

Holomorphic Functions

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Definition

Let $f: \Omega \rightarrow \mathbb{C}$ be a function defined on a non empty open set Ω .

1. We say that f is differentiable at $z \in \Omega$ if there exists $\ell \in \mathbb{C}$ such that

$$\lim_{z \rightarrow a} \frac{f(w) - f(z)}{w - z} = \lim_{h \rightarrow 0, h \in \mathbb{C}^*} \frac{f(z + h) - f(z)}{h} = \ell.$$

We denote $\ell = f'(z)$ and called the derivative of f at z .

2. We say that f is holomorphic on Ω if f is differentiable at any point of Ω .

We denote $\mathcal{H}(\Omega)$ the set of holomorphic functions on Ω .

Examples

1. The function $f(z) = z^n$ is holomorphic on \mathbb{C} , for every $n \in \mathbb{N}$.
2. The function $f(z) = \bar{z}$ is not differentiable at any point of \mathbb{C} because $\frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h}$ and $\lim_{h \rightarrow 0} \frac{\bar{h}}{h}$ does not exist.
3. If $f: \Omega \rightarrow \mathbb{C}$ is a holomorphic function, then the function f^* defined by $f^*(z) = \overline{f(\bar{z})}$ is holomorphic on $\bar{\Omega} = \{\bar{z}; z \in \Omega\}$, indeed if $z, a \in \bar{\Omega}$,

$$\lim_{z \rightarrow a} \frac{f^*(z) - f^*(a)}{z - a} = \lim_{z \rightarrow a} \overline{\left(\frac{f(\bar{z}) - f(\bar{a})}{\bar{z} - \bar{a}} \right)} = \overline{f'(\bar{a})}.$$

Proposition (Exercise)

- If f and g are holomorphic on Ω , then $f + g$, fg are also holomorphic on Ω .
The function $\frac{f}{g}$ is holomorphic on the open set where g does not vanishes.*
- If $f: \Omega_1 \rightarrow \mathbb{C}$ is a holomorphic function and $g: \Omega_2 \rightarrow \mathbb{C}$ is a holomorphic function such that $g(\Omega_2) \subset \Omega_1$, then $f \circ g$ is holomorphic on Ω_2 and $(f \circ g)'(z) = f'(g(z))g'(z)$.*

Theorem (Cauchy-Riemann conditions)

Let $f(z) = U(x, y) + iV(x, y)$ be a function defined on a neighborhood of $z_0 = x_0 + iy_0$. ($U = \operatorname{Re}f$ and $V = \operatorname{Im}f$). We assume that the functions U and V are differentiable at (x_0, y_0) . Then the function f of complex variable $z = x + iy$ is differentiable at z_0 , if and only if

$$\begin{cases} \frac{\partial U}{\partial x}(x_0, y_0) = \frac{\partial V}{\partial y}(x_0, y_0) \\ \frac{\partial U}{\partial y}(x_0, y_0) = -\frac{\partial V}{\partial x}(x_0, y_0) \end{cases} . \quad (1)$$

These conditions are called the Cauchy-Riemann conditions. They are equivalent to the following condition

$$\frac{\partial f}{\partial y}(z_0) = i \frac{\partial f}{\partial x}(z_0). \quad (2)$$

Proof

Necessary condition

If f is differentiable at $z_0 = x_0 + iy_0$, then

$\lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$. If we take the limit when h tends to 0, h real, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \rightarrow 0} \frac{U(x_0 + h, y_0) - U(x_0, y_0)}{h} + i \frac{V(x_0 + h, y_0) - V(x_0, y_0)}{h} \\ &= \frac{\partial U}{\partial x}(x_0, y_0) + i \frac{\partial V}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0 + iy_0). \end{aligned} \quad (3)$$

If we take the limit when $h = it$, with t real, we have

$$\begin{aligned} f'(z_0) &= \lim_{t \rightarrow 0} \frac{U(x_0, y_0 + t) - U(x_0, y_0)}{h} + i \frac{V(x_0, y_0 + t) - V(x_0, y_0)}{it} \\ &= -i \frac{\partial U}{\partial y}(x_0, y_0) + \frac{\partial V}{\partial y}(x_0, y_0) = -i \frac{\partial f}{\partial y}(z_0). \end{aligned} \quad (4)$$

We have then the Cauchy-Riemann conditions.

Sufficient condition

By definition of the differentiability of function of two real variables, we have

$$\begin{cases} U(x_0 + s, y_0 + t) - U(x_0, y_0) = s \frac{\partial U}{\partial x}(x_0, y_0) + t \frac{\partial U}{\partial y}(x_0, y_0) + |h| \varepsilon_1(h) \\ V(x_0 + s, y_0 + t) - V(x_0, y_0) = s \frac{\partial V}{\partial x}(x_0, y_0) + t \frac{\partial V}{\partial y}(x_0, y_0) + |h| \varepsilon_2(h) \end{cases}$$

with $h = s + it$, $\lim_{h \rightarrow 0} \varepsilon_1(h) = \lim_{h \rightarrow 0} \varepsilon_2(h) = 0$. Thus

$$f(z_0+h) - f(z_0) = s \frac{\partial U}{\partial x}(x_0, y_0) + t \frac{\partial U}{\partial y}(x_0, y_0) + i \left(s \frac{\partial V}{\partial x}(x_0, y_0) + t \frac{\partial V}{\partial y}(x_0, y_0) \right) + \varepsilon$$

with $\varepsilon = \varepsilon_1 + i\varepsilon_2$.

It follows by Cauchy-Riemann conditions that

$$f(z_0 + h) - f(z_0) = Ah + |h|\eta(h),$$

with $A = \frac{\partial U}{\partial x}(x_0, y_0) + i\frac{\partial V}{\partial x}(x_0, y_0)$ and $\eta(h)$ tends to 0 when h tends to 0. Thus f is differentiable at z_0 and $f'(z_0) = A$.

Remark (Riemann 1851)

If U and V are two functions twice continuously differentiable on an open subset Ω of \mathbb{C} and if $f = U + iV$ is holomorphic on Ω , then $\Delta U = \Delta V = 0$, with $\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$. (Δ is called the Laplace operator). We say that U and V are harmonic.

Corollary

If f is holomorphic on Ω , then

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x}(x, y) = -i \frac{\partial f}{\partial y}(x, y) \\ &= \frac{\partial U}{\partial x}(x, y) + i \frac{\partial V}{\partial x}(x, y) = \frac{\partial V}{\partial y}(x, y) - i \frac{\partial U}{\partial y}(x, y) \\ &= \frac{\partial U}{\partial x}(x, y) - i \frac{\partial U}{\partial y}(x, y) = \frac{\partial V}{\partial y}(x, y) + i \frac{\partial V}{\partial x}(x, y), \end{aligned}$$

where $z = x + iy$.

Example

Determine the holomorphic functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $\operatorname{Re}f(z) = U(x, y) = x^2 - y^2 - 2xy$.

Let $V = \operatorname{Im}f$, then by Cauchy-Riemann conditions,

$\frac{\partial V}{\partial y} = 2x - 2y$, thus $V = 2xy - y^2 + g(x)$. Furthermore

$\frac{\partial V}{\partial x} = 2y + g'(x) = -(-2y - 2x) = 2x + 2y$, thus

$V = x^2 - y^2 + 2xy + C$ and $f(z) = (1 + i)z^2 + iC$.

Example

Determine the holomorphic functions $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $\operatorname{Re}f(z) = x^3 - 3xy^2 - 2xy - 1$.

If $V = \operatorname{Im}f$, then $\frac{\partial V}{\partial y} = 3x^2 - 3y^2 - 2y$, thus

$V = 3x^2y - y^3 - y^2 + g(x)$. Furthermore

$\frac{\partial V}{\partial x} = 6xy + g'(x) = 6xy + 2x$, thus $V = 3x^2y - y^3 - y^2 + x^2 + C$.

$f'(z) = \frac{\partial f}{\partial x} = 3x^2 - 3y^2 - 2y + i(6xy + 2x) =$

$3(x^2 - y^2 + 2ixy) + 2i(x + iy) = 3z^2 + 2iz$. Thus

$f(z) = z^3 + iz^2 + 1 + ic$, where $c \in \mathbb{R}$.

Proposition

Let f be a holomorphic function on a domain Ω . Then $f' \equiv 0$ on $\Omega \iff f$ is constant on Ω .

Proof

The sufficient condition is trivial. For the necessary condition, it suffices to show that f is locally constant. Let $z_0 \in \Omega$ and $r > 0$ such that $D(z_0, r) \subset \Omega$. Let $z_1 \in D(z_0, r)$, the complex number $z_2 = \operatorname{Re} z_1 + i \operatorname{Im} z_0 \in D(z_0, r)$ and $f(z_0) = f(z_2)$ because $\frac{\partial f}{\partial x} = 0$ and $f(z_2) = f(z_1)$ because $\frac{\partial f}{\partial y} = 0$, thus $f(z_0) = f(z_1)$.

Proposition

Let $f: \Omega \rightarrow \mathbb{C}$ be a holomorphic function on a domain Ω of \mathbb{C} . Then the following properties are equivalent.

1. f is constant on Ω .
2. $\operatorname{Re} f$ is constant on Ω .
3. $\operatorname{Im} f$ is constant on Ω .
4. $|f|$ is constant on Ω .
5. \bar{f} is holomorphic on Ω .

Proof

- It is obvious that 1) \Rightarrow 2).
- By The Cauchy-Riemann conditions (1), 2) \iff 3).

Since 2) \iff 3), then 3) \Rightarrow 4).

- If $|f| = 0$, then \bar{f} is holomorphic on Ω .

If $|f| = c \neq 0$, then $f\bar{f} = c$ and $\bar{f} = \frac{c^2}{f}$ is holomorphic on Ω .

- \bar{f} is holomorphic on Ω . In use the Cauchy-Riemann conditions for f and \bar{f} , we find f is constant.



Theorem

Let f be the holomorphic function defined by the power series $\sum_{n \geq 0} a_n z^n$ which admits $R > 0$ as radius of convergence, then the function g defined by the power series $\sum_{n \geq 1} n a_n z^{n-1}$ admits R as radius of convergence. The function f is holomorphic on $D(0, R)$ and $f'(z) = g(z)$.

For the proof of this theorem, we need the following lemma

Lemma

Let $z \in \mathbb{C}$ and $h \in \mathbb{C}$ such that $0 < |h| \leq r$, then for all $n \in \mathbb{N}^*$

$$|(z + h)^n - z^n - nhz^{n-1}| \leq \frac{|h|^2}{r^2} (|z| + r)^n \quad (5)$$

and

$$n|z|^{n-1} \leq \frac{1}{r} (2(|z| + r)^n + |z|^n) \quad (6)$$

Proof

By inequality (5)

$$\begin{aligned}
 |(z+h)^n - z^n - nhz^{n-1}| &= \left| \sum_{k=0}^n C_n^k h^k z^{n-k} - z^n - nhz^{n-1} \right| \\
 &= \left| \sum_{k=2}^n C_n^k h^k z^{n-k} \right| \\
 &\leq |h|^2 \sum_{k=2}^n C_n^k |z|^{n-k} |h|^{k-2} \\
 &\leq \frac{|h|^2}{r^2} \sum_{k=2}^n C_n^k |z|^{n-k} r^k \leq \frac{|h|^2}{r^2} (|z| + r)^n.
 \end{aligned}$$

We have $|(z+h)^n - z^n - nhz^{n-1}| \geq nr|z|^{n-1} - |z|^n - (|z| + r)^n$.

By (5), we deduce

$$nr|z|^{n-1} \leq |z|^n + (|z|+r)^n + |(z+r)^n - z^n - nrz^{n-1}| \leq |z|^n + 2(|z|+r)^n.$$

Proof of theorem 2.1

We denote R' the radius of convergence of the power series $\sum_{n \geq 1} na_n z^{n-1}$. It is obvious that $R' \leq R$. Let $r > 0$ such that

$|z| + r < R$. By lemma 2.2, we have

$$|na_n z^{n-1}| \leq \frac{1}{r} (2|a_n|(|z| + r)^n + |a_n||z|^n) \text{ and thus } \sum_{n \geq 1} na_n z^{n-1}$$

converges absolutely on $D(0, R)$. Thus the radius of convergence of the series defining g is greater than R . Thus $R = R'$.

By inequality (5),

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq \frac{|h|}{r} \sum_{n=1}^{+\infty} |a_n| (|z| + r)^n,$$

this proves that when h tends to 0, $f'(z) = g(z)$, for all $z \in D(0, R)$.

Corollary

If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, then f is infinitely continuously differentiable

on $D(0, R)$, $a_n = \frac{f^{(n)}(0)}{n!}$ and $f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$. (This series is called the Taylor's series of f at 0.)

The exponential function e^z is defined by the series $\sum_{n \geq 0} \frac{z^n}{n!}$,

$(e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!})$ and it fulfills the following properties

- $(e^z)' = e^z$,
- $e^{z+w} = e^z e^w$ this results by definition of the product series,
- $e^z e^{-z} = 1$ for all $z \in \mathbb{C}$,
- $e^x > 0$ for all $x \in \mathbb{R}$,
- $0 < e^x < 1$ for all $x \in \mathbb{R}_-$,
- $e^{\bar{z}} = \overline{e^z}$,
- $|e^{iy}| = 1$ for all $y \in \mathbb{R}$, thus $|e^{x+iy}| = e^x$ for all $(x, y) \in \mathbb{R}^2$.
- $e^z \neq 0, \forall z \in \mathbb{C}$.

These properties prove that the exponential function $z \mapsto e^z (\mathbb{C}, +) \rightarrow (\mathbb{C}^*, \times)$ is an homomorphism of groups.

We define

- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ the cosine function,
- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ the sine function,
- $\cosh z = \frac{e^z + e^{-z}}{2}$ the hyperbolic cosine function,
- $\sinh z = \frac{e^z - e^{-z}}{2}$ the hyperbolic sine function.

Thus $e^{iz} = \cos z + i \sin z$, $e^{x+iy} = e^x(\cos y + i \sin y)$, y is an argument of e^z (this yields the Euler formula $e^{i\theta} = \cos \theta + i \sin \theta$).

Properties

$\cos z + i \sin z = e^{iz}$, $\cos z - i \sin z = e^{-iz}$, then $\cos^2 z + \sin^2 z = 1$.

$\cosh z + \sinh z = e^z$, $\cosh z - \sinh z = e^{-z}$, then

$\cosh^2 z - \sinh^2 z = 1$.

$\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$.

Proposition

The function e^{iz} is periodic. We denote 2π the period of e^{iz} .

Proof

$e^z = e^{z+w} \iff e^w = 1$, thus $w = iy$, with $y \in \mathbb{R}$.

Since $\cos^2 y + \sin^2 y = 1$, then $(\sin y)' = \cos y \leq 1$.

Since $\sin 0 = 0$, then $\sin y \leq y$ for $y \geq 0$. In the same time, $(\cos y)' = -\sin y \geq -y$ and $\cos 0 = 1$, then $\cos y \geq 1 - \frac{y^2}{2}$. It follows that $\sin y \geq y - \frac{y^3}{3!}$ and $\cos y \leq 1 - \frac{y^2}{2} + \frac{y^4}{4!}$. These inequalities leads that $\cos \sqrt{3} < 0$, thus there exists $0 < y_0 < \sqrt{3}$ such that $\cos y_0 = 0$ and $\sin y_0 = \pm 1$. $e^{iy_0} = \pm i$ and $e^{4iy_0} = 1$. Thus $4y_0$ is a period of the function e^{iz} and it is the smallest period. Indeed, let $y_0 > y > 0$, $\sin y \geq y(1 - \frac{y^2}{6}) \geq \frac{y}{2} > 0$. This shows that the function $\cos y$ is strictly decreasing and $\sin y$ is strictly increasing on the interval $[0, y_0]$, thus $\sin y < \sin y_0 = 1$, $0 < \sin y < 1$, this yields that $e^{iy} \neq \pm 1$, $e^{iy} \neq \pm i$ and $e^{4iy} \neq 1$. It follows that $4y_0$ is the smallest period denoted w_0 . Let w be a period such that $nw_0 \leq w < (n+1)w_0$. If $w \neq nw_0$, then $w - nw_0$ is a period and $0 < w - nw_0 < w_0$, this is impossible, thus $w = nw_0$. We denote $w_0 = 2\pi$, thus $e^{i\frac{\pi}{2}} = i$, $e^{i\pi} = -1$ and $e^{2i\pi} = 1$.

Lemma

For all $z = x + iy \in \mathbb{C} \setminus \{0\}$, there exists $r > 0$ and a unique $\theta \in [0, 2\pi[$ such that $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$.

Proof

One has $\cos^2 \theta + \sin^2 \theta = 1$, thus $r = \sqrt{x^2 + y^2}$. We assume that $r = 1$.

• If $0 \leq x < 1$ and $y \geq 0$, we know that the function $f(\theta) = \cos \theta$ is decreasing and continuous of the interval $[0, \frac{\pi}{2}]$ with values on the interval $[0, 1]$ and as $x \in [0, 1[$, there exists a unique $\theta \in]0, \frac{\pi}{2}[$ such that $x = \cos \theta$, furthermore $x^2 + y^2 = 1$ and $y \geq 0$, thus $y = \sqrt{1 - x^2} = \sin \theta$, where $x + iy = \cos \theta + i \sin \theta$.

- If $x \geq 0$ and $y \leq 0$, $x \neq 1$. From which above, there exists a unique $\theta \in]0, \frac{\pi}{2}]$ such that $x - iy = \cos \theta + i \sin \theta$, where $\cos \theta - i \sin \theta = x + iy = \cos(2\pi - \theta) + i \sin(2\pi - \theta)$ and $2\pi - \theta \in [\frac{3\pi}{2}, 2\pi[$.
- If $x \leq 0$, $x \neq -1$ and $y \geq 0$. From which above, there exists a unique $\theta \in]\frac{3\pi}{2}, 2\pi]$ such that $-x - iy = \cos \theta + i \sin \theta$, this yields that $x + iy = \cos(-\pi + \theta) + i \sin(-\pi + \theta)$, where $\theta - \pi \in [\frac{\pi}{2}, \pi[$.

- If $x \leq 0$, $x \neq -1$ and $y \leq 0$. From above, there exists a unique $\theta \in]0, \frac{\pi}{2}]$ such that $-x - iy = \cos \theta + i \sin \theta$, this yields that $x + iy = \cos(\pi + \theta) + i \sin(\pi + \theta)$, where $\theta + \pi \in]\pi, \frac{3\pi}{2}]$.
- If $x = 1$ and $y = 0$, then $x + iy = \cos 0 + i \sin 0$.
- If $x = -1$ and $y = 0$, then $x + iy = \cos \pi + i \sin \pi$.

The uniqueness of θ in $[0, 2\pi[$ results from the fact that 2π is the smallest period of the function e^{iz} . □

We will prove that the branches of the logarithm defined as above are holomorphic.

Remark

From which above, for all $\alpha \in \mathbb{R}$ and all $z \in \mathbb{C} \setminus \{0\}$, there exists a unique $r > 0$ and a unique $\theta \in [\alpha, \alpha + 2\pi[$ such that $z = r(\cos \theta + i \sin \theta)$.

Proposition

The mapping $A: \mathbb{C} \setminus \mathbb{R}^+ \rightarrow]0, 2\pi[$ defined by $A(z) = A(r(\cos \theta + i \sin \theta)) = \theta$ is continuous.

Proof

$z = r(\cos \theta + i \sin \theta) = x + iy$, where $\theta \in]0, 2\pi[$.

$$x = r \cos \theta = -r \cos(\pi - \theta) = -2r \cos^2\left(\frac{\pi - \theta}{2}\right) + r.$$

$$y = r \sin \theta = r \sin(\pi - \theta) = 2r \cos\left(\frac{\pi - \theta}{2}\right) \sin\left(\frac{\pi - \theta}{2}\right).$$

$$r - x = 2r \cos^2\left(\frac{\pi - \theta}{2}\right).$$

$$y = 2r \cos\left(\frac{\pi - \theta}{2}\right) \sin\left(\frac{\pi - \theta}{2}\right)$$

$$\frac{y}{-x + \sqrt{x^2 + y^2}} = \operatorname{tg}\left(\frac{\pi - \theta}{2}\right) \Rightarrow \frac{\pi - \theta}{2} = \tan^{-1}\left(\frac{y}{\sqrt{x^2 + y^2} - x}\right).$$

Thus

$$\theta = \pi - 2 \tan^{-1} \left(\frac{y}{\sqrt{x^2 + y^2} - x} \right),$$

and A is continuous.

In the same way, the mapping $\mathbb{C} \setminus \{te^{i\alpha}, t \geq 0\} \rightarrow]\alpha, \alpha + 2\pi[$ defined by $A_\alpha(z) = A(r(\cos \theta + i \sin \theta)) = \theta$ is continuous.

To prove the holomorphy of the Logarithmic function, we need the following theorem

Theorem

Let Ω be an open subset of \mathbb{C} and let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. We assume that $f'(z) \neq 0 \forall z \in \Omega$, f is bijective of Ω on $f(\Omega) = \Omega'$, where Ω' is an open subset of \mathbb{C} .
If f^{-1} , the inverse function of f is continuous on Ω' , then f^{-1} is holomorphic on Ω' and

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}, \quad \forall w \in \Omega'.$$

Proof

Let $w_0 \in \Omega'$ and $z_0 = f^{-1}(w_0) \in \Omega$. For $w \in \Omega'$, there exists a unique $z \in \Omega$, such that $w = f(z)$. Since f^{-1} is continuous, if w tends to w_0 , then z tends to z_0 . Thus

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \xrightarrow{w \rightarrow w_0} \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

We will prove that the assumptions, f^{-1} is continuous, $f'(z) \neq 0$, $\forall z \in \Omega$ and Ω' is an open subset of \mathbb{C} are hold for any holomorphic function $f: \Omega \rightarrow \Omega'$ bijective unlike in the real case.

Corollary

The mapping $z \mapsto e^z$ is a holomorphic function, bijective of the strip $A_t = \{x + iy \in \mathbb{C}; t - \pi < y < \pi + t, x \in \mathbb{R}\}$ on $\mathbb{C} \setminus J_t$, with $J_t = \{re^{it}, r \leq 0\}$, $t \in \mathbb{R}$. It has an inverse function defined on $\mathbb{C} \setminus J_t$ with values on A_t . We denote by \log_t this function.

This corollary results by theorem 2.9 and the continuity of the mapping $A(z)$.

Definition

The principal determination (branch) of the logarithm is the inverse function of the exponential function defined on $\mathbb{C} \setminus J_0$ with values on A_0 . We denote this function by Log .

Exercise

Show that

1. $\text{Log}z = \ln \sqrt{x^2 + y^2} + 2i \tan^{-1} \frac{y}{x + \sqrt{x^2 + y^2}}, \quad \forall z = x + iy \in \mathbb{C} \setminus J_0.$
2. $(\log_t z)' = \frac{1}{z}$ for all $t \in \mathbb{R}$ and $z \in \mathbb{C} \setminus J_t.$

Corollary

The function $f(z) = \log_t(z) - \log_{t'}(z)$ is constant on each connected component of $(\mathbb{C} \setminus J_t) \cap (\mathbb{C} \setminus J_{t'})$.

Remark

The relation $\text{Log}(z_1 z_2) = \text{Log}z_1 + \text{Log}z_2$ is not always valid. It suffices to take $z_1 = e^{\frac{3i\pi}{4}} = z_2$, $z_1 z_2 = e^{\frac{3i\pi}{2}} = e^{-\frac{i\pi}{2}}$, $\text{Log}z_1 z_2 = -i\frac{\pi}{2}$ and $\text{Log}z_1 + \text{Log}z_2 = \frac{3i\pi}{2} \neq -\frac{i\pi}{2}$.

Proposition

For $|z| < 1$, $\text{Log}(1 - z) = -\sum_{n=1}^{+\infty} \frac{z^n}{n}$, where $\text{Log}(1 - z)$ is the principal determination (branch) of the logarithmic function.

Proof

Let $f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n}$, for $|z| < 1$, then $f'(z) = \sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}$. It follows that $f(z) + \text{Log}(1 - z) = 0$ because $f(0) = \text{Log}(1 - 0) = 0$.

Let Ω be an open subset of \mathbb{C}^* and let $\alpha \in \mathbb{C}^*$. We define a continuous determination of z^α on Ω by any continuous function $g: \Omega \rightarrow \mathbb{C}$ such that for all $z \in \Omega$, there exists a logarithm h of z such that

$$\begin{cases} g(z) = e^{\alpha h(z)} \\ z = e^{h(z)} \end{cases} .$$

In particular, if there exists on Ω a continuous determination (branch) of the logarithm \log , then the mapping $z \rightarrow e^{\alpha \log z}$ is a continuous determination of z^α on Ω . We use only this type of determination for z^α .

Definition

A function $f: \Omega \rightarrow \mathbb{C}$ is called analytic on Ω if, whenever $z_0 \in \Omega$ there exists a neighborhood V of z_0 and a power series

$$\sum_{n \geq 0} a_n(z - z_0)^n \text{ such that } f(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n \text{ for all } z \in V.$$

Theorem

Any analytic function is holomorphic.

Proof

This theorem results by theorem 2.1. The inverse will be proved in the chapter III. □

Theorem

If the power series $\sum_{n \geq 0} a_n z^n$ has a radius of convergence $R > 0$, its

sum $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is analytic on the disc $D(0, R)$.

For the proof of this theorem we need the following result on double sums.

Theorem

Let $(a_{n,m})_{n,m \in \mathbb{N}}$ be a sequence of complex numbers. If

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} |a_{n,m}| < +\infty, \text{ one has}$$

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} |a_{n,m}| = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} |a_{n,m}| = \sup_{N, M \in \mathbb{N}} \left[\sum_{m=0}^M \sum_{n=0}^N |a_{n,m}| \right],$$

and

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{n,m} = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} a_{n,m}.$$



Proof

This theorem results by Fubini's theorem. Indeed, we consider the countable measure μ on \mathbb{N} . If $f: \mathbb{N} \rightarrow [0, +\infty]$ is a μ -integrable

function, then $\int_{\mathbb{N}} f(x) d\mu(x) = \sum_{k=0}^{+\infty} f(k)$. Furthermore, if

$\sigma: \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, then

$$\sum_{k=0}^{+\infty} f(k) = \sum_{k=0}^{+\infty} f(\sigma(k)).$$

Let $(b_{m,n})$ be a sequence of non negative real numbers. By Beppo-Levi's theorem

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} b_{m,n} = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} b_{m,n}.$$

It suffices to take the function f defined on $\mathbb{N} \times \mathbb{N}$ by
 $f(n, m) = b_{n,m}$.

In our case the function g defined on $\mathbb{N} \times \mathbb{N}$ by $g(n, m) = a_{n,m}$ is integrable, and the theorem results by Fubini's theorem.

Proof of the theorem 3.3

Let $z_0 \in D(0, R)$, $|z_0| = r_0 < R$. We shall show that

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \text{ whenever } z \in D(z_0, R - r_0).$$

Let $z \in \mathbb{C}$ such that $|z - z_0| < r - r_0 < R - r_0$ and let

$$S(z) = \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} C_n^k a_n z_0^{n-k} (z - z_0)^k \quad (7)$$

We shall prove that the series $\sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} C_n^k a_n z_0^{n-k} (z - z_0)^k$ is absolutely convergent.

Let

$$R(z) = \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} C_n^k |a_n| |z_0|^{n-k} |z - z_0|^k \quad (8)$$

$$R(z) \leq \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} C_n^k |a_n| r_0^{n-k} (r - r_0)^k =$$

$$\sum_{n=0}^{+\infty} |a_n| \left(\sum_{k=0}^n C_n^k r_0^{n-k} (r - r_0)^k \right) = \sum_{n=0}^{+\infty} |a_n| r^n < +\infty.$$

Thus the series which defines S is absolutely convergent, thus it is commutatively convergent.

$$S(z) = \sum_{k=0}^{+\infty} \frac{1}{k!} (z - z_0)^k \left(\sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n z_0^{n-k} \right), \text{ this means that}$$

$$S(z) = \sum_{k \geq 0} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \text{ and}$$

$$S(z) = \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^n C_n^k z_0^{n-k} (z - z_0)^k \right) = \sum_{n=0}^{+\infty} a_n z^n = f(z).$$

It follows that $f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$, whenever

$z \in D(z_0, R - r_0)$.

Theorem (Principle of the Analytic Continuation)

Let f be an analytic function on a domain Ω of \mathbb{C} and let $z_0 \in \Omega$.
The following conditions are equivalent

1. f is identically zero on Ω .
2. There exists a neighborhood V of z_0 such that f is identically zero on V .
3. For all $n \geq 0$, $f^{(n)}(z_0) = 0$.

Proof

We shall show that 2) \Rightarrow 1) (the other properties are trivial).
Let $A = \{z \in \Omega; f \equiv 0 \text{ on a neighborhood of } z\}$. A is a non empty open subset by hypothesis. Then it suffices to show that it is closed in Ω . Let $(z_n)_n$ be a sequence of A which converges to $a \in \Omega$. Since $z_n \in A$, then $f^{(k)}(z_n) = 0$ for all $k \in \mathbb{N}$ and by continuity $f^{(k)}(a) = 0$. f is analytic, this yields that f is zero on a neighborhood of a .

Corollary

Let f and g be two analytic functions on a domain Ω of \mathbb{C} . If f and g coincide on a neighborhood of a point of Ω , then they coincide on Ω .

Theorem (Principle of Isolated Zeros)

Let f be an analytic function on a domain Ω of \mathbb{C} . If f is not identically zero on Ω , the set of zeros of f is a discrete closed subset of Ω .

Proof

Let $A = f^{-1}\{0\}$ be the set of zeros of f . A is closed since f is continuous. Let $z_0 \in A$, by theorem 3.5, there exists k such that $f^{(k)}(z_0) \neq 0$. We choose k the smallest integer such that $f^{(k)}(z_0) \neq 0$. So the power series of f at z_0 is

$$f(z) = a_k(z - z_0)^k + (z - z_0)^{k+1}g(z), \text{ where } a_k \neq 0 \text{ and } g(z_0) = 0,$$

$$g(z) = \sum_{n=1}^{+\infty} a_{n+k}(z - z_0)^n. \text{ Since } g(z_0) = 0, \text{ there exists a}$$

neighborhood V of z_0 such that $|g(z)| < |a_k|$ for all $z \in V$. Thus $|f(z)| \geq |z - z_0|^k(|a_k| - |g(z)|) > 0$ for $z \in V \setminus \{z_0\}$. It results that z_0 is the only zero of f in V .

Corollary

If f is an analytic function and non-identically zero on a domain Ω , then any compact subset of Ω contains only a finite number of zeros of f .

Corollary

The ring of the analytic functions on a domain Ω is integral.

Corollary (The Identity Theorem)

If f and g are two analytic functions on a domain Ω , which coincides on a set admitting a cluster point (or an accumulation point) in Ω , then they coincide on Ω .