The Determinants

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Definition of The Determinant

Definition

If $A = (a_{j,k})$ is a square matrix of type n. Denote $A_{j,k}$ the square matrix of type $n-1$ obtained by deleting the $j^{\rm th}-$row and the $k^{\rm th}-$ column of A.

Example: If $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \\ 2 & -3 & 4 \end{pmatrix}$, then $A_{2,3} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$.

Definition

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is defined by:
$|A| = \det(A) = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = ad - bc.$
2. If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant of A is defined by:

$$|A| = \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Definition

3. If

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ & & \vdots & \\ a_{n,1} & a_{m,2} & \dots & a_{n,n} \end{pmatrix},$$

the determinant of A is defined by:

$$\begin{aligned} |A| = \det(A) &= a_{1,1}\det A_{1,1} + \ldots + (-1)^{n+1} a_{1,n} \det A_{1,n} \\ &= \sum_{j=1}^{n} (-1)^{j+1} a_{1,j} \det A_{1,j}. \end{aligned}$$

Example

1. If $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, the determinant of A is

$$|A| = \det(A) = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 4.3 - 5.2 = 2.$$

2. If $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{pmatrix}$, the determinant of the matrix A is

$$|A| = det(A) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + 5\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix}$$

Example

3. If $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{pmatrix}$, the determinant of the matrix A is

$$|A| = \det(A) = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{vmatrix} = 1 \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} - 3 \begin{vmatrix} 2 & 6 \\ 0 & 4 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} = 0.$$

Definition

If A is a square matrix of order n, the determinant $\det A_{j,k}$ is called the minor of the entry $a_{j,k}$ or the $(j,k)^{\mathrm{th}}$ minor of A and the number $C_{j,k} = (-1)^{j+k}\det A_{j,k}$ is called the cofactor of the entry $a_{j,k}$ or the $(j,k)^{\mathrm{th}}$ cofactor of the matrix A.

Remark

1. If A is a square matrix of order n, the determinant of the matrix A is equal to
$$\det A = \sum_{j=1}^{n} a_{1,j} C_{1,j}.$$
2. By rearranging the boundaries we conclude to
$$\begin{aligned} \det A &= \sum_{j=1}^{n} a_{1,j} C_{1,j} = \sum_{j=1}^{n} a_{k,j} C_{k,j} \\ &= \sum_{k=1}^{n} a_{k,j} C_{k,j}. \end{aligned}$$

The Sarrus's Theorem

If $n = 3$ and the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\left| \begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{matrix} \right| \begin{matrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{matrix}$$

$$\begin{aligned} \det A &= a_{1,1}(a_{2,2} \cdot a_{3,3} - a_{2,3} \cdot a_{3,2}) \\ &\quad - a_{1,2}(a_{2,1} \cdot a_{3,3} - a_{2,3} \cdot a_{3,1}) \\ &\quad + a_{1,3}(a_{2,1} \cdot a_{3,2} - a_{2,2} \cdot a_{3,1} \end{aligned}$$

Example

If $A = \begin{pmatrix} 3 & -4 & 0 \\ 0 & 7 & 6 \\ 2 & -6 & 1 \end{pmatrix}$,

$$\begin{aligned} \det A &= \left| \begin{array}{ccc|cc} 3 & -4 & 0 & 3 & -4 \\ 0 & 7 & 6 & 0 & 7 \\ 2 & -6 & 1 & 2 & -6 \end{array} \right| \\ &= 3 \cdot 7 \cdot 1 + (-4) \cdot 6 \cdot 2 - (-6) \cdot 6 \cdot 3 = 81. \end{aligned}$$

Properties of the Determinants

Theorem

1. If A is a square matrix, $\det A^T = \det A$.
2. If a square matrix A contains a zero row or column, then its determinant is 0.
3. If the matrix $A = (a_{j,k})_{1 \leq j,k \leq n}$ is upper (lower) triangular, then its determinant is equal to:
$$a_{1,1} \dots a_{n,n}.$$
4. If a square matrix A contains a row which is a multiple of a different row, then its determinant is 0.

Theorem

5. If a matrix B is obtained by multiplying a row of a matrix A by a number c, then $|B| = c|A|$ (i.e. $|c.R_j A| = c|A|$).
6. If a matrix B is obtained by interchanging two rows of a matrix A, then $\det B = -\det A$ (i.e. $|R_{j,k}A| = -|A|$).
7. If a matrix B is obtained by adding a multiple of a row to another row of a matrix A, then $\det B = \det A$. (i.e. $|cR_{j,k}A| = |A|$).

Example

$$\begin{vmatrix} 1 & 3 & 2 & 2 \\ 2 & 3 & 3 & 1 \\ 3 & 3 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix} \underset{(-1)R_{1,4}}{\overset{(-2)R_{1,2},(-3)R_{1,3}}{=}} \begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & -3 & -1 & -3 \\ 0 & -6 & -2 & -4 \\ 0 & -2 & -1 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & 1 & 3 \\ 6 & 2 & 4 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\overset{(-2)R_{1,2}}{=} -\begin{vmatrix} 3 & 1 & 3 \\ 0 & 0 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -2.$$

Example

$$\begin{vmatrix} 1 & -2 & 5 & -2 & -1 \\ -2 & 3 & -1 & 1 & 0 \\ 3 & -3 & 2 & 0 & -1 \\ 1 & -1 & 2 & 1 & -4 \\ 1 & -2 & 4 & -3 & 1 \end{vmatrix} \underset{-1R_{1,4},-1R_{1,5}}{\overset{2R_{1,2},-3R_{1,3}}{=}} \begin{vmatrix} 1 & -2 & 5 & -2 & -1 \\ 0 & -1 & 9 & -3 & -2 \\ 0 & 3 & -13 & 6 & 2 \\ 0 & 1 & -3 & 3 & -3 \\ 0 & 0 & -1 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 9 & -3 & -2 \\ 3 & -13 & 6 & 2 \\ 1 & -3 & 3 & -3 \\ 0 & -1 & -1 & 2 \end{vmatrix}$$

$$\overset{3R_{1,2},1R_{1,3}}{=} \begin{vmatrix} -1 & 9 & -3 & -2 \\ 0 & 14 & -3 & -4 \\ 0 & 6 & 0 & -5 \\ 0 & -1 & -1 & 2 \end{vmatrix}$$

$$
\begin{aligned}
&= \begin{vmatrix} 14 & -3 & -4 \\ 6 & 0 & -5 \\ 1 & 1 & -2 \end{vmatrix} \\
&= \begin{vmatrix} 14 & 6 & 1 \\ -3 & 0 & 1 \\ -4 & -5 & -2 \end{vmatrix} \\
&\overset{(-1)R_{1,2},2R_{1,3}}{=} \begin{vmatrix} 14 & 6 & 1 \\ -17 & -6 & 0 \\ 24 & 7 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -17 & -6 \\ 24 & 7 \end{vmatrix} | \\
&\overset{1R_{1,2}}{=} \begin{vmatrix} -17 & -6 \\ 7 & 1 \end{vmatrix} \\
&= 42 - 17 = 25.
\end{aligned}
$$

Example

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = -3.
\end{aligned}$$

Example

$$\begin{aligned}
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} \\
&= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\
&= (b-a)(c-a)(c-b).
\end{aligned}$$

Theorem

If A and B are in $\mathscr{M}_n(\mathbb{R})$, then

$$\det(AB) = \det A \, \det B.$$

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Remarks

1. If A is a square matrix of order n, then $|cA| = c^n|A|$.
2. Let A be a square matrix and B a row echelon form of A. Then there is a finite elementary matrices $E_1, \dots, E_m$ such that $E_1 \dots E_m A = B$.
   Moreover
   $$\det(E_1)\dots\det(E_m)\det(A) = \det(B).$$

The adjoint matrix

Definition

Let A be a square matrix. The adjoint matrix associated to the matrix A is $\mathrm{adj}(A) = (C_{j,k})^T$, where $(C_{j,k})$ is the cofactor matrix of A.

Theorem

Let A be a square matrix of order n, then

$$(\mathrm{adj}(A))A = A(\mathrm{adj}(A)) = (\det A)I_n.$$

Theorem

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}(A).$$

$$\det A^{-1} = \frac{1}{\det A}.$$

Example

1. $n = 2$, $A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$, $\det A = 5$, $\operatorname{adj}(A) = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$ and $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$.
2. $n = 3$, $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 3 \\ -1 & 0 & 2 \end{pmatrix}$, $\det A = -13$

   $$\operatorname{adj}(A) = \begin{pmatrix} -4 & -5 & -2 \\ -2 & 4 & -1 \\ 3 & -6 & -5 \end{pmatrix}^T = \begin{pmatrix} -4 & -2 & 3 \\ -5 & 4 & -6 \\ -2 & -1 & -5 \end{pmatrix}$$

   and $A^{-1} = \frac{1}{13}\begin{pmatrix} 4 & 2 & -3 \\ 5 & -4 & 6 \\ 2 & 1 & 5 \end{pmatrix}$.

Example

3. $n = 4,\ A = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$, $\det A = 24$

$$\mathrm{adj}(A) = 2\begin{pmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix},$$

and $A^{-1} = \dfrac{1}{12}\begin{pmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix}$

Exercises

Let A be the following matrix $A = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & -1 \\ 1 & -2 & 3 \end{pmatrix}$.

1. Find the matrix $\mathrm{adj}(A)$ and the determinant of the matrix A.
2. Find the inverse of the matrix A if it exists.

Example

Let A, B be matrices of size $(3,3)$ such that A is not invertible and $|B| = 2$.

Find $\left|A\mathrm{adj}(A) + 2B^{-1}\right|$.

$\left|A\mathrm{adj}(A) + 2B^{-1}\right| = \dfrac{8}{|B|} = 4.$

Exercises

Let A and B be the following matrices:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ -2 & 1 & 4 \end{pmatrix}$$

Find the number a such that $A^2 - AB + aI_3 = 0$ and find the inverse matrix of A.

If $\mathrm{adj}(A) = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$, find the matrix A.

We have $A\mathrm{adj}(A) = I_3$, then $A = |A|(\mathrm{adj}(A))^{-1}$ and $|A|\ |\mathrm{adj}(A)| = |A|^3$, then $|\mathrm{adj}(A)| = |A|^2$. Therefore $A = \sqrt{|\mathrm{adj}(A)|}(\mathrm{adj}(A))^{-1}$.

$|\mathrm{adj}(A)| = 1$, the $|A| = 1$ and $A = \begin{pmatrix} -1 & 3 & -4 \\ -2 & 5 & -6 \\ 3 & -7 & 9 \end{pmatrix}$

Exercises

1. Prove that if a matrix $A \in \mathscr{M}_n(\mathbb{R})$, $n \geq 2$ has an inverse, then
$$\mathrm{adj}(\mathrm{adj}(A)) = (\mathrm{det} A)^{n-2} A.$$
2. Prove that a matrix A has an inverse if and only if the matrix $\mathrm{adj}(A)$ has an inverse.

Solution

1. From the relation $A\mathrm{adj}(A) = |A|I_n$ we conclude that $|\mathrm{adj}(A)| = |A|^{n-1}$ and $(\mathrm{adj}(A))^{-1} = \frac{1}{|A|}A$ if $|A| \neq 0$.
   If the matrix A has an inverse, then $A^{-1} = \dfrac{1}{|A|}\mathrm{adj}(A)$.
   Let $B = \mathrm{adj}(A)$, then $B\mathrm{adj}(B) = |B|I_n = |A|^{n-1}I_n$ and $\mathrm{adj}(B) = |A|^{n-1}B^{-1} = |A|^{n-2}A$.
   $\mathrm{adj}(\mathrm{adj}(A) = (\det A)^{n-2}A$.
2. From the relation $A\mathrm{adj}(A) = |A|I_n$ we conclude that if the matrix A has an inverse then the matrix $\mathrm{adj}(A)$ has an inverse. Also from the same relation, if the matrix $\mathrm{adj}(A)$ has an inverse and the matrix A do not has an inverse, then $A\mathrm{adj}(A) = 0$ and $A\mathrm{adj}(A)(\mathrm{adj}(A))^{-1} = 0$.
   Then $A = 0$ this is absurd, because if $A = 0$ then $\mathrm{adj}(A) = 0$.