Techniques of Integration

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October 6, 2021

Integration By Parts

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x).$$

Integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}(u(x)v(x))dx = \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)dx + \int_a^b v(x)\left(\frac{du(x)}{dx}\right)dx.$$

Using the fundamental theorem of calculus, we get

$$[u(x)v(x)]_a^b = (u(b)v(b) - u(a)v(a)) = \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx.$$

Theorem

(Integration by Parts)
If u and v are two continuously differentiable functions on the interval $[a, b]$, then we have

$$\int_a^b u(x)v'(x)dx = (u(b)v(b) - u(a)v(a)) - \int_a^b v(x)u'(x)dx$$

and for the indefinite integrals

$$\int udv = uv - \int vdu.$$

Example

Using integration by parts, we have

1. $\displaystyle\int \ln(x)dx \overset{u=\ln(x),v'=1}{=} x\ln(x) - x + c.$
2. $\displaystyle\int xe^x dx \overset{u=x,v'=e^x}{=} xe^x - e^x + c.$
3. $\displaystyle\int_0^{\pi} x\sin(x)\ dx \overset{u=x,v'=\sin(x)}{=} [-x\cos(x)]_0^{\pi} + \int_0^{\pi}\cos(x)\ dx = [-x\cos(x)]_0^{\pi} = \pi.$
4. $$\begin{aligned}\int e^x\sin(x)dx &\overset{u=e^x,v'=\sin(x)}{=} -e^x\cos(x) + \int e^x\cos(x)dx\\ &\overset{u=e^x,v'=\cos(x)}{=} -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)dx\\ &= \frac{e^x}{2}(\sin(x) - \cos(x)) + c.\end{aligned}$$

5.

$$\begin{aligned}\int \cosh(x)\cos(x)dx &\overset{u=\cosh(x),v'=\cos(x)}{=} \cosh(x)\sin(x) - \int \sinh(x)\sin(x)dx\\ &\overset{u=\sinh(x),v'=\sin(x)}{=} \cosh(x)\sin(x) + \sinh(x)\cos(x)\\ &\quad - \int \cosh(x)\cos(x)dx.\end{aligned}$$

Then $\int \cosh(x)\cos(x)dx = \frac{1}{2}(\sin(x)\cosh(x) + \cos(x)\sinh(x)) + c.$

6.

$$\begin{aligned}\int \sinh^{-1}(x)dx &\overset{u=\sinh^{-1}(x),v'=1}{=} x\sinh^{-1}(x) - \int \frac{x}{\sqrt{1+x^2}}dx\\ &= x\sinh^{-1}(x) - \sqrt{1+x^2} + c.\end{aligned}$$

7.

$$\int \sin^{-1}(x)dx \overset{u=\sin^{-1}(x), v'=1}{=} x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= x\sin^{-1}(x) + \sqrt{1-x^2} + c.$$

8.

$$\int \tan^{-1}(x)dx \overset{u=\tan^{-1}(x), v'=1}{=} x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx$$
$$= x\tanh^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c.$$

Integrals of Type $\int \cos^n(x)\sin^m(x)dx,\ m,n \in \mathbb{N}$

1. If $m = 2q+1$, we set $u = \cos(x)$, then $du = -\sin(x)dx$ and

$$\begin{aligned}
\int \cos^n(x)\sin^{2q+1}(x)dx &= \int \cos^n(x)\sin^{2q}(x).\sin(x)dx \\
&= \int \cos^n(x)\left(\sin^2(x)\right)^q.\sin(x)dx \\
&= -\int \cos^n(x)\left(1-\cos^2(x)\right)^q.(-\sin(x))dx \\
&= -\int u^n\left(1-u^2\right)^q du.
\end{aligned}$$

2. If $n = 2p + 1$, we set $u = \sin(x)$, then $du = \cos(x)dx$ and

$$\begin{aligned}\int \cos^{2p+1}(x)(x)\sin^m(x)dx &= \int \cos^{2p}(x)\sin^m(x).\cos(x)dx \\ &= \int \left(\cos^2(x)\right)^p \sin^m(x).\cos(x)dx \\ &= \int \left(1-\sin^2(x)\right)^p \sin^m(x).\cos(x))dx \\ &= \int \left(1-u^2\right)^p u^m du.\end{aligned}$$

3. If $n = 2p$ and $m = 2q$,

$$\int \cos^{2p}(x)(x)\sin^{2q}(x)dx = \int \cos^{2p}(x)(1-\cos^2(x))^q dx.$$

We compute the integral $J_n = \int \cos^{2n}(x)dx$ by induction and by parts:

Example

1. $$\begin{aligned}\int \sin^3(x)dx &\overset{u=\cos(x)}{=} -\int (1-u^2)du \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + c,\end{aligned}$$

2. $\int \sin^5(x)\cos^4(x)dx \overset{u=\cos(x)}{=} -\int u^4(1-u^2)^2 du = -\frac{\cos^5(x)}{5} - \frac{\cos^9(x)}{9} + \frac{2\cos^7(x)}{7} + c,$

3. $\int \sqrt{\sin(x)}\ \cos^3(x)dx \overset{u=\sin(x)}{=} \int u^{\frac{1}{2}}(1-u^2)\ du = \frac{2}{3}\sin^{\frac{3}{2}}(x) - \frac{2}{7}\sin^{\frac{7}{2}}(x) + c,$

4. $$\int \cos^2(x)dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + c = \frac{\sin(2x)}{4} + \frac{x}{2} + c,$$

5. $$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (1-\cos^2(x))\cos^2(x)dx \\ &= \frac{\sin(x)\cos(x)}{8} + \frac{x}{8} - \frac{\sin(x)\cos^3(x)}{4} + c.\end{aligned}$$

Integrals of Type $\int \sec^m(x)\tan^n(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \tan(x)$, then $du = \sec^2(x)dx$ and

$$\int \sec^{2q}(x)\tan^n(x)dx = \int u^n(1+u^2)^{q-1}du.$$

2. If $m = 0$,

$$\int \tan(x)dx = \ln|\sec(x)| + c.$$

$$\int \tan^2(x)dx = \int (\sec^2(x) - 1)dx = \tan(x) - x + c.$$

For $n \geq 3$,

$$\begin{aligned} L_n = \int \tan^n(x)dx &= \int \tan^{n-2}(x)\tan^2(x)dx \\ &= \int \tan^{n-2}(x)\sec^2(x)dx - L_{n-2} \\ &= \frac{\tan^{n-1}(x)}{n-1} - L_{n-2}. \end{aligned}$$

3. If $m \neq 0$, $n = 2p + 1$, we set $u = \sec(x)$, then $du = \sec(x)\tan(x)dx$.

$$\int \sec^m(x)\tan^{2p+1}(x)dx = \int u^{m-1}(u^2 - 1)^p du.$$

4. If $m = 2q + 1$ and $n = 2p$, the result is obtained using integration by parts and induction.

Example

1. $\displaystyle\int \sec^4(x)dx \overset{u=\tan(x)}{=} \int (1+u^2)du = \tan(x) + \frac{\tan^3(x)}{3} + c;$
2. $\displaystyle\int \sec^4(x)\tan^7(x)dx \overset{u=\tan(x)}{=} \int u^7(1+u^2)du = \frac{\tan^8(x)}{8} + \frac{\tan^{10}(x)}{10} + c;$
3. $\displaystyle\int \tan^3(x)dx = \int \tan(x)(\sec^2(x)-1)dx =$
$\displaystyle\frac{\tan^2(x)}{2} - \ln|\sec(x)| + c;$

4. $\displaystyle\int \tan^3(x)\ \sec^3(x)dx \overset{u=\sec(x)}{=} \int (u^2-1)u^2\ du = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + c;$
5. By integration by parts, $u(x)=\sec(x)$, $v'(x)=\sec^2(x)$, we get

$$\begin{aligned}\int \sec^3(x)dx &= \int \sec(x)\sec^2(x)dx\\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx\\ &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx;\end{aligned}$$

Therefore

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

Integrals of Type $\int \csc^m(x)\cot^n(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \cot(x)$, then $du = -\csc^2(x)dx$ and
$$\int \csc^{2q}(x)\cot^n(x)dx = -\int u^n(1+u^2)^{q-1}du.$$
2. If $m = 0$, $\int \cot(x)dx = \ln|\sin(x)| + c$.
$$\int \cot^2(x)dx = \int (\csc^2(x) - 1)dx = -\cot(x) - x + c.$$
For $n \geq 3$,
$$\begin{aligned} L_n = \int \cot^n(x)dx &= \int \cot^{n-2}(x)\cot^2(x)dx \\ &= \int \cot^{n-2}(x)\csc^2(x)dx - L_{n-2} = -\frac{\cot^{n-1}(x)}{n-1} - L_{n-2}. \end{aligned}$$

3. If $m \neq 0$ and $n = 2p + 1$, we set $u = \csc(x)$, then $du = -\csc(x)\cot(x)dx$

$$\int \csc^m(x)\cot^{2p+1}(x)dx = -\int u^{m-1}(u^2 - 1)^p du.$$

4. If $m = 2q + 1$ and $n = 2p$. The result is obtained by integration by parts and induction.

Example

1. $\int \csc(x)dx = \ln|\csc(x) - \cot(x)| + c,$
2. Let $u = \csc(x)$, $v' = \csc^2(x)$,
$\int \csc^3(x)dx = -\csc(x)\cot(x) + \ln|\csc(x) - \cot(x)| - \int \csc^3(x)dx,$
then
$\int \csc^3(x)dx = \frac{1}{2}\left(-\csc(x)\cot(x) + \ln|\csc(x) - \cot(x)|\right) + c.$

3.

$$\begin{aligned}\int \csc^4(x)\ \cot^4(x)dx &\overset{u=\cot(x)}{=} -\int (1+u^2)u^4\ du = -\int (u^4+u^6)\ du \\ &= -\frac{\cot^5(x)}{5} - \frac{\cot^7(x)}{7} + c.\end{aligned}$$

Integrals of Type $\int \sin(ax)\cos(bx)dx$, $\int \sin(ax)\sin(bx)dx$, $\int \cos(ax)\cos(bx)dx$

1. $\int \sin(ax)\sin(bx)dx = \frac{1}{2}\int \cos((a-b)x) - \cos((a+b)x)dx,$
2. $\int \sin(ax)\cos(bx)dx = \frac{1}{2}\int \sin((a+b)x) + \sin((a-b)x)dx,$
3. $\int \cos(ax)\cos(bx)dx = \frac{1}{2}\int \cos((a+b)x) + \cos((a-b)x)dx,$

   For $a, b \in \mathbb{R}$ such that $|a| \neq |b|$, we have
4. $2\int \sin(ax)\sin(bx)dx = \frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b} + c,$
5. $2\int \cos(ax)\cos(bx)dx = \frac{\sin((a+b)x)}{a+b} + \frac{\sin((a-b)x)}{a-b} + c,$
6. $2\int \sin(ax)\cos(bx)dx = -\frac{\cos((a+b)x)}{a+b} - \frac{\cos((a-b)x)}{a-b} + c,$

Example

1.
$$\begin{aligned}\int \sin(5x)\sin(3x)dx &= \frac{1}{2}\int \cos(2x) - \cos(8x)dx \\ &= \frac{\sin(2x)}{4} - \frac{\sin(8x)}{16} + c.\end{aligned}$$

2.
$$\begin{aligned}\int \cos(5x)\ \cos(2x)dx &= \int \frac{1}{2}[\cos(7x) + \cos(3x)]dx \\ &= \frac{1}{14}\sin(7x) + \frac{1}{6}\sin(3x) + c.\end{aligned}$$

Integrals of Type $\int \sinh(ax)\cosh(bx)dx$, $\int \sinh(ax)\sinh(bx)dx$, $\int \cosh(ax)\cosh(bx)dx$

$$\int \sinh(ax)\sinh(bx)dx = \frac{1}{2}\int \cosh((a+b)x) - \cosh((a-b)x)dx,$$

$$\int \sinh(ax)\cosh(bx)dx = \frac{1}{2}\int \sinh((a+b)x) + \sinh((a-b)x)dx,$$

$$\int \cosh(ax)\cosh(bx)dx = \frac{1}{2}\int \cosh((a+b)x) + \cosh((a-b)x)dx,$$

Example

1. $$\begin{aligned}\int \sinh(5x)\sinh(3x)dx &= \frac{1}{2}\int \cosh(8x) - \cosh(2x)dx \\ &= \frac{1}{16}\sinh(8x) - \frac{1}{4}\sinh(2x) + c,\end{aligned}$$

2. $$\begin{aligned}\int \sinh(4x)\cosh(3x)dx &= \frac{1}{2}\int \sinh(7x) + \sinh(x)dx \\ &= \frac{1}{2}\cosh(x) + \frac{1}{14}\cosh(7x) + c,\end{aligned}$$

Integrals of Type $\int \cosh^n(x)\sinh^m(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q + 1$, we set $u = \cosh(x)$, then $du = \sinh(x)dx$ and

$$\begin{aligned}\int \cosh^n(x)\sinh^{2q+1}(x)dx &= \int \cosh^n(x)\sinh^{2q}(x)\sinh(x)dx \\ &= \int \cosh^n(x)\Big(\cosh^2(x)-1\Big)^q \sinh(x)dx \\ &= \int u^n\Big(u^2-1\Big)^q du.\end{aligned}$$

2. If $n = 2p+1$, we set $u = \sinh(x)$, then $du = \cosh(x)dx$ and

$$\begin{aligned}
\int \cosh^{2p+1}(x)(x)\sinh^m(x)dx &= \int \cosh^{2p}(x)\sinh^m(x)\cosh(x)dx \\
&= \int \Big(1+\sinh^2(x)\Big)^p \sinh^m(x)\cosh(x))dx \\
&= \int \left(1+u^2\right)^p u^m du.
\end{aligned}$$

3. If $n = 2p$ and $m = 2q$,

$$\int \cosh^{2p}(x)(x)\sinh^{2q}(x)dx = \int \cosh^{2p}(x)(\cosh^2(x) - 1)^q dx.$$

We compute the integral $I_n = \displaystyle\int \cosh^{2n}(x)dx$ by induction and by parts: We set $u = \cosh^{2n-1}(x)$ and $v' = \cosh(x)$, then

$$\begin{aligned} I_n &= \sinh(x)\cosh^{2n-1}(x) - (2n-1)\int \cosh^{2n-2}(x)\sinh^2(x)dx \\ &= \sinh(x)\cosh^{2n-1}(x) - (2n-1)I_n + (2n-1)I_{n-1}. \end{aligned}$$

Thus $I_n = \frac{1}{2n}\sinh(x)\cosh^{2n-1}(x) - \frac{2n-1}{2n}I_{n-1}$.

In particular

$$I_1 = \int \cosh^2(x)dx = \frac{\sinh(x)\cosh(x)}{2} - \frac{x}{2} + c = \frac{\sinh(2x)}{4} - \frac{x}{2} + c.$$

$$I_2 = \int \cosh^4(x)dx = \frac{\sinh(x)\cosh^3(x)}{4} - \frac{3\sinh(x)\cosh(x)}{8} + \frac{3x}{8} + c.$$

Example

1. $$\int \sinh^5(x)\cosh^4(x)dx \overset{u=\cosh(x)}{=} \int u^4(u^2-1)^2 du = \frac{\cosh^5(x)}{5} + \frac{\cosh^9(x)}{9} - \frac{2\cosh^7(x)}{7} + c,$$
2. $$\int \sinh^4(x)\cosh^3(x)dx \overset{u=\sinh(x)}{=} \int u^4(u^2-1)du = \frac{\sinh^7(x)}{7} - \frac{\sinh^5(x)}{5} + c.$$
3. $$\int \sqrt{\sinh(x)}\ \cosh^3(x)dx \overset{u=\sinh(x)}{=} \int u^{\frac{1}{2}}(u^2-1)\ du = -\frac{2(\sinh(x))^{\frac{3}{2}}}{3} + \frac{2(\sinh(x))^{\frac{7}{2}}}{7} + c,$$

Integrals of Type $\int \operatorname{sech}^m(x)\tanh^n(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \tanh(x)$, then $du = \operatorname{sech}^2(x)dx$ and

$$\int \operatorname{sech}^{2q}(x)\tanh^n(x)dx = \int u^n(1-u^2)^{q-1}du.$$

2. If $m = 0$,

$$\int \tanh(x)dx = \ln|\cosh(x)| + c.$$

$$\int \tanh^2(x)dx = \int (1-\operatorname{sech}^2(x))dx = x - \tanh(x) + c.$$

For $n \geq 3$,

$$\begin{aligned} L_n = \int \tanh^n(x)dx &= \int \tanh^{n-2}(x)\tanh^2(x)dx \\ &= L_{n-2} - \int \tanh^{n-2}(x)\mathrm{sech}^2(x)dx \\ &= L_{n-2} - \frac{\tanh^{n-1}(x)}{n-1}. \end{aligned}$$

3. If $m = 2q+1$ and $n = 2p+1$, we set $u = \mathrm{sech}(x)$, then $du = -\mathrm{sech}(x)\tanh(x)dx$.

$$\int \mathrm{sech}^{2q+1}(x)\tanh^{2p+1}(x)dx = -\int u^{2q}(1-u^2)^p du.$$

4. If $m = 2q + 1$ and $n = 2p$. The result is obtained using integration by parts and induction.

Example

1. $\int \tanh^4(x)\text{sech}^2(x)dx = \int (\tanh(x))^4\text{sech}^2(x)dx = \frac{\tanh^5(x)}{5} + c$

2. $$\begin{aligned}\int \tanh^3(x)\,\text{sech}(x)dx &= \int \tanh^2(x)\,\text{sech}(x)\tanh(x)dx \\ &= \int (1-\text{sech}^2(x))\,\text{sech}(x)\tanh(x)dx \\ &\overset{u=\text{sech}(x)}{=} \int \tanh^3(x)\,\text{sech}(x)dx \\ &= -\text{sech}(x) + \frac{\text{sech}^3(x)}{3} + c,\end{aligned}$$

Integrals of Type $\int \operatorname{csch}^m(x)\coth^n(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \coth(x)$, then $du = -\operatorname{csch}^2(x)dx$ and

$$\int \operatorname{csch}^{2q}(x)\coth^n(x)dx = -\int u^n(u^2 - 1)^{q-1}du.$$

2. If $m = 0$,

$$\int \coth(x)dx = \ln|\cosh(x)| + c.$$

$$\int \coth^2(x)dx = \int (\operatorname{csch}^2(x) + 1)dx = -\coth(x) + x + c.$$

For $n \geq 3$,

$$\begin{aligned} L_n = \int \coth^n(x)dx &= \int \coth^{n-2}(x)\coth^2(x)dx \\ &= \int \coth^{n-2}(x)\operatorname{csch}^2(x)dx + L_{n-2} \\ &= -\frac{\coth^{n-1}(x)}{n-1} + L_{n-2}. \end{aligned}$$

3. If $m = 2q+1$ and $n = 2p+1$, we set $u = \operatorname{csch}(x)$, then $du = -\operatorname{csch}(x)\coth(x)dx$.

$$\int \operatorname{csch}^{2q+1}(x)\coth^{2p+1}(x)dx = -\int u^{2q}(u^2+1)^p du.$$

4. If $m = 2q + 1$ and $n = 2p$. The result is obtained using integration by parts and induction.

In this section, we study the integrals of the form $\int F(x)dx$, where F is a rational function:

$$F(x) = \frac{P(x)}{Q(x)}, \; P, \; Q \in \mathbb{R}[X].$$

We shall describe a method for computing this type of integrals. The method is to decompose a given rational function into a sum of simpler fractions (called partial fractions) which is easier to integrate.

The Irreducible Polynomials in $\mathbb{R}[X]$

Definition

1. A rational function has the form $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.
2. A rational function is called proper if the degree of the numerator is less than the degree of the denominator, and improper otherwise.
3. A rational function $R(x) = \dfrac{P(x)}{Q(x)}$ is called irreducible if there is no polynomial $S \in \mathbb{R}[X]$ which divide P and Q.

Remark

Let $F = \dfrac{P}{Q}$ be rational functions with $\deg Q \leq \deg P$, then by using polynomial long division, there are two polynomials R and S such that

$$\frac{P}{Q} = R + \frac{S}{Q},$$

and $\deg S < \deg Q$. (i.e. the rational function $\dfrac{S(x)}{Q(x)}$ is proper).

For example $\dfrac{x^4 + 5x^2 + 3}{x^3 - x} = x + \dfrac{6x^2 + 3}{x^3 - x}$.

In what follows, the rational functions are taken irreducible and proper.

Definition

1. The irreducible linear polynomials are the polynomials of the form
$$R(x) = x - \alpha, \quad \alpha \in \mathbb{R}.$$
2. The irreducible quadratic polynomials are the polynomials of the form
$$R(x) = ax^2 + bx + c, \quad a,\ b,\ c \in \mathbb{R} : \ b^2 - 4ac < 0.$$

Example

1. $x^2 + 9$ and $x^2 + x + 1$ are examples of irreducible quadratic polynomials.
2. $x^2 - x$ and $x^2 - 1$ are reducible quadratics polynomials since $x^2 - x = x(x-1)$ and $x^2 - 1 = (x-1)(x+1)$.

Theorem

The only irreducible polynomials in $\mathbb{R}[X]$ are the irreducible linear polynomials and the irreducible quadratic polynomials.
Any polynomial $Q \in \mathbb{R}[X]$ has the following decomposition:

$$Q(x) = \prod_{j=1}^{m} L_j^{m_j}(x) \prod_{k=1}^{n} Q_k^{n_k}(x),$$

where $L_j(x) = a_j x + b_j$ and $Q_k(x) = c_k x^2 + d_k x + e_k$, where $a_k \neq 0$, $d_k^2 - 4c_k e_k < 0$ for all $j = 1, \dots, m$ and $k = 1, \dots, n$.

Example

1. $$\frac{3x+11}{x^2-2x-3} = \frac{5}{x-3} - \frac{2}{x-1}.$$
2. $$\begin{aligned} \frac{2x^2-5x-8}{x^3-x^2-8x+12} &= \frac{2x^2-5x-8}{(x-2)^2(x+3)} \\ &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3} \\ &= \frac{1}{x-2} - \frac{2}{(x-2)^2} + \frac{1}{x+3} \end{aligned}$$

3.

$$\begin{aligned}\frac{3x+1}{(x^2+x+2)(x+3)} &= \frac{Ax+B}{x^2+x+2}+\frac{C}{x+3}\\ &= \frac{x+1}{x^2+x+2}-\frac{1}{x+3}\end{aligned}$$

4.

$$\begin{aligned}\frac{1}{(x-1)(x+1)(x^2-x+1)} &= \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}\\ &= \frac{1}{2(x-1)}-\frac{1}{6(x+1)}-\frac{1}{3}\frac{x+1}{x^2-x+1}.\end{aligned}$$

5. $$\frac{2x-1}{(x+2)^2(x-3)} = \frac{-1}{5(x+2)} + \frac{1}{(x+2)^2} + \frac{1}{5(x-3)}.$$
6. $$\frac{x+3}{(x^2-1)(x+5)} = \frac{1}{3}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} + \frac{1}{12}\frac{1}{x+5}.$$
7. $$\frac{2x-2}{(x^2+x+4)(x+2)} = \frac{x+1}{x^2+x+4} - \frac{1}{x+2}.$$
8. $$\frac{x^2+1}{x^4+4x^2} = \frac{x^2+1}{x^2(x^2+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x+C_1}{x^2+4} = \frac{1}{4(x^2)} - \frac{1}{4(x^2+4)}.$$

Integral of Rational Functions

1.
$$\begin{aligned}\int \frac{6x^2+3}{x(x-1)(x+1)}dx &= -3\int \frac{dx}{x} + \frac{9}{2}\int \frac{dx}{x-1} + \frac{9}{2}\int \frac{dx}{x+1} \\ &= -3\ln|x| + \frac{9}{2}\ln|x-1| + \frac{9}{2}\ln|x+1| + c.\end{aligned}$$

2.
$$\begin{aligned}\int \frac{x^4+5x^2+3}{x^3-x}dx &= \int \left(x - \frac{3}{x} + \frac{9}{2(x-1)} + \frac{9}{2(x+1)}\right)dx \\ &= \frac{x^2}{2} - 3\ln|x| + \frac{9}{2}\ln|x-1| + \frac{9}{2}\ln|x+1| + c.\end{aligned}$$

Case of Irreducible Quadratic Factor of Denominator:

1.
$$\begin{aligned}\int \frac{8}{(x^2+1)(x^2+9)}dx &= \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+9}\\ &= \tan^{-1}(x) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c,\end{aligned}$$

2.
$$\begin{aligned}\int \frac{2x+5}{x^2+x+1}dx &= \int \frac{2x+1}{x^2+x+1}dx + \int \frac{4}{x^2+x+1}dx\\ &= \ln(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\\ &= \ln(x^2+x+1) + \frac{8}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c,\end{aligned}$$

Integral Involving $\sqrt{a^2 - x^2}$

We use the substitution $x = a\sin(\theta)$ where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ to solve the integral. We can also take $x = a\cos(\theta)$, with $\theta \in]0, \pi[$.

Integral Involving $\sqrt{a^2 + x^2}$

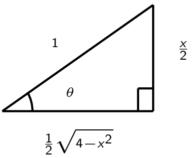
We use the substitution $x = a\tan(\theta)$ where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ to solve the integral. We can also use the substitution $x = a\sinh(t)$, with $t \in \mathbb{R}$ to solve the integral.

Integral Involving $\sqrt{x^2 - a^2}$

We use the substitution $x = a\sec(\theta)$ where $\theta \in \left[0, \frac{\pi}{2}\right[$ to solve the integral. We can also use the substitution $x = a\cosh(t)$, with $t \in]0, +\infty[$ to solve the integral.

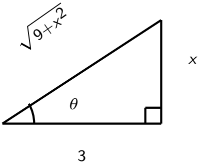
Example

1.
$$\begin{aligned}\int \frac{dx}{x\sqrt{4-x^2}} &= \int \frac{d\theta}{2\sin(\theta)} \quad (x = 2\sin(\theta))\\ &= -\frac{1}{2}\ln(\csc(\theta)+\cot(\theta)) + c\\ &= -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x}) + c.\end{aligned}$$

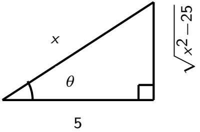


We know that
$$\int \frac{dx}{x\sqrt{4-x^2}} = -\frac{1}{2}\mathrm{sech}^{-1}(\frac{x}{2}) + c = -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x}) + c.$$

2.
$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+9}} &= \int \frac{\sec(\theta)}{9\tan^2(\theta)}d\theta \quad (x = 3\tan(\theta))\\ &= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta\\ &= -\frac{\csc(\theta)}{9} + c = -\frac{\sqrt{9+x^2}}{9x} + c.\end{aligned}$$



3.
$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-25}} &= \int \frac{d\theta}{25\sec(\theta)} \quad (x = 5\sec(\theta)) \\ &= \frac{1}{25}\int \cos(\theta)d\theta \\ &= \frac{1}{25}\sin(\theta) + c = \frac{\sqrt{x^2-25}}{25x} + c.\end{aligned}$$



Integrals Involving Quadratic Expressions

1. $x^2+2x+5=(x^2+2x+1)-1+5=(x+1)^2+4.$
2. $x^2+x+1=(x^2+2.\frac{1}{2}.x+(\frac{1}{2})^2)-(\frac{1}{2})^2+1,=(x+\frac{1}{2})^2+\frac{3}{4}.$
3. $x^2-x+1=(x^2-2.\frac{1}{2}.x+(-\frac{1}{2})^2)-(-\frac{1}{2})^2+1=(x-\frac{1}{2})^2+\frac{3}{4}.$
4. $x^2+5x=(x^2+2.\frac{5}{2}.x+(\frac{5}{2})^2)-(\frac{5}{2})^2=(x+\frac{5}{2})^2-\frac{25}{4}.$

Example

We use completing the square method to evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{x^2-2x+2} \overset{u=x-1}{=} \int \frac{du}{u^2+1} = \tan^{-1}(u)+c = \tan^{-1}(x-1)+c.$

2. $$\begin{aligned}\int \frac{1}{\sqrt{7+6x-x^2}}dx &= \int \frac{1}{\sqrt{16-(x-3)^2}}dx \\ &\overset{u=x-3}{=} \int \frac{1}{\sqrt{4^2-u^2}}du \\ &= \sin^{-1}\left(\frac{u}{4}\right)+c = \sin^{-1}\left(\frac{x-3}{4}\right)+c.\end{aligned}$$

In this section, we treat the integrals of the following form

$$\int \frac{P\big(\cos(x), \sin(x)\big)}{Q\big(\cos(x), \sin(x)\big)} dx$$

where $P(X, Y)$ and $Q(X, Y)$ are two polynomial functions in X, Y.

Method: Generally we use the following substitution $u = \tan(\frac{x}{2})$, $\quad du = \frac{1}{2}\sec^2(\frac{x}{2})dx$, then $dx = \frac{2du}{1+u^2}$. We have

$$\sin(x) = \frac{2u}{1+u^2}, \qquad \cos(x) = \frac{1-u^2}{1+u^2}.$$

Example

1.

$$\begin{aligned}\int \frac{dx}{2+\sin(x)} &\overset{u=\tan(\frac{x}{2})}{=} \int \frac{1}{2+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du \\ &= \int \frac{du}{u^2+u+1} = \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan(\frac{x}{2})+1}{\sqrt{3}}\right) + c.\end{aligned}$$

2.

$$\int \frac{1}{2+\cos(x)}dx \overset{u=\tan\left(\frac{x}{2}\right)}{=} \int \frac{1}{2+\left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{3+u^2}\, du = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c.$$

Integrals Involving Fraction Powers of x

Evaluation of the integrals

1.

$$\begin{aligned}\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx &\overset{x=u^6}{=} \int \frac{u^3}{1+u^2}6u^5du = 6\int \frac{u^8}{1+u^2}du \\ &= 6\int u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}du \\ &= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6\tan^{-1}(x^{\frac{1}{6}}) + c.\end{aligned}$$

2

$$\int \frac{2x+3}{\sqrt{1+2x}}dx \overset{x=\frac{u-1}{2}}{=} \frac{1}{2}\int \frac{2(\frac{u-1}{2})+3}{u^{\frac{1}{2}}}du = \frac{1}{2}\int \frac{u+2}{u^{\frac{1}{2}}}du = \frac{1}{3}(1+2x)^{\frac{3}{2}} + (1+2x)^{\frac{1}{2}} + c.$$

Therefore

$$\int_0^4 \frac{2x+3}{\sqrt{1+2x}}dx = \left[\frac{1}{3}(1+2x)^{\frac{3}{2}} + (1+2x)^{\frac{1}{2}}\right]_0^4 = \frac{32}{3}.$$

3.
$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx \overset{u=x^{\frac{1}{6}}}{=} \int \frac{6u^5}{u^3+u^2}\ du = \int \frac{6u^3}{u+1}\ du.$$

Using long division of polynomials,

$$\begin{aligned}\int \frac{6u^3}{u+1}\ du &= \int \left(6u^2 - 6u + 6 - \frac{6}{u+1)}\right)\ du \\ &= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c.\end{aligned}$$

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln\left|x^{\frac{1}{6}}+1\right| + c.$$

4.

$$\begin{aligned}
\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}}+1}dx &\overset{u=x^{\frac{1}{6}}}{=} \int \frac{u\ 6u^5}{u^2+1}\ du = \int \frac{6u^6}{u^2+1}\ du \\
&= \int \left(6u^4 - 6u^2 + 6 - \frac{6}{u^2+1}\right)\ du \\
&= \frac{6u^5}{5} - 2u^3 + 6u - 6\tan^{-1} u + c \\
&= \frac{6}{5}x^{\frac{5}{6}} - 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} - 6\tan^{-1}\left(x^{\frac{1}{6}}\right) + c.
\end{aligned}$$

Definition

Let f be a continuous function on the interval $[a, b[$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.
We say that the integral of f on the interval $[a, b[$ is convergent if the function

$$F(x) = \int_a^x f(t)dt$$

defined on $[a, b[$ has a finite limit when x tends to b $(x < b)$. This limit is called the improper integral of f on $[a, b[$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

Example

1. $$\int_0^1 \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx = \lim_{t\to 1^-}\int_0^t \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx \overset{u=\sin^{-1}(x)}{=} \lim_{t\to 1^-}\int_0^{\sin^{-1}(t)} e^u du = e^{\frac{\pi}{2}} - 1.$$

   This integral is convergent.

2. $\displaystyle\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-(x-1)^2}}dx = \sin^{-1}(x-1)\Big]_0^1 = \frac{\pi}{2}$. The integral is convergent.

3. $$\begin{aligned}\int_0^\infty xe^{-x}\,dx &= \lim_{t\to\infty}\left([-xe^{-x}]_0^t - \int_0^t -e^{-x}\,dx\right)\\ &= \lim_{t\to\infty}\left([-xe^{-x}]_0^t - [e^{-x}]_0^t\right)\\ &= \lim_{t\to\infty}\left([-te^{-t}] - [(e^{-t}-1]\right) = 1.\end{aligned}$$

This integral is convergent.

4. $\displaystyle\int_0^{+\infty} \frac{x}{1+x^2}dx = \lim_{t\to+\infty}\int_0^t \frac{x}{1+x^2}dx = \lim_{t\to+\infty}\frac{1}{2}\ln(1+t^2) = +\infty.$ This integral is divergent.
5. $\displaystyle\int_0^{+\infty} x^n e^{-x}dx = \lim_{t\to+\infty}\int_0^t x^n e^{-x}dx.$ By induction we prove that $\displaystyle\int_0^{+\infty} x^n e^{-x}dx = n!.$ This integral is convergent.

6. $\int_1^{+\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to+\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to+\infty} \frac{(\ln t)^2}{2} = +\infty.$

Therefore, $\int_1^{\infty} \frac{\ln x}{x}\, dx$ diverges.

Definition

Let f be a continuous function on the interval $]a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$.
We say that the integral of f on the interval $]a, b]$ is convergent if the function $G(x) = \int_x^b f(t)dt$ defined on $]a, b]$ has a finite limit when x tends to a $(x > a)$. This limit is called the improper integral of f on $]a, b]$ and will be denoted by: $\int_a^b f(x)dx$.

Example

1. $\displaystyle\int_0^1 \ln(x)dx = \lim_{t\to 0^+}\int_t^1 \ln(x)dx = \lim_{t\to 0^+}[x\ln(x) - x]_t^1 = -1.$
This integral is convergent.
2. $\displaystyle\int_{-\infty}^0 \frac{dx}{(x-3)^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{(x-3)^2} = \lim_{t\to-\infty} = \left[\frac{-1}{x-3}\right]_t^0 = \frac{1}{3}.$ This integral is convergent.
3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2.$ This integral is convergent.

Definition

Let f be a continuous function on the interval $]a, b[$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$. We say that the integral of f on the interval $]a, b[$ is convergent if the integral of f is convergent on $]a, c]$ and on $[c, b[$ for any c in $]a, b[$.

Example

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\tan^{-1}(x)}}{1+x^2}dx \overset{u=\tan^{-1}(x)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^u du = 2\sinh(\frac{\pi}{2})$. This integral is convergent.
2. $\displaystyle\int_0^1 \frac{\ln x}{(1-x)^{\frac{3}{2}}}dx \overset{x=1-t^2}{=} 2\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = 2 - 2\ln 2$. This integral is convergent.
3. $\displaystyle\int_{-\infty}^{+\infty} e^x dx = [e^x]_{-\infty}^{+\infty} = +\infty$. This integral is divergent.

4. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi$. This integral is convergent.
5. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^a \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.
6. The integral $\displaystyle\int_0^{+\infty} \sin(x)dx$ is divergent since $\displaystyle\int_0^x \sin(t)dt = 1 - \cos(x)$ doesn't have a limit when x tends to $+\infty$.

7. The integral $\displaystyle\int_0^1 \frac{\sin(x)}{x}dx$ is convergent since the function $\dfrac{\sin(x)}{x}$ can be considered as a continuous function on the interval $[0, 1]$.