Fundamental Properties of Holomorphic Functions

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Theorem

Let f be a holomorphic function on an open subset Ω of $\mathbb{C}$. For $z_0 \in \Omega$ and $r > 0$ such that $\overline{D(z_0, r)} \subset \Omega$, there exists a power series $\sum_{k \geq 0} a_k (z - z_0)^k$ which converges to f on $D(z_0, r)$ and if $M_f(z_0, r) = \sup_{z \in \overline{D(z_0, r)}} |f(z)|$, we have

$$|a_n| \leq \frac{M_f(z_0, r)}{r^n}, \quad \forall n \in \mathbb{N}_0. \tag{1}$$

These inequalities are called the Cauchy's inequalities.

Proof

By theorem **??** (chapter IV)

$$a_n = \frac{1}{2\mathrm{i}\pi}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{\mathrm{i}\theta})e^{-\mathrm{i}n\theta}d\theta.$$

Thus $|a_n| \leq \dfrac{M_f(z_0,r)}{r^n}$, with $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}\theta},\ \theta \in [0, 2\pi]$.

□

Corollary

Any bounded holomorphic function on $\mathbb{C}$ is constant.

Proof

Let f be a bounded holomorphic function on $\mathbb{C}$ and let $M = \sup_{z\in\mathbb{C}} |f(z)|$. If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, then by Cauchy's inequalities $|a_n| \leq \dfrac{M}{r^n}$ for all $r > 0$ and all $n \geq 1$. Since for $n \geq 1$, $\lim_{r\to+\infty} \frac{M}{r^n} = 0$, then $a_n = 0$ for all $n \geq 1$ and f is constant. $\square$

Theorem (The fundamental Theorem of algebra, or D'Alembert's Theorem)

Every non constant polynomial has at least one zero.

This theorem is rephrased as "$\mathbb{C}$ is algebraically closed". For the proof, we need the following lemma:

Lemma (Growth Lemma)

Let P be a polynomial of degree $n \geq 1$, $P(z) = a_0 + a_1 z + \ldots + a_n z^n$, then there exists R large enough such that

$$\frac{|a_n||z|^n}{2} \leq |P(z)| \leq \frac{3|a_n||z|^n}{2}, \quad \forall\, z \in \mathbb{C} \text{ and } |z| \geq R. \tag{2}$$

Proof

For $z \neq 0$, $P(z) = z^n \left(\sum_{k=0}^{n} \frac{a_k}{z^{n-k}} \right)$. In use of the triangle inequality, we have

$$|z|^n \left(|a_n| - \left| \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \right) \leq |P(z)| \leq |z|^n \left(|a_n| + \left| \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \right).$$

So $\lim_{|z| \to +\infty} \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} = 0$, then there exists R large enough such that for $|z| \geq R$,
$\left| \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} \right| \leq \frac{|a_n|}{2}$, the result now follows. □

Proof of theorem 1.3

Let $P \in \mathbb{C}[X]$ be a non constant polynomial. If P never vanishes, then the function $f(z) = \dfrac{1}{P(z)}$ is holomorphic on $\mathbb{C}$ and is bounded because $\lim_{|z|\to+\infty} |P(z)| = +\infty$, thus f is constant and P is constant, this contradicting our assumption.

$\square$

Corollary

Every polynomial of degree n has exactly n (not necessarily distinct) zeros.

Proof

The proof is given by induction on the degree of the polynomial.

$\square$

Corollary

Every polynomial of degree n takes every complex number exactly n times.

Proof

If P is a polynomial of degree n and $a \in \mathbb{C}$, then the polynomial $Q = P - a$ is also a polynomial of degree n. By Corollary 1.5, Q

Definition

We say that a continuous function f on an open set Ω fulfills the Mean Property on Ω, if for all $a \in \Omega$ and all $r > 0$ such that $\overline{D(a,r)} \subset \Omega$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + r\mathrm{e}^{\mathrm{i}\theta}) d\theta.$$

Remark 1 :
If f satisfies the Mean Property, then $\mathrm{Re} f$ and $\mathrm{Im} f$ also satisfies the Mean Property.

Proposition

Any holomorphic function on an open set Ω satisfies the Mean Property.

Proof

Let $f \in \mathcal{H}(\Omega)$, $a \in \Omega$ and $r > 0$ such that $\overline{D(a,r)} \subset \Omega$, the Cauchy formula on a circle yields

$$f(a) = \frac{1}{2\mathrm{i}\pi}\int_\gamma \frac{f(w)}{w-a}dw = \frac{1}{2\pi}\int_0^{2\pi} f(a + r\mathrm{e}^{\mathrm{i}\theta})d\theta.$$

where $\gamma(\theta) = a + r\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$. $\square$

Definition

1. *Let f be a continuous function on an open set Ω. We say that f has a relative maximum at a point $a \in \Omega$ if there exists a neighborhood $V \subset \Omega$ of a such that $|f(z)| \leq |f(a)|$ for all $z \in V$.*
2. *We say that f satisfies the Maximum Modulus Principle on Ω if for any relative maximum a of f, f is constant in a neighborhood of a.*

Theorem

Any function which satisfies the Mean Property on Ω (in particular $f \in \mathcal{H}(\Omega)$) satisfies the Maximum Modulus Principle.

Proof

Let a be a relative maximum of f and let $r > 0$ such that $|f(z)| \leq |f(a)|$ for all $z \in D(a, r)$.

- If $f(a) = 0$, the result is trivial.
- If $f(a) \neq 0$, we can suppose that $f(a) > 0$ (it suffices to take the function $g(z) = f(z)\dfrac{\overline{f(a)}}{|f(a)|^2}$). For all $s < r$,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+se^{i\theta})d\theta \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} f(a) - \mathrm{Re} f(a+se^{i\theta})d\theta = 0.$$

Since $\theta \longmapsto f(a) - \mathrm{Re} f(a + s e^{\mathrm{i}\theta})$ is a non negative continuous function and s is arbitrary, then $f(a) = \mathrm{Re} f(z)$, for all $z \in D(a, r)$. And since $|f(a)| \geq |f(z)|$ on the disc $D(a, r)$, then $\mathrm{Im} f = 0$ on the disc $D(a, r)$, which proves that f is constant on the disc $D(a, r)$. Therefore, $|f|$ cannot reaches a relative maximum at a point of Ω unless f is constant. $\square$

Theorem

[Maximum Modulus Principle (second form)]
Let Ω be a bounded domain and let $f : \overline{\Omega} \longrightarrow \mathbb{C}$ be a continuous function on $\overline{\Omega}$ and holomorphic on Ω. If $M = \sup\limits_{z \in \overline{\Omega} \setminus \Omega} |f(z)|$, then $|f(z)| \leq M$ for every $z \in \Omega$, and if there exists $a \in \Omega$ such that $|f(a)| = M$, then f is constant on Ω. (Furthermore, $|f|$ does not attains a maximum at an interior point unless f is constant.)

Proof

Let $M' = \sup_{z\in\overline{\Omega}} |f(z)|$. Since f is continuous on the compact $\overline{\Omega}$, there exists $a \in \overline{\Omega}$ such that $|f(a)| = M'$.

• If $a \in \Omega$, f is constant in a neighborhood of a, thus f is constant on Ω.

• If $a \notin \Omega$ and $|f(z)| < M'$ $\forall z \in \Omega$. M' is reached on $\overline{\Omega} \setminus \Omega$, then $M' = M$ and $|f(z)| < M$, $\forall$ $z \in \Omega$.

$\square$

Remarks 2 :

1. If f is holomorphic on the annulus $\Omega = \{z \in \mathbb{C};\ \frac{1}{r} < |z| < R\}$ and continuous on $\overline{\Omega}$, then f reaches its maximum on the boundary $\mathscr{C}(0, r) \cup \mathscr{C}(0, R)$.
   For example the function $f(z) = z$ reaches its maximum on the outer boundary $\mathscr{C}(0, R)$, whereas the function $g(z) = \dfrac{1}{z}$ reaches its maximum on the inner boundary $\mathscr{C}(0, r)$.
2. Theorem 2.5 is not true if Ω is not bounded. For example, if $f(z) = e^z$ and $\Omega = \{z \in \mathbb{C};\ \mathrm{Re} z > 0\}$, then $|f(\mathrm{i}y)| = |e^{\mathrm{i}y}| = 1$, i.e., $f(\partial\Omega) \subset \mathscr{C}(0, 1)$. But $f(x) > 1$ along the positive real axis. Thus, the hypothesis that Ω is bounded is essential in theorem 2.5.

Theorem (The Open Mapping Theorem)

Any non constant holomorphic function on a domain of $\mathbb{C}$ is open.

Proof

Let f be a non constant holomorphic function on a domain Ω. Assume that $0 \in \Omega$ and $f(0) = 0$. (If $a \in \Omega$ and $f(a) = \alpha$, we take the function $g(z) = f(a+z) - \alpha$). It suffices to prove that $f(\Omega)$ is a neighborhood of 0.

Let $r > 0$ be such that $\overline{D(0,r)} \subset \Omega$ and $f(z) \neq 0$ for all z such that $|z| = r$. (A such r exists if not 0 will be a cluster point (accumulation point) of the set of zeros of f, and then f is constant.) Let $m = \inf_{|z|=r} |f(z)| > 0$.

If $D(0,m) \subset f(\Omega)$ this yields the result, if not let $w \in \mathbb{C}$ such that $|w| < m$ and $w \notin f(\Omega)$. The function $\psi(z) = \dfrac{1}{f(z) - w}$ is holomorphic on Ω and

$$|\psi(0)| = \frac{1}{|w|} \leq \sup_{|z|=r} |\psi(z)| \leq \frac{1}{m - |w|}.$$

Thus $|w| \geq \frac{m}{2}$. Then if $|w| < \frac{m}{2}$, $w \in f(\Omega)$ and $D(0, \frac{m}{2}) \subset f(\Omega)$.
$\square$

Theorem (Schwarz's Lemma)

Let f be a holomorphic function on the unit disc D with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Then

$$|f(z)| \leq |z|,\ \forall z \in D \text{ and } |f'(0)| \leq 1.$$

Furthermore if there exists $z \in D \setminus \{0\}$ such that $|f(z)| = |z|$ or if $|f'(0)| = 1$, then f is a rotation, i.e. there exists some unimodular complex number ($|\lambda| = 1$) such that $f(z) = \lambda z$ for all $z \in D$.

Proof

The function g defined on D by: $\begin{cases} g(z) = \dfrac{f(z)}{z} \quad \text{if} \quad z \neq 0 \\ g(0) = f'(0) \end{cases}$ is holomorphic on $D \setminus \{0\}$ and continuous on D, thus g is holomorphic on the disc D. By maximum modulus principle, for $|z| \leq r < 1$, $|g(z)| \leq \sup\limits_{|w|=r} |g(w)| = \dfrac{1}{r} \sup\limits_{|w|=r} |f(w)| \leq \dfrac{1}{r}$. This is for all positive real number $r < 1$. Now, since r can come arbitrarily close to 1, we have

$$|g(z)| \leq \lim_{r \to 1} \frac{1}{r} = 1, \ \forall z \in D.$$

This proves that $|f(z)| \leq |z|$ and therefore, $|f'(0)| \leq 1$.
In case either $|f'(0)| = 1$ or $|f(a)| = |a|$ for some $a \in D \setminus \{0\}$, we get $|g(a)| = 1$ or $|g(0)| = 1$, so $|g|$ reaches its maximum in an interior point of D, then g is a constant function by the Maximum Modulus Principle and the result follows. $\square$

Corollary

Let $f : \mathcal{D}(0, R) \longrightarrow \mathbb{C}$ *be a holomorphic function with* $f^{(k)}(0) = 0$ *for all* $0 \leq k \leq n-1$. *If* $|f(z)| \leq M$ *for* $z \in D(0, R)$, *then*

$$|f(z)| \leq M \left(\frac{|z|}{R} \right)^n, \ \forall z \in D(0, R)$$

Furthermore, if there exists $a \in D(0, R) \setminus \{0\}$ *such that* $|f(a)| = M \left(\frac{|a|}{R} \right)^n$, *there exists* $\alpha \in \mathbb{R}$ *such that* $f(z) = M\mathrm{e}^{\mathrm{i}\alpha} \left(\frac{z}{R} \right)^n$, *for all* $z \in D(0, R)$.

Proof

There exists a holomorphic function g on $D(0, R)$ such that $f(z) = z^n g(z)$. The result is deduced by maximum modulus principle for the function $h(z) = \dfrac{g(Rz)R^n}{M}$.

□

Corollary

Let f be an automorphism of the unit disc D (i.e. a biholomorphic function of the unit disc), such that $f(0) = 0$, then there exists $\alpha \in \mathbb{R}$ such that $f(z) = \mathrm{e}^{\mathrm{i}\alpha} z$, for all $z \in D$.

Proof

Let $g = f^{-1}$, then $g(0) = 0$, $g'(0)f'(0) = 1$ and by Schwarz's lemma $|g'(0)| \leq 1$ and $|f'(0)| \leq 1$, thus $|g'(0)| = |f'(0)| = 1$, this yields that $f(z) = \mathrm{e}^{\mathrm{i}\alpha} z$, with $\alpha \in \mathbb{R}$. $\square$

Remark 3 :
For all $a \in D$, we set $h_a(z) = \dfrac{a-z}{1-\bar{a}z}$. $h_a(a) = 0$, $h_a(0) = a$ and $|h_a(e^{i\theta})| = \left|\dfrac{a-e^{i\theta}}{1-\bar{a}e^{i\theta}}\right| = \left|\dfrac{a-e^{i\theta}}{e^{-i\theta}-\bar{a}}\right| = 1$. Then h_a is an automorphism of the unit disc. The function $h_a \circ h_a$ is an automorphism of the unit disc and $h_a \circ h_a(0) = 0$, $h_a \circ h_a(a) = a$, then $h_a \circ h_a = \mathrm{Id}$. Furthermore if g is an automorphism of the unit disc with $g(a) = 0$, for some $a \in D$, the function $f = g \circ h_a$ is so an automorphism of the unit disc with $f(0) = 0$. Thus $g(z) = \mathrm{e}^{i\alpha} h_a(z)$. This characterizes the group of automorphisms of the unit disc.

Lemma

Let Ω be an open subset of $\mathbb{C}$ and K a compact subset of Ω. If $r < \delta(K, \Omega^c)$, then for any holomorphic function f on Ω

$$\sup_{z \in K} |f'(z)| \leq \frac{1}{r} \sup_{z \in K_r} |f(z)|,$$

with $K_r = \{z \in \Omega;\ \delta(z, K) \leq r\}$.

This lemma is deduced by Cauchy's integral formula.

Theorem

Let $(f_n)_n$ be a sequence of holomorphic functions on Ω which converges uniformly on compact subsets of Ω to a function f, then f is holomorphic on Ω. The sequence $(f'_n)_n$ converges uniformly on compact subsets of Ω to f'.

Corollary

Under the same hypotheses, for all $k \in \mathbb{N}$, the sequence $(f_n^{(k)})$ converges uniformly on compact subsets of Ω to $f^{(k)}$.

Proof of theorem 3.2

The uniform convergence theorem yields that f is continuous. To prove f is holomorphic, we use Morera's theorem. For any closed triangle Δ in Ω, $\displaystyle\int_{\partial\Delta} f_n(z)\ dz = 0$ and by the uniform convergence $\displaystyle\lim_{n\longrightarrow+\infty}\int_{\partial\Delta} f_n(z)\ dz = \int_{\partial\Delta} f(z)\ dz$, then $\displaystyle\int_{\partial\Delta} f(z)dz = 0$.
From the previous lemma 3.1, the sequence $(f_n')_n$ converges uniformly on compact subsets K of Ω to f'. $\square$

In this section, we are interesting to study the isolated singularities of holomorphic functions.

Definition

Let Ω be an open subset of $\mathbb{C}$ and $z_0 \in \Omega$. If $f \in \mathcal{H}(\Omega \setminus \{z_0\})$, we say that z_0 is an isolated singularity of f.

Theorem

Let Ω be an open subset of $\mathbb{C}$ and f a holomorphic function on $\Omega \setminus \{z_0\}$, $z_0 \in \Omega$. Assume that f is bounded in some deleted neighborhood of z_0, then f can be extended on Ω to a holomorphic function.

Proof

Let g be the function defined on Ω by
$g(z) = \begin{cases} (z-z_0)f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$. Since f is bounded on some deleted neighborhood of z_0, g is continuous. Thus g is holomorphic on Ω. There exists a neighborhood V of z_0 such that $g(z) = \sum_{n=1}^{+\infty} a_n(z-z_0)^n$, for all $z \in V$. Thus f can be extended on V by $f(z) = \sum^{+\infty} a_n(z-z_0)^{n-1}$, $a_1 = g'(z_0)$.

Corollary

Let f be a holomorphic function on $\Omega \setminus \{z_0\}$. If f has an isolated singularity at z_0 and bounded in some deleted neighborhood of z_0, then $\lim\limits_{z \to z_0} f(z)$ exists.

Definition

Let f be a holomorphic function on $\Omega \setminus \{z_0\}$. If f can be extended to a holomorphic function on a neighborhood of z_0, we say that z_0 is a removable singularity of f.

Theorem (Classification of Isolated Singularities of Holomorphic Functions)

Let f be a holomorphic function on $\Omega \setminus \{z_0\}$, $(z_0 \in \Omega)$. Then f satisfies one of the following properties

1. *z_0 is a removable singularity of f.*
2. *There exist $a_{-1}, \dots, a_{-m}$ in $\mathbb{C}$, with $a_{-m} \neq 0$ such that z_0 is a removable singularity of the function $f(z) - \displaystyle\sum_{j=1}^{m} \frac{a_{-j}}{(z-z_0)^j}$.*
3. *f comes arbitrarily close to every complex value in each deleted neighborhood of z_0. In other words, for all $r > 0$ such that $D(z_0, r) \subset \Omega$, $f(D(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Remark 4 :

In the second case we say that z_0 is a pole of order m of f. The polynomial of $\frac{1}{z-z_0}$, $\displaystyle\sum_{j=1}^{m} \frac{a_{-j}}{(z-z_0)^j}$ is called the principal part of f at z_0. In this case $\displaystyle\lim_{z\to z_0} |f(z)| = +\infty$.

In a neighborhood of z_0, the function $f(z) - \displaystyle\sum_{j=1}^{m} \frac{a_{-j}}{(z-z_0)^j}$ has a power series representation.

$$f(z) - \sum_{j=1}^{m} \frac{a_{-j}}{(z-z_0)^j} = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

The series $\displaystyle\sum_{k=-m}^{\infty} a_k (z-z_0)^k$ is called the Laurent series expansion of f at z_0

Remark 5 :

In the third case we say that z_0 is an essential singularity of f. In this case $\lim\limits_{z\to z_0} f(z)$ does not exist as a finite value or as an infinite value, and generally the function f which is holomorphic in a deleted neighborhood of z_0 has an essential singularity at z_0 if there exists no non negative integer n for which $\lim\limits_{z\to z_0}(z-z_0)^n f(z)$ exists (either as a finite value or as an infinite value).

Proof of theorem 4.5

Let $D^*(z_0, r) = D(z_0, r) \setminus \{z_0\} \subset \Omega$ and assume that the property (3) is not valid. There exists $b \in \mathbb{C}$ and $\varepsilon > 0$ such that $f(D^*(z_0, r)) \cap D(b, \varepsilon) = \emptyset$, which is equivalent to $|f(z) - b| \geq \varepsilon$, $\forall z \in D^*(z_0, r)$. The function $g(z) = \dfrac{1}{f(z) - b}$ is holomorphic on $D^*(z_0, r)$ and bounded by $\frac{1}{\varepsilon}$, thus it can be extended to a holomorphic function on $D(z_0, r)$. We denote this extension also by g.

If $g(z_0) \neq 0$, then z_0 is a removable singularity of the function $f(z) = b + \dfrac{1}{g(z)}$.

If z_0 is a zero of g of multiplicity m, then $g(z) = (z - z_0)^m g_1(z)$, with g_1 a holomorphic function on $D(z_0, r)$ and $g_1(z_0) \neq 0$. Then $f(z) = b + \dfrac{h(z)}{(z - z_0)^m}$, with h holomorphic on $D(z_0, r)$. Let $h(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$, the power series expansion of h.

Thus $f(z) = b + \dfrac{b_0}{(z - z_0)^m} + \ldots + \dfrac{b_m}{z - z_0} + \sum_{k=0}^{+\infty} b_{m+k} (z - z_0)^k$. $\square$

Corollary

Suppose f has an essential singularity at z_0, then for any complex number a, there exists a sequence $(z_n)_n$ such that $\lim_{n \to +\infty} z_n = z_0$ and $\lim_{n \to +\infty} f(z_n) = a$.

Remarks 6 :

We conclude that if f is a holomorphic function on the open set $\Omega \setminus \{z_0\}$, $z_0 \in \Omega$, then

1. z_0 is a removable singularity if and only if f is bounded in a deleted neighborhood of z_0.
2. z_0 is a pole of f if and only if $\lim\limits_{z \to z_0} |f(z)| = +\infty$.
3. z_0 is a pole of f of order m if and only if $\lim\limits_{z \to z_0} |(z - z_0)^m f(z)| = c$, with $c \in \mathbb{C}^*$.
4. z_0 is an essential singularity of f, if and only if, f is not bounded in any neighborhood of z_0 and $\lim\limits_{z \to z_0} |f(z)|$ does not exists on $\mathbb{C} \cup \{+\infty\}$.

Definition

A mapping f is called a meromorphic function on an open subset Ω, if there exists a closed subset $A \subset \Omega$, such that f is holomorphic on $\Omega \setminus A$ and each point $a \in A$ is a pole of f.

If $A = \emptyset$, f is holomorphic on Ω.
The set A is at most countable without cluster points (accumulation points) in Ω.

Example

Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function on a domain Ω and f is not the zero function, then $\dfrac{1}{f}$ is a meromorphic function on Ω. $(A = f^{-1}\{0\})$.

Exercise 1 :

Prove that the set $\mathcal{M}(\Omega)$ of the meromorphic functions on Ω is a field.

Proposition

Let f be a meromorphic function on an open subset Ω, then f' is also a meromorphic function, and f and f' have the same set of poles in Ω.
If a is a pole of order m for f, then a is a pole of order $(m+1)$ for f'.

Exercise 2 :
If f is a meromorphic on Ω, then $\dfrac{f'}{f}$ is meromorphic and its poles are simple.