The Transcendental Functions

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The Natural Logarithmic Function

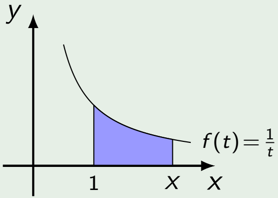
For $\alpha \in \mathbb{Q}$, the function $x \longmapsto x^\alpha$ is continuous on $(0, +\infty)$, then it is Riemann integrable on any interval $[a, b] \subset (0, +\infty)$.

For $\alpha \in \mathbb{Q}$, $\alpha \neq -1$, $\displaystyle\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha + 1} + c$.

Definition

For $x > 0$, the function $\ln(x) = \int_1^x \frac{dt}{t}$ represents the algebraic area of the region between the graph of the function $f(t) = \frac{1}{t}$, the $x-$ axis and the straight lines $t = 1$ and $t = x$.

The function $x \longmapsto \ln(x)$ is called **the Natural Logarithmic Function.**



Theorem

For all x, y in $]0, +\infty[$, we have

1. $\ln xy = \ln x + \ln y$.
2. $\ln \frac{1}{x} = -\ln x$.
3. $\ln x^n = n \ln x$, for all $n \in \mathbb{N}$.
4. $\ln x^r = r \ln x$, for all $r \in \mathbb{Q}$.

Example

Simplification of $\frac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right]$.

$$\begin{aligned}\frac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right] &= \frac{1}{5}\left[\ln|x(x+1)^2| - \ln|x^2-2|\right] \\ &= \ln\left|\left(\frac{x(x+1)^2}{x^2-2}\right)^{\frac{1}{5}}\right|\end{aligned}$$

Theorem

1. $\lim\limits_{h \to 0} \dfrac{\ln(1+h)}{h} = 1,$
2. $\lim\limits_{x \to +\infty} \ln x = +\infty,$
3. $\lim\limits_{x \to 0^+} \ln x = -\infty,$
4. $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x} = 0,$
5. $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x^s} = 0; \ \forall\ s \in \mathbb{Q}_+^*.$

corollary

The Logarithmic function $\ln :]0, +\infty[\longrightarrow \mathbb{R}$ is bijective. There exists a unique real number which will be denoted by e such that $\ln(e) = 1$, $(2 < e < 3)$, e is called the base of the Natural Logarithmic function. $(e \approx 2.71828)$

Remark

1. $\ln(x) > 0, \ \forall x > 1,$
2. $\ln(x) < 0, \ \forall 0 < x < 1,$
3. $\ln(x) = 0 \iff x = 1,$
4. $\dfrac{d^2}{dx^2}(\ln(x)) = -\dfrac{1}{x^2} > 0; \ \forall x > 0,$ (i.e. The function $x \longmapsto \ln(x)$ is concave on $(0, \infty)$).

The Logarithmic Differentiation

In some cases, the derivative of the function $\ln|f|$ is used to compute the derivative of f.

Theorem

(The Logarithmic Differentiation)
Let $u\colon I \longrightarrow \mathbb{R}\setminus\{0\}$ be a differentiable function, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$

Examples

1. $f(x) = \ln\left(\sqrt{\frac{4+x^2}{4-x^2}}\right) = \frac{1}{2}\ln(4+x^2) - \frac{1}{2}\ln(4-x^2)$. Then $f'(x) = \frac{1}{2}\frac{2x}{4+x^2} - \frac{1}{2}\frac{(-2x)}{4-x^2} = \frac{8x}{(4+x^2)(4-x^2)}$.
2. If $y = \sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}$, $\ln y = \frac{1}{2}\left[4\ln|x+1| + 3\ln|x+2| - 2\ln|x-1|\right]$.
   Differentiate both sides, we get $\frac{y'}{y} = \frac{1}{2}\left[\frac{4}{x+1} + \frac{3}{x+2} - \frac{2}{x-1}\right]$.
   Hence $y' = \frac{1}{2}\sqrt{\frac{(x+1)^4(x+2)^3}{(x-1)^2}}\left[\frac{4}{x+1} + \frac{3}{x+2} - \frac{2}{x-1}\right]$.

The Exponential Function

The natural logarithmic function $\ln\colon]0,+\infty[\longrightarrow \mathbb{R}$ is increasing and bijective, then it has an inverse function.

Definition

The natural exponential function is the inverse of the natural logarithmic function. It is denoted by e^x.

Properties

1. The exponential function is bijective and increasing.
2. $\frac{d}{dx}e^x = e^x$,
3. $e^{x+y} = e^x e^y$,
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$,
5. $\lim_{x \to -\infty} e^x = 0$,
6. $\lim_{x \to +\infty} e^x = +\infty$,
7. $\lim_{x \to +\infty} \frac{e^x}{x} = +\infty$,
8. $\lim_{x \to -\infty} xe^x = 0$.

corollary

If $u\colon I \longrightarrow \mathbb{R}$ is a differentiable function, then

$$\frac{d}{dx}(e^{u(x)}) = u'(x)e^{u(x)}.$$

Examples

1. $\dfrac{d}{dx}e^{1-x^2} = -2xe^{1-x^2}$.
2. $\dfrac{d}{dx}e^{x\ln(x)} = (\ln(x)+1)e^{x\ln(x)}$.
3. $\dfrac{d}{dx}\left(e^{5x}+\dfrac{1}{e^x}\right) = 5e^{5x} - e^{-x}$.
4. If $xe^y + 2x - \ln(y+1) = 3$, then using implicit differentiation, we get $e^y + xy'e^y + 2 - \dfrac{y'}{y+1} = 0$ and $y' = -\dfrac{2+e^y}{xe^y - \frac{1}{y+1}}$.

Integration Using “ln” and “exp” Functions

Theorem

Using the last properties of the logarithmic and exponential functions, we have

1. $$\int \frac{dx}{x} = \ln|x| + c,$$
2. $$\int e^x dx = e^x + c,$$
3. $$\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c,$$
4. $$\int u'(x) e^{u(x)} dx = e^{u(x)} + c.$$

Examples

Evaluation of the following integrals:

1. $$\int \frac{e^{-x}}{(1-e^{-x})^2}dx \overset{u=e^{-x}}{=} -\int \frac{du}{(1-u)^2} = \frac{-1}{(1-e^{-x})} + c,$$
2. $$\int \frac{e^{\frac{3}{x}}}{x^2}dx \overset{u=e^{\frac{3}{x}}}{=} -\frac{1}{3}\int du = -\frac{1}{3}e^{\frac{3}{x}} + c,$$
3. $$\int \frac{e^{\sin(x)}}{\sec(x)}dx = \int e^{\sin(x)}\cos(x)dx = e^{\sin(x)} + c,$$
4. $$\int e^{(x^2+\ln x)}dx = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c.$$

5. $\displaystyle\int_1^e \frac{\sqrt[3]{\ln x}}{x}\,dx \overset{u=\ln x}{=} \int_0^1 u^{\frac{1}{3}}\,du = \frac{3}{4}(\ln e)^{\frac{4}{3}} = \frac{3}{4}.$
6. $\displaystyle\int \tan(x)dx \overset{u=\cos x}{=} -\int \frac{du}{u} = -\ln|\cos(x)| + c =$ $\ln|\sec(x)| + c,$
7. $\displaystyle\int \cot(x)dx \overset{u=\sin x}{=} \int \frac{du}{u} = \ln|\sin(x)| + c,$
8. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}} \overset{u=\ln x}{=} \int u^{-\frac{1}{2}}\,du = 2(\ln x)^{\frac{1}{2}} + c.$

Theorem

$$\begin{aligned}
\int \sec(x)dx &= \ln|\sec(x)+\tan(x)| + c \\
\int \csc(x)dx &= \ln|\csc(x)-\cot(x)| + c \\
&= -\ln|\csc(x)+\cot(x)| + c.
\end{aligned}$$

The General Exponential Functions

Definition

For $a > 0$, the function $f(x) = e^{x\ln(a)}$ defined for $x \in \mathbb{R}$ is called the exponential function with base a and denoted by a^x.

Theorem

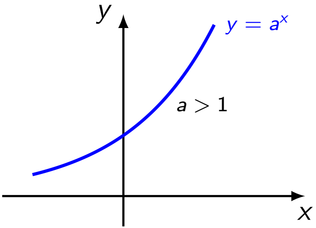
Let $a > 0$ and $b > 0$, x and y two real numbers, then

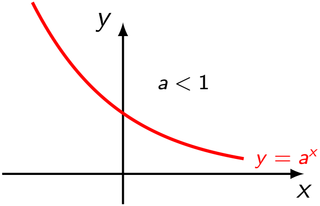
1. $a^{x+y} = a^x a^y$,
2. $a^{x-y} = \dfrac{a^x}{a^y}$,
3. $(a^x)^y = a^{xy}$,
4. $(ab)^x = a^x b^x$,
5. $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$,
6. $\dfrac{d}{dx}(a^x) = a^x \ln(a)$,
7. $\dfrac{d}{dx}(a^{u(x)}) = a^{u(x)} \ln(a) u'(x)$, if u is differentiable.

Properties

If $a > 1$, $\frac{d}{dx}(a^x) = a^x \ln(a) > 0$, and the function a^x is increasing on $\mathbb{R}$.

If $0 < a < 1$, $\frac{d}{dx}(a^x) = a^x \ln(a) < 0$, and the function a^x is decreasing on $\mathbb{R}$.





If $a > 0$ and $a \neq 1$, $\displaystyle\int a^u du = \frac{a^u}{\ln(a)} + c.$

Examples

1. $\frac{d}{dx}(5^x) = 5^x \ln(5),$
2. $\frac{d}{dx}(6^{\sqrt{x}}) = 6^{\sqrt{x}} \ln(6) \frac{1}{2\sqrt{x}}.$
3. $\int 3^x dx = \frac{3^x}{\ln(3)} + c,$
4. $\int_{-1}^{0} 3^x dx = \left[\frac{3^x}{\ln(3)}\right]_{-1}^{0} = \frac{1 - \frac{1}{3}}{\ln(3)} = \frac{2}{3\ln(3)},$
5. $\int \frac{5^{\tan(x)}}{\cos^2(x)} dx = \int 5^{\tan(x)} \sec^2(x) dx = \frac{5^{\tan(x)}}{\ln(5)} + c.$

Theorem

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{x\to\infty}(1+\frac{1}{x})^x = e.$$

Exercise Find $f'(x)$ if

1. $2x = 4^{f(x)}$,
2. $f(x) = 7^{\sqrt[3]{x}}$,
3. $f(x) = \pi^{3x}$,
4. $f(x) = |\sin(x)|^x$,
5. $f(x) = \left(1+x^2\right)^{2x+1}$,

Examples

1. $$\int x^2 6^{x^3}\,dx \overset{t=x^3}{=} \frac{6^{x^3}}{3\ln 6} + c.$$
2. $$\int \frac{2^x}{2^x+1}\,dx \overset{t=2^x}{=} \frac{1}{\ln 2}\int \frac{dt}{t+1} = \frac{\ln(2^x+1)}{\ln 2} + c.$$
3. $$\int \frac{3^{-\cot(x)}}{\sin^2(x)}\,dx \overset{t=-\cot(x)}{=} \int 3^t\,dt = \frac{3^{-\cot(x)}}{\ln 3} + c$$
4. $$\int 2^{x\ln x}(1+\ln x)\,dx \overset{t=x\ln x}{=} \int 2^t\,dt = \frac{2^{x\ln x}}{\ln 2} + c$$
5. $$\int 4^x 5^{4^x}\,dx \overset{t=4^x}{=} \frac{1}{\ln 4}\int 5^t\,dt = \frac{5^{4^x}}{\ln 4\ \ln 5} + c$$

The General Logarithmic Function

Definition

If $a \in (0, \infty)$ and $a \neq 1$, the function $f : \mathbb{R} \longrightarrow (0, \infty)$ defined by $f(x) = a^x$ is bijective. Its inverse function f^{-1} is denoted by $\log_a$ and called the logarithmic function with base a. For $y \in (0, \infty)$ and $x \in \mathbb{R}$,

$$x = \log_a(y) \iff y = a^x. \tag{1}$$

Examples

1. $9 = 3^2 \Longleftrightarrow 2 = \log_3(9)$,
2. $16 = 4^2 \Longleftrightarrow 2 = \log_4(16)$,
3. $64 = 4^3 \Longleftrightarrow 3 = \log_4(64)$.
4. $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8$.
5. $\log_a 125 = 3 \Leftrightarrow 125 = a^3 \iff a = \sqrt[3]{125} = 5$.

Theorem

For all $a \in (0, \infty) \setminus \{1\}$,

1. $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$,
2. $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, $\forall x > 0$,
3. $\log_e(x) = \ln(x)$.

Notation. For $a = 10$ the function $\log_{10}$ is denoted by Log.

Properties

For $a > 0$, $b > 0$, $a \neq 1$ and $b \neq 1$, we have

1. $\log_b(b) = 1$, $\log_b(1) = 0$, and $\log_b(b^x) = x$, $\forall x \in \mathbb{R}$,
2. $\log_b(xy) = \log_b x + \log_b y$, $\forall x > 0,\ y > 0$,
3. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, $\forall x > 0,\ y > 0$,
4. $\log_b(x^y) = y \log_b x$, $x > 0,\ x \neq 1$, $\forall y \in \mathbb{R}$,
5. $(\log_b x)(\log_a b) = \log_a x$.
6. $y = b^{(\log_b y)}$, for $y > 0$,
7. $b^{\ln a} = a^{\ln b}$.

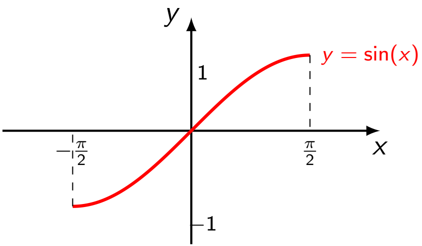
Theorem

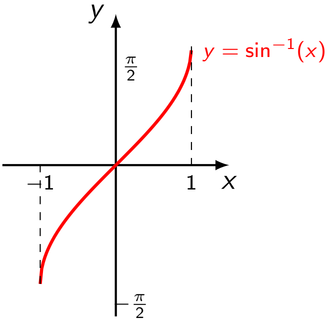
Let $f: I \longrightarrow J$ be a bijective function where I and J are intervals, then

1. If f is continuous, then f^{-1} is also continuous.
2. If f is differentiable and $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is differentiable on J and $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$.

The Sine Function

The function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$ defined by $f(x) = \sin(x)$ is continuous and bijective. The inverse function f^{-1} is denoted by $\sin^{-1}(x)$ or $\mathrm{Arcsin} x$. The inverse function is continuous on $[-1, 1]$.



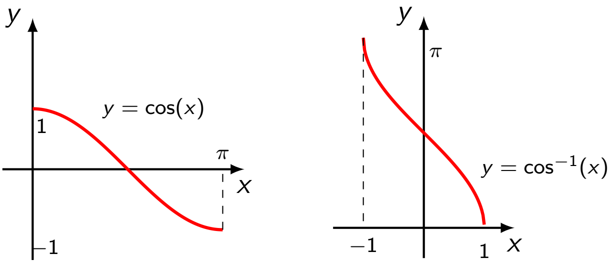


Remark

1. $\sin^{-1}(\sin(x)) = x$ only for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. $\sin\left(\sin^{-1}(x)\right) = x; \ \forall x \in [-1, 1]$.
3. Since $\sin^{-1}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $x \in [-1, 1]$, then
   $\cos(\sin^{-1}(x)) = \sqrt{1 - \sin^2(\sin^{-1}(x))} = \sqrt{1 - x^2}$.
4. $\dfrac{d}{dx}(\sin^{-1})(x) = \dfrac{1}{\cos(\sin^{-1}(x))} = \dfrac{1}{\sqrt{1 - x^2}}$, for all $x \in]-1, 1[$.

The Cosine Function

The function $f : [0, \pi] \longrightarrow [-1, 1]$ defined by $f(x) = \cos(x)$ is continuous and bijective. The inverse function f^{-1} is denoted by $f^{-1}(x) = \cos^{-1}(x)$ or $f^{-1}(x) = \mathrm{Arccos}(x)$.

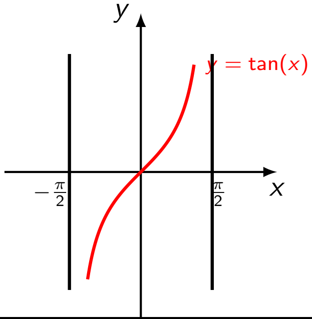


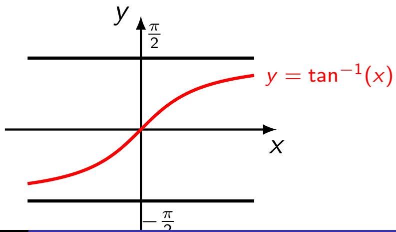
Remark

1. $\cos(\cos^{-1}(x)) = x$, if $x \in [-1, 1]$,
2. $\cos^{-1}(\cos(x)) = x$, if $x \in [0, \pi]$.
3. $\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}$, if $x \in [-1, 1]$.
4. $\dfrac{d}{dx}(\cos^{-1})(x) = \dfrac{-1}{\sin(\cos^{-1}(x))} = \dfrac{-1}{\sqrt{1 - x^2}}$, for $x \in]-1, 1[$.

The Tangent Function

The function $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ defined by $f(x) = \tan(x)$ is increasing, continuous and differentiable, $(f'(x) = 1 + \tan^2(x) = \sec^2(x))$. The inverse function f^{-1} is denoted by $\tan^{-1}(x)$, for $x \in \mathbb{R}$.





Remark

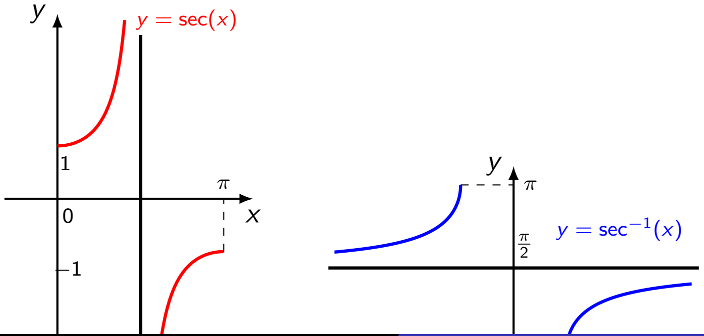
1. $y = \tan^{-1}(x) \iff x = \tan y$, $\forall x \in \mathbb{R}$ and $\forall y \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,
2. $\tan(\tan^{-1}(x)) = x$, $\forall x \in \mathbb{R}$,
3. $\tan^{-1}(\tan(x)) = x$; $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,
4. $\frac{d}{dx}(\tan^{-1})(x) = \frac{1}{1+\tan^2(\tan^{-1}(x))} = \frac{1}{1+x^2}$, for all $x \in \mathbb{R}$.

In the same way we define the function $\cot^{-1} : \mathbb{R} \longrightarrow]0, \pi[$, as the inverse function of $\cot :]0, \pi[\longrightarrow \mathbb{R}$.

$$(\cot^{-1})'(x) = \frac{-1}{1+\cot^2(\cot^{-1}(x))} = \frac{-1}{1+x^2}.$$

The Secant Function

The function $f:[0,\frac{\pi}{2}[\cup]\frac{\pi}{2},\pi]$ defined by $f(x)=\dfrac{1}{\cos(x)}=\sec(x)$ is increasing and $\mathcal{C}^\infty$. Its inverse function is denoted by $f^{-1}(x)=\sec^{-1}(x)$, for $x\in]-\infty,-1]\cup[1,+\infty[$.

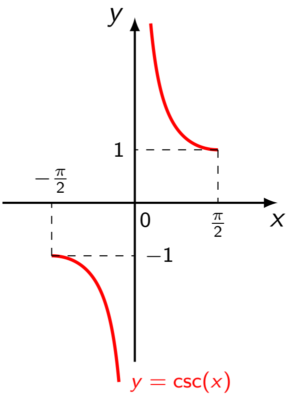


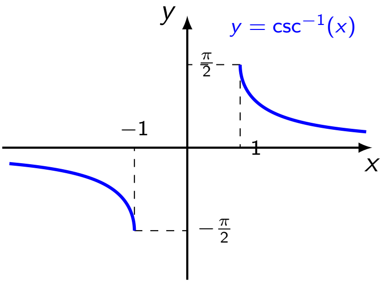
Remark

1. $\sec'(x) = \sec(x)\tan(x)$, $\sec^2(x) = 1 + \tan^2(x)$,
2. $\tan^2(\sec^{-1}(x)) = x^2 - 1$ and $\tan(\sec^{-1}(x)) = \sqrt{x^2-1}$, if $x \in]1, +\infty[$
3. $\tan(\sec^{-1}(x)) = -\sqrt{x^2-1}$, if $x \in]-\infty, -1[$,
4. $\dfrac{d}{dx}(\sec^{-1})(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, for all $x \in]-\infty, -1[\cup]1, +\infty[$.

The Cosecant Function

The function $f : [-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}]$ defined by $f(x) = \dfrac{1}{\sin(x)} = \csc(x)$ is decreasing and $\mathcal{C}^{\infty}$, $(f'(x) = -\csc(x)\cot(x) = -\dfrac{\cos(x)}{\sin^2(x)})$. Its inverse function is denoted by $f^{-1}(x) = \csc^{-1}(x)$ for $x \in]-\infty, -1] \cup [1, +\infty[$.


y
1
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
x
−1
$y = \csc(x)$


y
$y = \csc^{-1}(x)$
$\frac{\pi}{2}$
−1
1
x
$-\frac{\pi}{2}$

Remark

1. $\csc'(x) = -\csc(x)\cot(x)$, $\csc^2(x) = 1 + \cot^2(x)$,
2. $\cot^2(\csc^{-1}(x)) = x^2 - 1$,
3. $\cot(\csc^{-1}(x)) = \sqrt{x^2 - 1}$, if $x \in]1, +\infty[$,
4. $\cot(\csc^{-1}(x)) = -\sqrt{x^2 - 1}$, if $x \in]-\infty, -1[$,
5. $\dfrac{d}{dx}(\csc^{-1})(x) = \dfrac{-1}{|x|\sqrt{x^2 - 1}}$, for all $x \in]-\infty, -1[\cup]1, +\infty[$.

Theorem

1. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}, \ \forall\ |x| < 1,$
2. $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}, \ \forall\ |x| < 1,$
3. $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}, \ \forall\ x \in \mathbb{R},$
4. $\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}, \ \forall\ x \in \mathbb{R},$
5. $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}, \ \forall\ |x| > 1.$
6. $\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}, \ \forall\ |x| > 1.$

Theorem

For $a > 0$,

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (|x| < a)$$
2. $$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (|f| < a))$$
3. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$
4. $$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{f(x)}{a}\right) + c$$
5. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (x > a)$$
6. $$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (f > a))$$

The Hyperbolic Functions

Definition

1. The function $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, for $x \in \mathbb{R}$ is called the hyperbolic sine function.
2. The function $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$, for $x \in \mathbb{R}$, is called the hyperbolic cosine function.
3. The function $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, for $x \in \mathbb{R}$, is called the hyperbolic tangent function.

4. The function $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, for $x \in \mathbb{R} \setminus \{0\}$, is called the hyperbolic cotangent function.
5. The function $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$, for $x \in \mathbb{R}$, is called the hyperbolic secant function:
6. The function $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$, for $x \in \mathbb{R} \setminus \{0\}$, is called the hyperbolic cosecant function:

Some properties of the hyperbolic functions:

Theorem

1. $\cosh^2(x) - \sinh^2(x) = 1, \quad \forall x \in \mathbb{R},$
2. $1 - \tanh^2(x) = \mathrm{sech}^2(x), \ \forall x \in \mathbb{R},$
3. $\coth^2(x) - 1 = \mathrm{csch}^2(x), \ \forall x \in \mathbb{R} \setminus \{0\},$
4. $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$
5. $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$

Theorem

(Derivative of Hyperbolic Functions)

1. $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
2. $\frac{d}{dx}(\cosh(x)) = \sinh(x)$
3. $\frac{d}{dx}(\tanh(x)) = \mathrm{sech}^2(x)$
4. $\frac{d}{dx}(\coth(x)) = -\mathrm{csch}^2(x)$
5. $\frac{d}{dx}(\mathrm{sech}(x)) = -\mathrm{sech}(x)\tanh(x)$
6. $\frac{d}{dx}(\mathrm{csch}(x)) = -\mathrm{csch}(x)\coth(x).$

Theorem

(Integration of Hyperbolic Functions)

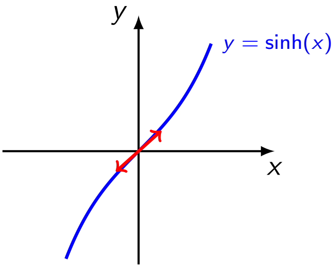
1. $\int \sinh(x)dx = \cosh(x) + c$
2. $\int \cosh(x)dx = \sinh(x) + c$
3. $\int \operatorname{sech}^2(x)dx = \tanh(x) + c$
4. $\int \operatorname{csch}^2(x)dx = -\coth(x) + c$
5. $\int \operatorname{sech}(x)\tanh(x)dx = -\operatorname{sech}(x) + c$
6. $\int \operatorname{csch}(x)\coth(x)dx = -\operatorname{csch}(x) + c$

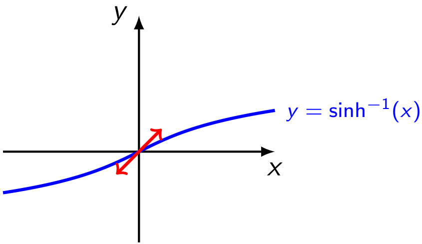
Examples

1. $\displaystyle\int \frac{\sinh(\sqrt{x})}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} 2\int \sinh(u)du = 2\cosh(u) + c = 2\cosh(\sqrt{x}) + c.$
2. $\displaystyle\int \cosh(x)\mathrm{csch}^2(x)dx = \int \frac{\cosh(x)}{\sinh^2(x)}dx = -\frac{1}{\sinh(x)} + c.$

The Sine Hyperbolic Function and its Inverse

1. The function $f(x) = \sinh(x)$ is odd and $f'(x) = \cosh(x) > 0$,
2. $\lim_{x \to +\infty} \sinh(x) = +\infty$ and $\lim_{x \to +\infty} \dfrac{\sinh(x)}{x} = +\infty$.
3. f is continuous and bijective. The inverse function f^{-1} is denoted by
   $f^{-1} = \sinh^{-1}$ and it is continuous,
4. $x, y \in \mathbb{R}$, $y = \sinh^{-1}(x) \iff x = \sinh(y)$.


y
$y = \sinh(x)$
x

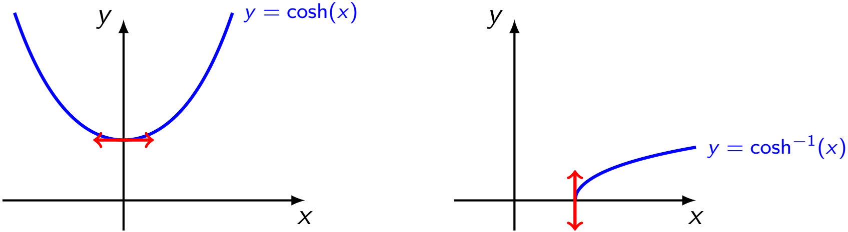

y
$y = \sinh^{-1}(x)$
x

Theorem

1. $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}, \ \forall x \in \mathbb{R},$
2. $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \ \forall x \in \mathbb{R}.$

The Cosine Hyperbolic Function

1. The function $f(x) = \cosh(x)$ defined on $\mathbb{R}$ is even and $f'(x) = \sinh(x)$,
2. $f(x) = \cosh(x)$, $f'(x) = \sinh(x)$,
3. The restriction of the function f on the interval $[0, +\infty[$ is continuous and increasing. Then $f : [0, +\infty[\longrightarrow [1, +\infty[$ is bijective. The inverse function $f^{-1} : [1, +\infty[\longrightarrow [0, +\infty[$ is denoted by $\cosh^{-1}$. The function $\cosh^{-1}$ is continuous on $[1, +\infty[$.
4. $\lim\limits_{x \to +\infty} \cosh(x) = +\infty$ and $\lim\limits_{x \to +\infty} \dfrac{\cosh(x)}{x} = +\infty$,
5. If $x \in [1, \infty)$ and $y \in [0, \infty)$, $y = \cosh^{-1}(x) \iff x = \cosh(y)$.

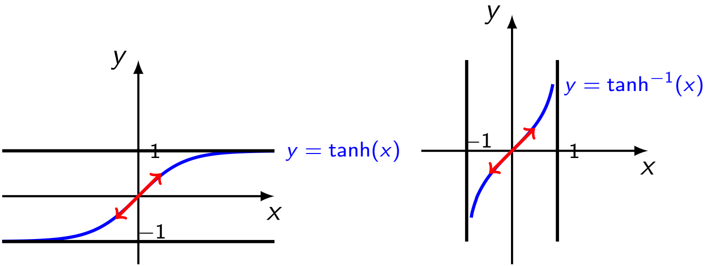

y
y = cosh(x)
x
y
y = cosh^{-1}(x)
x

Theorem

1. $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}, \ \forall x \in]1, +\infty[,$
2. $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1}), \ \forall x \in [1, +\infty[.$

The Tangent Hyperbolic Function

1. The function $f(x) = \tanh(x)$ defined on $\mathbb{R}$ is odd and $f'(x) = 1 - \tanh^2(x) = \mathrm{sech}^2(x) > 0$,
2. The function $f : \mathbb{R} \longrightarrow]-1, 1[$ is continuous and increasing. Then f is bijective. The inverse function f^{-1} denoted by $\tanh^{-1}$ is continuous on $]-1, 1[$.
3. $\lim_{x \to +\infty} \tanh(x) = 1$,
4. $y = \tanh^{-1}(x) \iff x = \tanh(y)$ for all $y \in \mathbb{R}$ and $x \in]-1, 1[$.


y
1
x
−1
y = tanh(x)
y
y = tanh^{-1}(x)
−1
1
x

Theorem

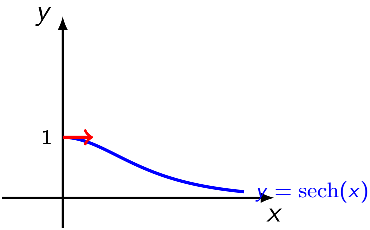
1. $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}, \ \forall x \in]-1,1[,$
2. $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \ \forall x \in]-1,1[.$

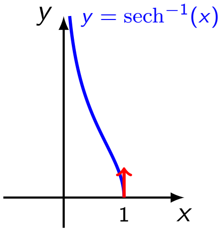
The Inverse Hyperbolic Cotangent Function

1. The function $f(x) = \coth(x)$ defined on $\mathbb{R}^*$ is odd and $f'(x) = 1 - \coth^2(x) = -\mathrm{csch}^2(x) < 0$. The function f is continuous and decreasing, then f is bijective. The inverse function f^{-1} is denoted by $f^{-1} = \coth^{-1}$ and it is also continuous. $\coth^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$.
2. $\lim\limits_{x\to+\infty} \coth(x) = 1$,
3. $y = \coth^{-1}(x) \iff x = \coth(y)$ for all $y \in]0, +\infty[$ and $x \in]0, 1[$.
4. $(f^{-1})'(x) = \dfrac{-1}{1-x^2}$.
5. $\displaystyle\int \frac{dx}{1-x^2} = -\coth^{-1}(x) + c$ for $|x| > 1$.

The Inverse Hyperbolic Secant Function

1. The function $f: [0, +\infty[\longrightarrow]0, 1]$ defined by: $f(x) = \mathrm{sech}(x) = \dfrac{2}{e^x + e^{-x}}$ is bijective and decreasing since $f'(x) = -\mathrm{sech}(x)\tanh(x) < 0$.
2. The inverse function f^{-1} is denoted by $f^{-1} = \mathrm{sech}^{-1}$ and it is continuous,
3. $\lim_{x \to +\infty} \mathrm{sech}(x) = 0$,
4. For all $x \in]0, 1]$ and $y \in [0, +\infty[$, $y = \mathrm{sech}^{-1}(x) \iff x = \mathrm{sech}(y)$.


y
1
x
y = sech(x)

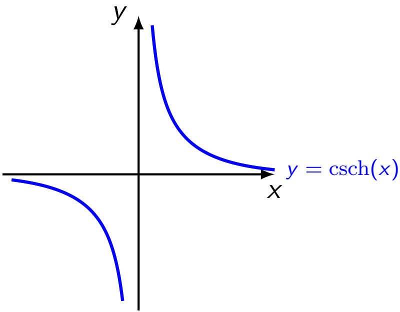

y
y = sech⁻¹(x)
1
x

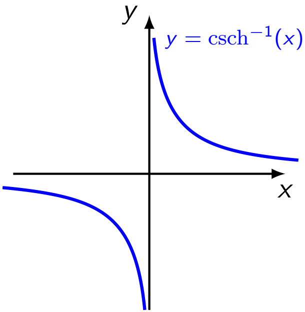
Theorem

1. $(\operatorname{sech}^{-1})'(x) = \dfrac{-1}{x\sqrt{1-x^2}}, \forall x \in]0,1[,$
2. $\operatorname{sech}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \forall x \in]0,1[.$

The Inverse Cosecant Hyperbolic Function

1. The function $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ defined by: $f(x) = \operatorname{csch}(x) = \dfrac{2}{e^x - e^{-x}}$ is bijective and decreasing since $f'(x) = -\operatorname{csch}(x)\coth(x) < 0$.
2. If $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$, $y = \operatorname{csch}^{-1}(x) \iff x = \operatorname{csch}(y)$.
3. $\lim_{x \to +\infty} \operatorname{csch}(x) = 0$,
4. For all $x, y \in \mathbb{R} \setminus \{0\}$, $y = \operatorname{csch}^{-1}(x) \iff x = \operatorname{csch}(y)$.


y
x
$y = \operatorname{csch}(x)$


y
x
$y = \operatorname{csch}^{-1}(x)$

Theorem

1. $(\operatorname{csch}^{-1})'(x) = \dfrac{-1}{x\sqrt{1+x^2}}, \qquad \forall x \in]0, +\infty[,$
2. $\operatorname{csch}^{-1}(x) = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), \qquad \forall x \in]0, +\infty[.$

Indeterminate Forms

The indeterminate forms arise from the fact that $(\bar{\mathbb{R}}, +, .)$ is not a field, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The only operations that are wrong are $0.\infty$ and $+\infty + (-\infty)$. These operations are obtained for example within the real sequences or the limits of functions. For example if a sequence $(u_n)_n$ converges to 0 and the sequences $(v_n)_n$ tends to ∞, we can not decide if the limit of the sequence $(u_n.v_n)_n$ exists.

The only indeterminate forms are $0.\infty$ and $+\infty + -\infty$. The other indeterminate forms can be transformed to these two forms. For examples we have

$$\frac{0}{0} = 0.\infty, \quad \frac{\infty}{\infty} = 0.\infty, \quad 1^{\infty} = e^{\infty \ln(1)} = e^{0.\infty}, \quad 0^0 = e^{0 \ln(0)} = e^{0.\infty}.$$

Example

$$"\frac{0}{0}" = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)} = \lim_{x\to 2}(x-2) = 0$$

$$"\frac{0}{0}" = \lim_{x\to 2}\frac{3(x-2)}{(x-2)} = \lim_{x\to 2} 3 = 3$$

$$"\frac{0}{0}" = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)^4} = \lim_{x\to 2}\frac{1}{(x-2)^2} = \infty.$$

In each of above cases the functions are undefined at $x = 2$. And both numerator and denominator in each example approach to 0 as $x \to 0$.

Example

$\lim_{x\to 0} \frac{\ln(\cos(x))}{\sin(x)}$, $\lim_{x\to\infty} e^{3x}\ln(1+\frac{1}{x})$, $\lim_{x\to\infty}(1+x)^2-\sqrt{x^4+x+2}$, $\lim_{x\to\infty}(1+\frac{1}{x})^x$ are all indeterminate forms.

The Hôpital's Rule

Theorem

Let f and g be two continuous functions on the interval $[a, b]$ and differentiable on $]a, b[$. We assume that $g'(x) \neq 0$ for all $x \in]a, b[$. Then there exists $c \in]a, b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Theorem

[The Hôpital's Rule]
Let f, g be two differentiable functions on $]a,b[\setminus\{c\}$. Assume that $g'(x)\neq 0$ for all $x\in]a,b[\setminus\{c\}$ and $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=0$.
If $\lim_{x\to c}\dfrac{f'(x)}{g'(x)}=\ell\in\mathbb{R}\cup\{-\infty,+\infty\}$, then $\lim_{x\to c}\dfrac{f(x)}{g(x)}=\ell$.

Remark

1. The theorem is valid for one-sided limits as well as the two sided limit. This theorem is also true if $c = +\infty$ or $c = -\infty$.
2. The theorem is valid for the case, $\lim_{x \to c} f(x) = \infty$ or $-\infty$ and $\lim_{x \to c} g(x) = \infty$ or $-\infty$.

Examples

1. $$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \cos(x) = 1,$$
2. $$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2},$$
3. $$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} -x = 0,$$
4. $$\lim_{x\to 1} \frac{\sqrt{x}-1}{\ln x} = \lim_{x\to 1} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2},$$
5. $$\lim_{x\to 0} \frac{\int_0^x \sqrt{1+\sin(t)}\, dt}{x} = \lim_{x\to 0} \frac{\sqrt{1+\sin(x)}}{1} = \frac{\sqrt{1+0}}{1} = 1,$$

6. $\lim_{x\to 1} \dfrac{\tan^{-1} x - \dfrac{\pi}{4}}{x-1} = \lim_{x\to 1} \dfrac{\left(\dfrac{1}{1+x^2}\right)}{1} = \dfrac{1}{2},$
7. $\lim_{x\to\infty} \dfrac{x}{\ln(x)} = \lim_{x\to\infty} x = +\infty,$
8. $\lim_{x\to\infty} (1 + \dfrac{1}{x})^{5x} = \lim_{x\to\infty} e^{5\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}} = e^5,$
9. $\lim_{x\to\infty} x^x = \lim_{x\to\infty} e^{x\ln(x)} = +\infty,$
10. $\lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1}{x} = 0.$

11. $\lim_{x\to\infty}(x^2-1)e^{-x^2} = \lim_{x\to\infty}\frac{x^2-1}{e^{x^2}} = \lim_{x\to\infty}\frac{2x}{2x\ e^{x^2}} = \lim_{x\to\infty}\frac{1}{e^{x^2}} = 0,$

12. $\lim_{x\to\infty}\left(1+e^{2x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln\left(1+e^{2x}\right)}{x}} = \lim_{x\to\infty} e^{\frac{2e^{2x}}{1+e^{2x}}} = \lim_{x\to\infty} e^{\frac{4e^{2x}}{2e^{2x}}} = e^2.$