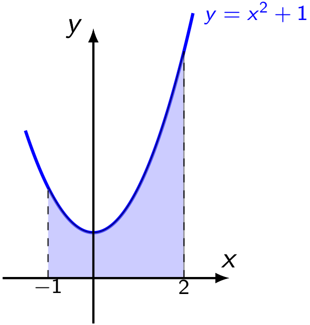


$$V = \int_a^b \pi f^2(x) dx.$$

Example

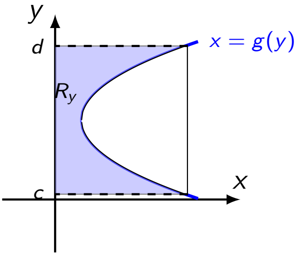
Let f be the function defined on the interval $[-1, 2]$ by $f(x) = x^2 + 1$. The volume of the solid obtained by revolving the region under the graph of f around the x-axis is

$$\pi \int_{-1}^{2} (x^2 + 1)^2 dx = \frac{78\pi}{5}.$$



Remark

Let g be a positive continuous function on the interval $[c, d]$ and R_y the region bounded by: the graph of the function $x = g(y)$, the axis $y = c$, $y = d$ and $y-$axis. The volume of the solid of revolution of the region R_y around the y-axis is:

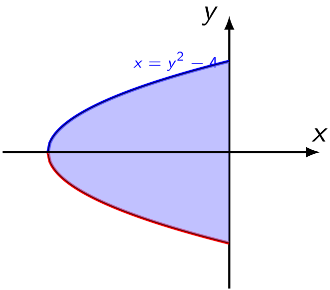


$$V = \pi \int_c^d g^2(y)\,dy$$

Example

If $g(y) = y^2 - 4$ defined on the interval $[0, 2]$. The volume of the solid obtained by revolving the region under the graph of g around the y-axis is:

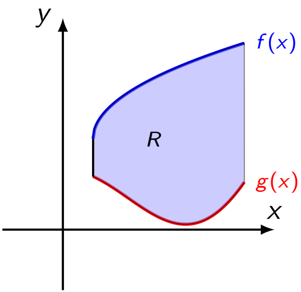
$$V = \pi \int_0^2 (y^2 - 4)^2 dy = \frac{256}{15}\pi.$$



Washer Method

Theorem

Let $f, g\colon [a,b] \longrightarrow \mathbb{R}^+$ be two continuous functions such that $f(x) \geq g(x) \geq 0$, $\forall x \in [a,b]$. If R is the region between the graph of f and the graph of g. The volume of the solid obtained by revolving the region R around the x-axis is equal to

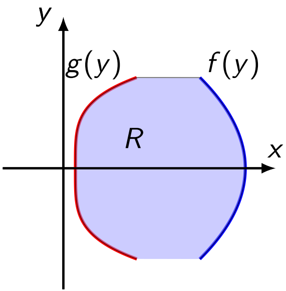


$$\pi \int_a^b \left(f^2(x) - g^2(x)\right) dx.$$

This formula can be interpreted as:

$$V = \pi \int_a^b (\text{outer radius})^2 - (\text{inner radius})^2 dx.$$

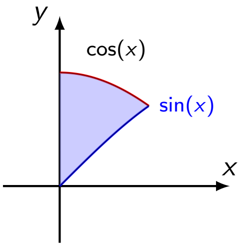
If R is the region bounded by the graphs of $x = f(y)$ and $x = g(y)$, where $f(y)$ and $g(y)$ continuous functions defined on the interval $[c, d]$ and satisfies $0 \leq g \leq f$. The volume of the solid of revolution generated by revolving the region R around the $y-$axis is



$$V = \pi \int_c^d \left[f^2(y) - g^2(y)\right] \, dy.$$

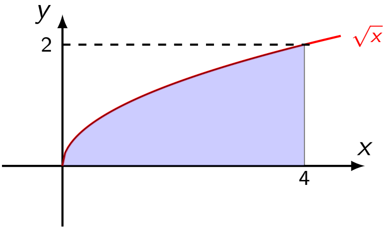
Examples

1. If $f(x) = \cos(x)$ and $g(x) = \sin(x)$ on the interval $[0, \frac{\pi}{4}]$. The volume of the solid of revolving R between the graph of f and g around the x-axis is



$$V = \pi \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x)\right) dx = \pi \int_0^{\frac{\pi}{4}} \cos(2x) dx = \frac{\pi}{2}.$$

2. Let $f(x) = \sqrt{x}$ defined on the interval $[0,4]$. If R is the region under the graph of f and S the solid of revolution of R around the axis $y = 2$. The volume of S is:



$$V = \pi \int_0^4 \left(2^2 - (2-\sqrt{x})^2\right) dx = \frac{40\pi}{3}.$$

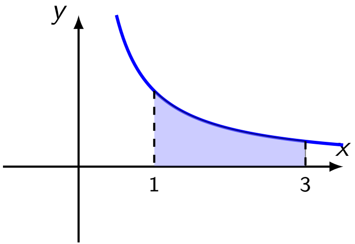
In this example, the outer radius is 2, the inner radius is $2 - y = 2 - \sqrt{x}$.

Examples

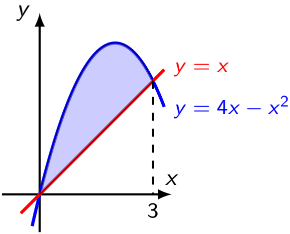
Use disk or washer method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the following curves

1. $y = \dfrac{1}{x}$, $x = 1$, $x = 3$ and $y = 0$, around the $x-$axis.

$$V = \pi \int_1^3 \frac{dx}{x^2} = \frac{2\pi}{3}.$$



2. $y = 4x - x^2$ and $y = x$, around the $x-$axis.
$4x - x^2 = 4 - (x-2)^2$ is a parabola opens downward with vertex (2,4) and $y = x$ is a straight line passing through the origin.
$x = 4x - vx^2 \iff x = 0\ ,\ x = 3$. The points of intersection of $y = 4x - x^2$ and $y = x$ are $(0,0)$ and $(3,3)$. Using Washer Method, we get

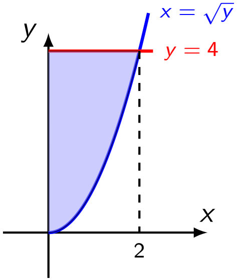


$$V = \pi \int_0^3 \left[(4x - x^2)^2 - x^2 \right] \, dx$$
$$= \pi \int_0^3 \left[x^4 - 8x^3 + 15x^2 \right] \, dx = \frac{108}{5}\pi.$$

3. $x = \sqrt{y}$, $x = 0$ and $y = 4$, around the $y-$axis

Using Disk Method, we get

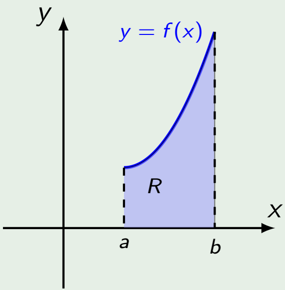
$$V = \pi \int_0^4 (\sqrt{y})^2 \; dy = \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi.$$



The Cylindrical Shells Method

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function and R the region under the graph of f on the interval $[a, b]$. The volume V of the solid of revolution generated by revolving the region R around the y-axis is given by



$$V = 2\pi \int_a^b x f(x) dx.$$

Example

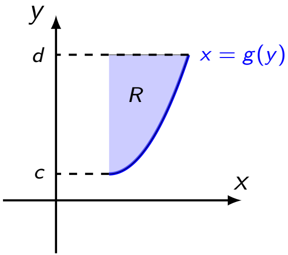
Let $f\colon [2,11] \longrightarrow \mathbb{R}^+$ be the function defined by $\sqrt{x-2}$. The volume of the solid of revolution generated by revolving the region under the graph of f around the y-axis is

$$V = 2\pi \int_2^{11} x\sqrt{x-2}dx \overset{x-2=t^2}{=} 4\pi \int_0^3 (2t^2 + t^4)dt = 12\pi \frac{111}{5}.$$

Remark

Consider the region R bounded by the graphs of the curves of $g(y)$, $y = d$, $y = c$ and the $y-$axis. Using cylindrical shells method, the volume of the solid of revolution generated by revolving the region R around the $x-$axis is

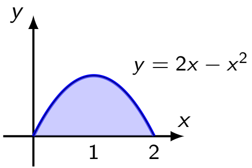
$$V = 2\pi \int_c^d y\ g(y)\ dy.$$



Examples

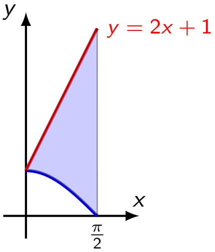
We use cylindrical shells method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the following curves:

1. $y = 2x - x^2$ and $y = 0$, around the $y-$axis.
   $y = 2x - x^2 = 1 - (x-1)^2$ is a parabola opens downward with vertex $(1,1)$. $2x - x^2 = 0 \iff x = 0$, $x = 2$, then the points of intersection between $y = 2x - x^2$ and $y = 0$ are $(0,0)$ and $(2,0)$. Using Cylindrical shells method, we get



$$V = 2\pi \int_0^2 x(2x - x^2)\ dx = 2\pi \int_0^2 (2x^2 - x^3)\ dx = \frac{8}{3}\pi.$$

2. $y = \cos x$, $y = 2x + 1$ and $x = \frac{\pi}{2}$, around the $y-$axis.



The line $y = 2x + 1$ passes through the point $(0, 1)$. The desired region is under the line $y = 2x + 1$ and above the curve of $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.
Using Cylindrical shells method, we get

$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} x\left[(2x+1) - \cos x\right]\ dx \\ &= 2\pi \int_0^{\frac{\pi}{2}} (2x^2 + x)\ dx - 2\pi \int_0^{\frac{\pi}{2}} (x\cos(x))\ dx \\ &= 2\pi \left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - 2\pi\left[x\sin(x) + \cos(x)\right]_0^{\frac{\pi}{2}} \\ &= 2\pi\left(\frac{\pi^3}{12} + \frac{\pi^2}{8}\right) - 2\pi\left(\frac{\pi}{2} - 1\right). \end{aligned}$$

Arc Length

Definition

Let $f: I \longrightarrow \mathbb{R}$ be a function. We say that f is continuously differentiable if f is differentiable and f' is itself continuous on I.

Definition

Let $f: [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. The length of the curve $(x, f(x))$, for $x \in [a, b]$ is defined by:

$$L_a^b = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Example

Let $f : [0, \frac{\pi}{4}] \longrightarrow \mathbb{R}$ defined by: $f(x) = \ln(\cos(x))$. The length of the curve defined by f is given by:

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}dx = \int_0^{\frac{\pi}{4}} \sec(x)dx = \ln(\sqrt{2} + 1).$$

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. Then the arc length function "s" for the graph of f on $[a, b]$ is defined by:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt.$$

We have

$$ds = \sqrt{1 + (f'(x))^2} dx.$$

Examples

1. The arc length of the curve defined by the function $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ on the interval $[1, 2]$ is given by:

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx = \int_1^2 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\, dx \\ &= \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right)\, dx = \frac{13}{12} \end{aligned}$$

2. The arc length of the curve defined by the function $f(x) = \cosh(x)$ on the interval $[0, 2]$ is given by:

$$L = \int_0^2 \sqrt{1 + \sinh^2(x)}\, dx = \int_0^2 \cosh(x)\, dx = \sinh(2).$$

3. Let g be the function defined by: $g(y) = \sqrt{25 - y^2}$ on the interval $[-5, 5]$. The arc length of the curve defined by the function g is equal to half of the perimeter of the circle $x^2 + y^2 = 25$, the arc length is equal to 5π.
$g'(y) = \dfrac{-y}{\sqrt{25 - y^2}}$. Then the arc length of the curve defined by the function g on the interval $[-5, 5]$ is given by:

$$\begin{aligned} L &= \int_{-5}^{5} \sqrt{1 + \frac{y^2}{25 - y^2}}\, dy = 5 \int_{-5}^{5} \frac{dy}{\sqrt{25 - y^2}} \\ &= 5 \left[\sin^{-1}\left(\frac{y}{5}\right)\right]_{-5}^{5} = 5\pi. \end{aligned}$$

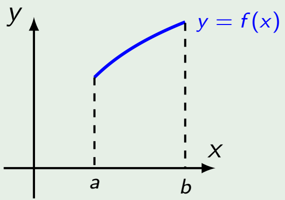
4. The arc length of the curve defined by the function $f(x) = 1 + \frac{2}{3}x^{\frac{3}{2}}$ on the interval $[0, 3]$ is:

$$L = \int_0^3 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\, dx = \int_0^3 \sqrt{1+x}\, dx = \int_0^3 (1+x)^{\frac{1}{2}}\, dx$$
$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$$

Surfaces of Revolution

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. The area of the surface generated by revolving the curve $y = f(x)$ around the x-axis denoted by S is given by



$$S = \int_a^b 2\pi |f(x)| \sqrt{1 + (f'(x))^2}\, dx.$$

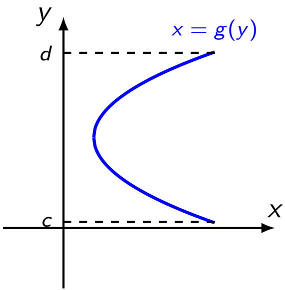
Example

Let f be the function defined on the interval $[0, 1]$ by: $f(x) = \frac{x^3}{3}$.
The surface of revolution of the graph of f around the $x-$axis is

$$S = 2\pi \int_0^1 \frac{x^3}{3}\sqrt{1+x^4}dx \overset{t^2=1+x^4}{=} \frac{\pi}{3}\int_1^{\sqrt{2}} t^2 dt = \frac{\pi}{9}(2\sqrt{2}-1).$$

Remark

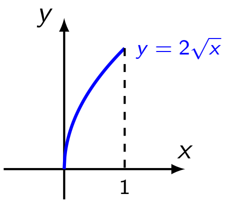
If $x = g(y)$, $y \in [c, d]$ and g continuously differentiable, the surface area generated by revolving the curve of g around the y-axis is given by



$$S = \int_c^d 2\pi |x| ds = \int_c^d 2\pi |g(y)| ds = \int_c^d 2\pi |g(y)| \sqrt{1 + (g'(y))^2} dy.$$

Examples

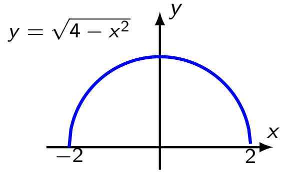
1. Consider the function $f(x) = 2\sqrt{x}$ defined on the interval $[0,1]$. The surface area generated by revolving the curve defined by the graph of the function f around the $x-$axis is:



$$S = 2\pi \int_0^1 2\sqrt{x}\sqrt{1+\left[\frac{1}{\sqrt{x}}\right]^2}\,dx = 4\pi \int_0^1 \sqrt{x+1}\,dx$$
$$= 4\pi \left[2\frac{(x+1)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{8\pi}{3}\left(2\sqrt{2}-1\right).$$

2. Consider the function $f(x) = \sqrt{4 - x^2}$ defined on the interval $[-2, 2]$.

The surface area generated by revolving the curve defined by the graph of the function f around the $x-$axis is:



$$\begin{aligned} S &= 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2} \, dx \\ &= 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \sqrt{\frac{(4 - x^2) + x^2}{4 - x^2}} \, dx \\ &= 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}} \, dx \\ &= 4\pi \int_{-2}^{2} dx = 4\pi \left[x\right]_{-2}^{2} = 16\pi. \end{aligned}$$

Note: It is the surface area of the sphere with radius 2, and it is equal to $4\pi(2)^2 = 16\pi$