# Linear System Equations

Mongi BLEL

King Saud University

January 2, 2024

# Table of contents

# Introduction to Linear System Equations

## Definition

A linear system of equations with $m$ equations and $n$ unknowns is defined as follows:

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ \vdots & \vdots & \ddots & + & \vdots & + & \vdots & & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m. \end{cases}$$

where $b_1, \dots, b_n$, $(a_{j,k})$ are real numbers with $(1 \leq j \leq m, 1 \leq k \leq n)$ called the data of the system and $x_1, \dots, x_n$ the unknowns or the variables of the system.

This linear system can be represented in matrix form:
$AX = B$ where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

# Example

The following linear system with two variables

$$\begin{cases} 4x & - & y & = & 5 \\ -7x & + & 2y & = & 3 \end{cases}$$

can be interpreted as the intersection in the plane of the straight lines of equations respectively $4x - y = 5$ and $-7x + 2y = 3$.

# Example

The following linear system with three variables

$$\begin{cases} x & + & y & -3 & z = & 1 \\ 2 & + & y & - & z = & 3 \end{cases}$$

can be interpreted as the intersection in the space of the planes of equations respectively $x + y - 3z = 1$ and $2x + y - z = 3$.
The solution of this system is $\{(2, 1, 0)+z(-2, -5, 1);\ z \in \mathbb{R}\}$. This is the equation of the line passing through the point $A$ of coordinates $(2, 1, 0)$ and parallel of the vector $v$ of coordinates $(-2, -5, 1)$.

# Example

Let the matrices $A = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and we look for a matrix of order $(2, 3)$ such that $AB = C$.

If $B = \begin{pmatrix} x & y & z \\ t & u & v \end{pmatrix}$ we find the following linear system:

$$\begin{cases} x - t &= 0 \\ x - 2t &= 1 \\ y - u &= 1 \\ y - 2u &= 2 \\ z - v &= 2 \\ z - 2v &= 3 \end{cases}$$

The solution of this system is $(-1, 0, 1, -1, -1, -1)$

$$x = t = -1, y = 0, u = -1, z = 1, v = -1.$$

**Definition**

1. We say that two linear systems are equivalent if they have the same set of solutions.
2. We say that a linear system is consistent if it has solutions and we call that it is inconsistent if it has no solutions.

# Gauss And Gauss Jordan Method

The augmented matrix of the linear system $AX = B$ is the matrix $[A|B]$.
The elementary row operations on the augmented matrix of a system produce the augmented matrix of an equivalent system.

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use elementary row operations to transform the augmented matrix in a reduced row echelon form.
3. Solve the obtained triangular system.

## The Gauss and Gauss Jordan Method

• The Gauss Jordan method consists to take the reduced row echelon form of the augmented matrix $[A|B]$ and solve the obtained system.

## Examples

Consider the following linear system

$$\begin{cases} x + 2y - z & = & -4 \\ -x + y & = & -2 \\ y - z & = & -4 \end{cases}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 1 & 0 & -2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

and the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is a row echelon form of this matrix.

Using Gauss method, the system has a unique solution which is $x = 1, y = -1, z = 3$.
The reduced row echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Using Gauss Jordan method, the system has a unique solution which is $x = 1, y = -1, z = 3$.

Consider the following linear system

$$\begin{cases} x+2y-z+t &= 1 \\ 3x-y+5z-t &= 2 \\ 5x+3y+3z+t &= m \end{cases} \quad ; \quad m \in \mathbb{R}.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{8}{7} & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 & m-4 \end{array}\right]$$

If $m \neq 4$, the system, is inconsistent.
If $m = 4$, the system has infinite solutions :

$$\{(\frac{5}{7} - \frac{9}{7}z + \frac{1}{7}t, \frac{1}{7} + \frac{8}{7}z - \frac{4}{7}t) : \ z, t \in \mathbb{R}^2\}.$$

Consider the following linear system $\begin{cases} -2y + 3z &= 0 \\ 2x - 4y + 2z &= 1 \\ -x - 2y + 5z &= 0 \\ x - 2y &= 1 \end{cases}$,

The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 0 \\ 2 & -4 & 2 & 1 \\ -1 & -2 & 5 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$

The system is inconsistent.

Give the relations between the numbers $a$, $b$ and $c$ such that the following linear system is consistent.

$$\begin{cases} x+y+2z &= a \\ x+z &= b \\ 2x+y+3z &= c \end{cases}$$

The augmented matrix of the system is: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right]$.

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{array}\right]$.

The system is consistent if and only if $c-a-b=0$.

Consider the following linear system

$$\begin{cases} x + my + (m-1)z & = & m+1 \\ 3x + 2y + mz & = & 3 \\ (m-1)x + my + (m+1)z & = & m-1 \end{cases}$$

The determinant of the system is $m^2(m-4)$.

If $m = 0$, the augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 3 & 2 & 0 & 3 \\ -1 & 0 & 1 & -1 \end{array}\right]$$

A row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The system has an infinite many solutions $\{(1+z, -\frac{3}{2}z, z);\ z \in \mathbb{R}\}$.

If $m = 4$, a row echelon form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 4 & 3 & 5 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 12 \end{array}\right]$.

The system is inconsistent.

# Homogeneous Linear Systems

## Definition

We say that a linear system $AX = B$ is homogeneous if $B = 0$.

## Remarks

1. Any homogeneous linear system is consistent. 0 is a solution of the system.
2. If $X_1$ and $X_2$ are solutions of the homogeneous system $AX = 0$, then $X_1 + \lambda X_2$ is also a solution of the linear system for all $\lambda \in \mathbb{R}$.
3. If the homogeneous linear system $AX = 0$ has a non zero solution, it has an infinite number of solutions.

**Theorem**

If $X_0$ is a solution of the linear system $AX = B$, then any solution $X$ of the system is in the following form: $X = X_0 + X_1$ with $X_1$ is a solution of the homogeneous system. $AX = 0$.

**Conclusion** Any consistent linear system can has only one solution or an infinite number of solutions.

# Crammer Method

**Theorem**

If $A$ is a square matrix of order $n$ and has an inverse, then the linear system $AX = B$ has the following unique solution

$$x_1 = \frac{\det A_1}{\det A}, \ldots, x_n = \frac{\det A_n}{\det A}.$$

with $A_j$ is the matrix obtained by replace the $j^{\text{th}}$ column in the matrix $A$ by the column matrix $B$.

# Example

Use Crammer method to solve the following system:

$$\begin{cases} 3x - 2z = 2 \\ -2x + 3y - 2z = 3 \\ -5x + 4y - z = 1 \end{cases}$$

$$\begin{vmatrix} 3 & 0 & -2 \\ -2 & 3 & -2 \\ -5 & 4 & -1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 0 & -2 \\ 3 & 3 & -2 \\ 1 & 4 & -1 \end{vmatrix} = -8 = x,$$

$$\begin{vmatrix} 3 & 2 & -2 \\ -2 & 3 & -2 \\ -5 & 1 & -1 \end{vmatrix} = -13 = y, \qquad \begin{vmatrix} 3 & 0 & 2 \\ -2 & 3 & 3 \\ -5 & 4 & 1 \end{vmatrix} = -13 = z.$$