The Riemann Integral

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Anti-Derivative

Definition

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval I. A function $F : I \longrightarrow \mathbb{R}$ is called an anti-derivative of f on I, if F is differentiable on I and

$$F'(x) = f(x), \quad \forall x \in I.$$

Example

1. The function $F(x) = x^2 + 1$ is an anti-derivative of the function $f(x) = 2x$ on $\mathbb{R}$.
2. The function $2\sqrt{x}$ is an anti-derivative of the function $\dfrac{1}{\sqrt{x}}$ on $(0, +\infty)$.

Theorem

Let F and G be two anti-derivatives of a function f on an interval I, then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c, \quad \forall x \in I.$$

The Indefinite Integral

Definition

If a function $f : I \longrightarrow \mathbb{R}$ has an anti-derivative on I, $\displaystyle\int f(x)dx$ denotes any anti-derivative of f. The function $\displaystyle\int f(x)dx$ is called an indefinite integral of f on I. Therefore,

$$\frac{d}{dx}\int f(x)dx = f(x), \quad \forall x \in I.$$

Example

1. $\displaystyle\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \in \mathbb{Q} \setminus \{-1\},$
2. $\displaystyle\int \cos(x) dx = \sin(x) + c,$
3. $\displaystyle\int \sin(x) dx = -\cos(x) + c,$
4. $\displaystyle\int \sec^2(x) dx = \tan(x) + c,$
5. $\displaystyle\int \csc^2(x) dx = -\cot(x) + c,$
6. $\displaystyle\int \sec(x)\tan(x) dx = \sec(x) + c,$
7. $\displaystyle\int \csc(x)\cot(x) dx = -\csc(x) + c,$

Theorem

Important formulas

Let $f, g\colon I \longrightarrow \mathbb{R}$ be two functions.

1. If f is differentiable and $\frac{d}{dx}f(x)$ has an anti-derivative, then
$$\int \frac{d}{dx}f(x)dx = f(x) + c.$$
2. If f has an anti-derivative, then
$$\frac{d}{dx}\int f(x)dx = f(x).$$

3. If f has an anti-derivative on I, then for all $\lambda \in \mathbb{R}$,

$$\int \lambda f(x)dx = \lambda \int f(x)dx.$$

4. If f and g have anti-derivatives, then the functions $f \pm g$ have anti-derivatives and

$$\int (f(x) \pm g(x))\, dx = \int f(x)dx \pm \int g(x)dx.$$

Example

1. $$\begin{aligned}\int \left(\frac{3}{x^4} - 5x\right) dx &= \int (3x^{-4} - 5x)\, dx = \int 3x^{-4} dx - \int 5x dx \\ &= -x^{-3} - \frac{5}{2}\, x^2 + c.\end{aligned}$$

2. $$\begin{aligned}\int \frac{2x^2 + 3}{\sqrt{x}} dx &= \int x^{\frac{-1}{2}}\, (2x^2 + 3)\, dx = 2 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{-1}{2}} dx \\ &= \frac{4}{5}\, x^{\frac{5}{2}} + 6\, x^{\frac{1}{2}} + c.\end{aligned}$$

Substitution Method

Theorem

Integration by Substitution
Let $g\colon I \longrightarrow J$ be a continuously differentiable function and $f\colon J \longrightarrow \mathbb{R}$ be a function which has an anti-derivative F on J, then $F(g(x))$ is an anti-derivative of $f(g(x))g'(x)$ and

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This formula is obtained by the chain rule formula. The substitution method is also called the changing variable method.

Example

1. $\int \cos(2x)dx \overset{(u=2x)}{=} \int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(2x) + c.$
2. $\int (x^2+1)^n 2x dx \overset{(u=x^2+1)}{=} \int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{(x^2+1)^{n+1}}{n+1} + c$, for $n \neq -1$.
3. $$\begin{aligned} \int \sin(2x+3)dx &\overset{(u=2x+3)}{=} \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(u) + c \\ &= -\frac{1}{2}\cos(2x+3) + c. \end{aligned}$$
4. $\int \sec^2(\pi x)dx \overset{(u=\pi x)}{=} \frac{1}{\pi}\int \sec^2(u)du = \frac{1}{\pi}\tan(\pi x) + c.$

Theorem

Let I be an interval, $r \in \mathbb{Q} \setminus \{-1\}$ and $f : I \longrightarrow \mathbb{R}$ a continuously differentiable function. Assume also that the function f^r is continuous on I. Then

$$\int f^r(x) f'(x) dx = \frac{1}{r+1} f^{r+1}(x) + c.$$

Example

1. $$\int (2x^3+1)^7 6x^2 dx \overset{(u=2x^3+1)}{=} \int u^7 du = \frac{1}{8}(2x^3+1)^8 + c,$$
2. $$\int (7-6x^2)^{\frac{1}{2}} x dx \overset{(u=7-6x^2)}{=} -\frac{1}{12}\int u^{\frac{1}{2}} du = -\frac{1}{18}(7-6x^2)^{\frac{3}{2}} + c,$$
3. $$\int \frac{x^2-1}{(x^3-3x+1)^6} dx \overset{(u=x^3-3x+1)}{=} \frac{1}{3}\int \frac{du}{u^6} = -\frac{1}{15}(x^3-3x+1)^{-5} + c,$$
4. $$\int \cos(3x+4) dx \overset{u=3x+4}{=} \frac{1}{3}\int \cos(u) du = \frac{1}{3}\sin(3x+4) + c,$$
5. $$\int \left(1+\frac{5}{x}\right)^3 \frac{1}{x^2} dx \overset{u=1+\frac{5}{x}}{=} \frac{-1}{5}\int u^3 du = \frac{-1}{20}\left(1+\frac{5}{x}\right)^4 + c,$$
6. $$\int \frac{\cos(\sqrt{x})}{\sqrt{x}\ \sin^2(\sqrt{x})} dx \overset{u=\sqrt{x}}{=} 2\int \frac{\cos(u)}{\sin^2(u)} du = -\frac{2}{\sin(\sqrt{x})} + c.$$

Summation Notation

Definition

Given a set of real numbers $\{a_1, a_2, ..., a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

Theorem

For $m, n \in \mathbb{N}$, the following summation properties hold:

1. $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$
2. $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k.$
3. $\displaystyle\sum_{k=1}^{n} Ca_k = C\sum_{k=1}^{n} a_k$, where $C \in \mathbb{R}$
4. For $1 \leq m \leq n$, $\displaystyle\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k.$

Theorem

For $n \in \mathbb{N}$ and $C \in \mathbb{R}$, the following properties hold:

1. $$\sum_{k=1}^{n} C = \underbrace{C + \cdots + C}_{n \text{ times}} = nC,$$
2. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$
3. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
4. $$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example

Evaluation of the following sums

1. $\displaystyle\sum_{k=1}^{100} k = \frac{100 \times (100+1)}{2} = 5050,$
2. $\displaystyle\sum_{k=1}^{20} k^2 = \frac{20.(20+1).(2.20+1)}{6} = 2870,$
3.

$$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= 3\sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1\\ &= \frac{n(n+1)(2n+1)}{2} - n(n+1) + n\\ &= \frac{n}{2}(2n^2+n+1).\end{aligned}$$

Example

Find the following limits:

1. $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} 5k,$
2. $\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2.$

The Riemann Integral

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a **bounded** function on a **closed and bounded** interval. The aim of the section is to define the Riemann integral of the function f on $[a, b]$ if it is possible.
The integral of f on $[a, b]$ is a real number whose geometrical interpretation is the signed area under the graph of the function f on $[a, b]$. This number is also called the definite integral of f.
By integrating the function f over the interval $[a, x]$ with varying x in $[a, b]$, we get a function F of x. The most important result about integration is the fundamental Theorem of calculus, which states that if the function f is continuous, the function F is an anti derivative of f.

Definition

1. A partition P of the closed interval $[a, b]$ is a finite set of points $P = \{a_0, a_1, \ldots, a_n\}$ such that $a = a_0 < \ldots < a_n = b$. Each $[a_{j-1}, a_j]$ is called a sub-interval of the partition and the number $h_j = a_j - a_{j-1}$ is called the amplitude of this interval.
2. The norm of a partition $P = \{a_0, a_1, \ldots, a_n\}$ is the length of the longest sub-interval $[a_j, a_{j+1}]$, that is: $||P|| = \max\{h_j, j = 1, \ldots, n\}$.
3. A partition $P = \{a_0, a_1, \ldots, a_n\}$ of the closed interval $[a, b]$ is a called uniform if $a_{k+1} - a_k = \dfrac{b-a}{n}$. In this case

$$a_k = a + k\frac{b-a}{n},\ 0 \leq k \leq n.$$

4. A mark on the partition $P = \{a_0, a_1, \dots, a_n\}$ is a set of points $w = \{x_1, \dots, x_n\}$ such that $x_j \in [a_{j-1}, a_j]$ for all $1 \leq j \leq n$.
5. A pointed partition of the interval $[a, b]$ is a partition of the interval together with a mark $w = \{x_1, \dots, x_n\}$ on this partition. This pointed partition will be denoted by

$$(P, w) = \{([a_{j-1}, a_j], x_j)\}_{1 \leq j \leq n}.$$

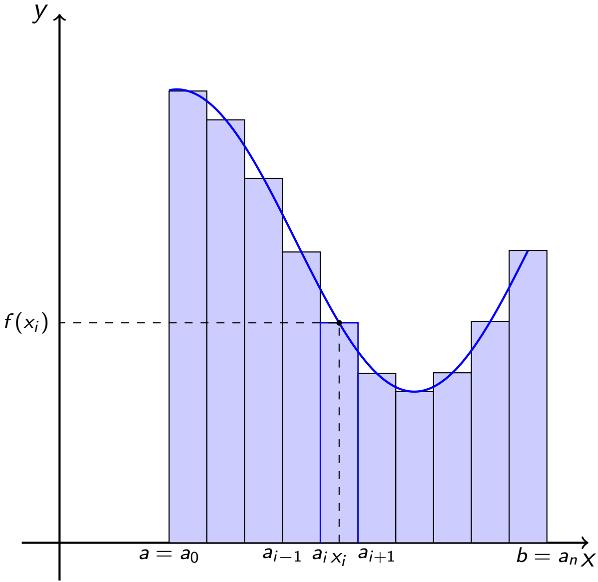
Definition

Let $(P, w) = \{([a_{j-1}, a_j], x_j)\}_{1 \le j \le n}$ be a pointed partition of the interval $[a, b]$. The Riemann sum of f with respect to the pointed partition P is the number

$$R(f, P, w) = \sum_{j=1}^{n} f(x_j)(a_j - a_{j-1}) = \sum_{j=1}^{n} f(x_j)h_j.$$

Each term in the sum is the product of the value of the function at a given point by the length of an interval. Consequently, each term represents the area of a rectangle with height $f(x_j)$ and length $a_j - a_{j-1}$.

The Riemann sum $R(f, P, w)$ is the algebraic area of the union of the rectangles of width h_j and height $f(x_j)$. This is an algebraic area since $f(x_j)h_j$ is counted positively if $f(x_j) > 0$ and negatively if $f(x_j) < 0$.


y
f(x_i)
a = a_0
a_{i−1}
a_i
x_i
a_{i+1}
b = a_n
X

Example

Let $f : [0,1] \longrightarrow \mathbb{R}$ the function defined by $f(x) = 2x - 2x^2$.
If $P = \{a_k = \frac{k}{10},\ 0 \le k \le 10\}$ is the uniform partition of the interval $[0,1]$ and the mark $w = \{x_k = a_k,\ 1 \le k \le 10\}$, we get the Riemann sum

$$\begin{aligned} R(f,P,w) &= \frac{1}{10}\sum_{k=1}^{10} f(x_k) = \frac{1}{10}\sum_{k=1}^{10}(2x_k - 2x_k^2) \\ &= \frac{1}{10}[0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0] \\ &= 0.33. \end{aligned}$$

Example

Consider the function $f(x) = x$ on the interval $[0, 1]$ and the uniform partition $P = \{\frac{k}{n},\ 0 \leq k \leq n\}$, for $n \geq 1$. Presenting three principal cases of Riemann sums, as we put the x_k at the left, the middle or the right end point of the intervals $[a_{k-1}, a_k]$, where $a_k = \dfrac{k}{n}$, for $1 \leq k \leq n$.

1. $x_k = a_{k-1}$:

$$R(f, P, w) = \frac{1}{n}\sum_{k=1}^{n}\frac{k-1}{n} = \frac{1}{n^2}\sum_{k=0}^{n-1}k = \frac{n-1}{2n}.$$

2. $x_j = \dfrac{a_{k-1} + a_k}{2}$

$$R(f, P, w) = \frac{1}{n}\sum_{k=1}^{n}\frac{2k-1}{2n} = \frac{1}{2n^2}\sum_{k=1}^{n}2k - 1 = \frac{1}{2}.$$

3. $x_k = a_k$:

$$R(f,P,w) = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} = \frac{n+1}{2n}.$$

The second sum is equal to $\frac{1}{2}$ for every n, the other sums tend to $\frac{1}{2}$ when n tends to infinity. The area of the triangle under the graph of the function is equal to $\frac{1}{2}$.

Example

Let $f : [1, 3] \longrightarrow \mathbb{R}$ be the function defined by: $f(x) = 3x + 1$, the uniform partition $P = \{a_k,\ 0 \leq k \leq n\}$ of the interval $[1, 3]$ and the mark $w = \{x_k,\ 1 \leq k \leq n\}$, where x_k is the middle point of the sub-interval $[a_{k-1}, a_k]$, $x_k = 1 + \dfrac{2k-1}{n}$. The Riemann sum is

$$
\begin{aligned}
R(f, P, w) &= \frac{2}{n}\sum_{k=1}^{n} f(x_k) = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(1 + \frac{2k-1}{n}\right) + 1\right) \\
&= \frac{2}{n}\sum_{k=1}^{n}\left(4 + \frac{6k}{n} - \frac{3}{n}\right) \\
&= 8 + 6(1 + \frac{1}{n}) - \frac{6}{n}.
\end{aligned}
$$

Example

Referring to the last example with x_k the right end point of the sub-interval $[a_{k-1}, a_k]$, $x_k = 1 + \frac{2k}{n}$.

$$\begin{aligned} R(f, P, w) &= \frac{2}{n}\sum_{k=1}^{n} f(x_k) = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(1+\frac{2k}{n}\right)+1\right) \\ &= \frac{2}{n}\sum_{k=1}^{n}\left(4+\frac{6k}{n}\right) = 8+6(1+\frac{1}{n}). \end{aligned}$$

Fundamental Properties

Theorem

Let $f, g\colon [a, b] \to \mathbb{R}$ be two functions and $\alpha, \beta \in \mathbb{R}$ and (P, w) a pointed partition of the interval $[a, b]$.

1. Linearity: $R(\alpha f + \beta g, P, w) = \alpha R(f, P, w) + \beta R(g, P, w)$.
2. Monotony: If $f \leq g$, then $R(f, P, w) \leq R(g, P)$. In particular, if $f \geq 0$, then $R(f, P, w) \geq 0$.
3. Chasles's Formula: Let $c \in (a, b)$, (P_1, w_1) a pointed partition of $[a, c]$ and (P_2, w_2) a pointed partition of $[b, c]$, then
$(P_1 \cup P_2, w = w_1 \cup w_2)$ is a pointed partition of $[a, b]$ and

$$R(f, P_1 \cup P_2, w) = R(f, P_1, w_1) + R(f, P_2, w_2).$$

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function, the Riemann integral of f on the interval $[a, b]$ is

$$\lim_{\|P\| \to 0} R(f, P, w).$$

whenever the limit exists. (The limit is over all pointed partitions $P = \{([x_{j-1}, x_j], w_j)\}_{1 \le j \le n}$).
If the limit exists, it is said that f is Riemann integrable (or integrable) on $[a, b]$. This limit if it exists, is denoted by: $\int_a^b f(x)dx$ and called the definite integral of f on the interval $[a, b]$.

Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + k\frac{b-a}{n}).$$

Theorem

Let $f, g\colon [a,b] \longrightarrow \mathbb{R}$ be two Riemann integrable functions and $\alpha, \beta \in \mathbb{R}$. Then

1. $\displaystyle\int_a^b \alpha dx = \alpha(b-a)$.
2. The function αf is Riemann integrable on $[a,b]$ and
$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$
3. The functions $f \pm g$ are Riemann integrable on $[a,b]$ and
$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

4. For all $c \in (a, b)$ the function f is Riemann integrable on $[a, c]$, on $[c, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5. If $f \geq 0$, then $\displaystyle\int_a^b f(x)dx \geq 0$.
6. If $f \leq g$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable, the function $F(x) = \int_a^x f(t)dt$ is continuous.

Definition

A function $f : [a, b] \longrightarrow \mathbb{R}$ is called piecewise continuous if there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that f is continuous on every interval $]x_k, x_{k+1}[$, $\lim_{x \to x_k^+} f(x)$ and $\lim_{x \to x_{k+1}^-} f(x)$ exist in $\mathbb{R}$, for all $k = 0, \ldots, n-1$.

Theorem

Any piecewise continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable.

Example

Evaluation of the following definite integrals

1. $\int_{-1}^{3} 4dx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} 4 = \lim_{n\to+\infty} \frac{16n}{n} = 16,$
2. $\int_{0}^{4} xdx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} \frac{4k}{n} = \lim_{n\to+\infty} \frac{8(n+1)}{n} = 8,$
3. $$\begin{aligned}\int_{0}^{1} (3x+7)dx &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{3k}{n} + 7 \\ &= \lim_{n\to+\infty} \frac{3(n+1)}{2n} + 7 = \frac{3}{2} + 7 = \frac{17}{2}\end{aligned}$$

4. $\displaystyle\int_{-1}^{4}|x|dx = -\int_{-1}^{0} xdx + \int_{0}^{4} xdx = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} + 8 = \frac{1}{2} + 8 = \frac{17}{2},$

5.

$$\begin{aligned}\int_{1}^{4}(x^2+x+2)dx &= \lim_{n\to+\infty}\frac{3}{n}\sum_{k=1}^{n}\left(1+3\frac{k}{n}\right)^2 + (1+3\frac{k}{n}) + 2\\ &= \lim_{n\to+\infty}\frac{3}{n}\sum_{k=1}^{n}\left(1+6\frac{k}{n}+9\frac{k^2}{n^2}+1+3\frac{k}{n}+2\right)\\ &= \lim_{n\to+\infty}\frac{3}{n}\left(4n+\frac{9(n+1)}{2}+\frac{3(n+1)(2n+1)}{2n}\right)\\ &= \frac{69}{2}.\end{aligned}$$

Example

Using the definition of the Riemann integral, the following limits can be expressed as definite integrals

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}((2+\frac{k}{n})^2-4)$, and

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}((-4+\frac{k}{n})^{\frac{1}{3}}+4(-4+\frac{k}{n}))$,

If $f(x)=x^2-4$ on the interval $[2,3]$,

$$\int_2^3 (x^2-4)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}((2+\frac{k}{n})^2-4).$$

If $f(x)=x^{\frac{1}{3}}+4x$ on the interval $[-4,-3]$,

$$\int_{-4}^{-3} (x^{\frac{1}{3}}+4x)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}((-4+\frac{k}{n})^{\frac{1}{3}}+4(-4+\frac{k}{n})).$$

Theorem

If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $f(x) \geq 0, \quad \forall x \in [a, b]$, then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx.$$

The Fundamental Theorem of Calculus

Theorem

The Mean Value Theorem
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. There exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = (b - a)f(c).$$

Remark

If f is a non negative continuous function on $[a, b]$. The integral $\int_a^b f(x)dx$ represents the area under the graph of f and $(b-a)f(c)$ represents the area of the rectangle with side measurements $f(c)$ and $b-a$.

Definition

Let f be a continuous function on $[a, b]$. The average value of f is defined by:

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example

1. The average value of the function $f(x) = 3x + 7$ on the interval $[0, 1]$ is $\displaystyle\int_0^1 (3x + 7)dx = \frac{17}{2}$. The number c where f reaches its average value verifies $3c + 7 = \dfrac{17}{2}$, then $c = \dfrac{1}{2}$.
2. The average value of the function $f(x) = x^2 + x + 2$ on the interval $[1, 4]$ is $\displaystyle\int_0^1 (x^2 + x + 2)dx = \frac{17}{6}$. The number c where f reaches its average value verifies $c^2 + c + 2 = \frac{17}{6}$, then $c = \dfrac{\sqrt{13} - \sqrt{3}}{2\sqrt{3}}$.

Example

Let f be a continuous function on $[a, b]$ such that $\int_a^b f(x)dx = 0$, then the equation $f(x) = 0$ has a solution in $[a, b]$. The average value of f on $[a, b]$ is 0. Then by the Mean Value Theorem, f reaches this value at some point $c \in [a, b]$.

Theorem

(**First Fundamental Theorem of Calculus)**
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, then the function F defined by $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$ and $F'(x) = f(x)$.

Remark

1. The continuity of the function f is important. It is possible that a discontinuous function never equals its average value. We can take the function $f(x) = 0$ on the interval $[0, 1]$ and $f(x) = 1$ on the interval $[1, 2]$. The average value of f on the interval $[0, 2]$ is $\frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}$. But $f(x) \neq \frac{1}{2}$, for all $x \in [0, 2]$.
2. Let f be a continuous function on a closed interval $[a, b]$. For any $c \in [a, b]$, the function $G(x) = \int_c^x f(t)dt; \quad x \in [a, b]$ is an anti derivative of f i.e. $G'(x) = f(x); \quad \forall x \in [a, b]$ because $G(x) = \int_a^x f(t)dt - \int_a^c f(t)dt$.

Theorem

(Second Fundamental Theorem of Calculus)
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and F an anti-derivative of f on $[a, b]$, then $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$

Theorem

Let f be a continuous function on an interval I. If u and v are two differentiable functions on an interval J such that $v(J) \subset I$ and $u(J) \subset I$, then the function

$$x \longmapsto \int_{u(x)}^{v(x)} f(t)dt$$

is differentiable on the interval J. Moreover

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Example

1. $$\frac{d}{dx}\left(\int_{3x}^{x^2}(t^3+1)^7dt\right) = 2x(x^6+1)^7 - 3(27x^3+1)^7$$

2. $$\begin{aligned}\frac{d}{dx}\int_{1-x}^{x^2}\frac{1}{4+3t^2}\,dt &= \frac{1}{4+3\left(x^2\right)^2}(2x) - \frac{1}{4+3\left(1-x\right)^2}(-1)\\ &= \frac{2x}{4+3x^4} + \frac{1}{4+3\left(1-x\right)^2}.\end{aligned}$$

3. $\frac{d}{dx}\int_0^5\sqrt{t^2+3}\,dt = 0$ since $\int_0^5\sqrt{t^2+3}\,dt$ is constant,

4. $\frac{d}{dx}\int_x^1 u^2\cos(u)\,du = -x^2\cos(x),$

Numerical Integration

Very often definite integration cannot be done in closed form. When this happens some simple and useful techniques are needed to approximate the definite integrals. This section discuss two such simple and useful methods.

Trapezoidal Rule

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a non negative continuous function. To approximate the area under the graph of f, the function f on $[x_j, x_{j+1}]$ is replaced by the polynomial P of degree 1 such that $P(x_j) = f(x_j)$ and $P(x_{j+1}) = f(x_{j+1})$. It is said that the polynomial P interpolates the function f on the points x_j and x_{j+1}. Then

$$P(x) = f(x_j)\frac{x_{j+1} - x}{x_{j+1} - x_j} + f(x_{j+1})\frac{x - x_j}{x_{j+1} - x_j}.$$

The area under the graph of P on the interval $[x_j, x_{j+1}]$ is the area of a trapezoid equal to

$$\frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_{j+1})).$$

The area under the graph of f is approximated by

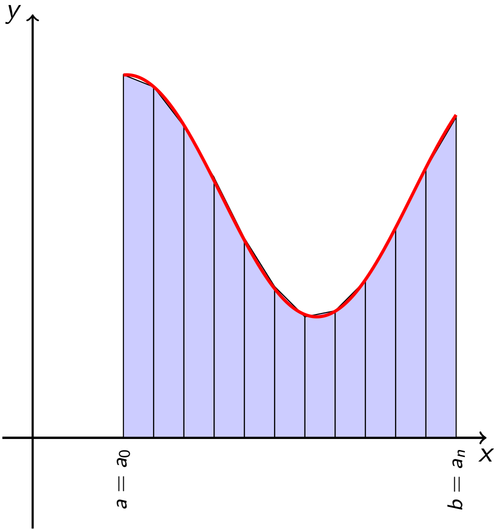
$$\sum_{j=1}^{n} \frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_j)).$$

In the case where $x_{j+1} - x_j = \dfrac{b-a}{n}$, this area is approximated by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) \quad + f(b)\right).$$

This formula is called the **trapezoidal rule**.

This formula is exact for polynomials of degree at most 1.



Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. The remainder for this method is approximated as follows

$$|R_n| \leq \frac{(b-a)^3 M_2}{12n^2}, \quad \text{where } M_2 = \sup_{x \in [a,b]} |f^{(2)}(x)|.$$

Example

Let $f(x) = 2x - 1$ and $g(x) = x^2 + 3x - 1$ defined on the interval $[1, 3]$. Using trapezoidal method for $n = 5$. An approximation of the integrals $\int_1^3 f(x)dx$ and $\int_1^3 g(x)dx$ is given by:

$x_k = 1 + \frac{2k}{5}$, $f(x_k) = 1 + \frac{4k}{5}$ and $g(x_k) = 3 + 2k + \frac{4k^2}{25}$.

$$\int_1^3 (2x-1)dx \approx \frac{1}{5}\left(1 + 5 + 2\sum_{k=1}^{4}(1 + \frac{4k}{5})\right) = 6.$$

$$\int_1^3 (2x-1)dx = \left[x^2 - x\right]_1^3 = 6.$$

The reminder $R = 0$.

$$\begin{aligned}\int_1^3 (x^2+3x-1)dx &\approx \frac{1}{5}\left(3+17+2\sum_{k=1}^{4}(1+\frac{4k}{5})^2+3(1+\frac{4k}{5})-1\right)\\ &= \frac{1}{5}\left(20+2\sum_{k=1}^{4}(\frac{4k^2}{25}+2k+3)\right)=\frac{1}{5}(93+\frac{3}{5})\\ &= 18.72.\end{aligned}$$

$$\int_1^3 (x^2+3x-1)dx = \left[\frac{x^3}{3}+\frac{3x^2}{2}-x\right]_1^3 = \frac{37}{2}.$$

The reminder $|R| \leq 0.06$.

Example

Approximation of the integral $\int_0^1 \sqrt{1+x+x^2}\, dx$ using trapezoidal rule with $n = 4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	1	1	1
1	0.25	1.1456	2	2.2913
2	0.5	1.3228	2	2.6457
3	0.75	1.5207	2	3.0414
4	1	1.73205	1	3.4641
				12.44248

$$\int_0^1 \sqrt{1+x+x^2}\, dx \approx$$
$$\frac{1-0}{2(4)}\left[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)\right],$$
$$\int_0^1 \sqrt{1+x+x^2}\, dx \approx \frac{1}{8}[12.44248] \approx 1.5553.$$

The reminder R fulfills $|R| \leq \frac{1}{4^4} = \frac{1}{256}$.

Example

Approximation of the integral $\int_2^4 \frac{1}{x-1}\, dx$ using trapezoidal rule with $n = 4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	2	1	1	1
1	2.5	0.6666	2	1.333
2	3	0.5	2	1
3	3.5	0.4	2	0.8
4	4	0.3333	1	0.3333
				4.1666

$\int_2^4 \frac{1}{x-1}\, dx \approx 1.0415$. The reminder R fulfills $|R| \leq \frac{1}{12}$.

The Simpson Method

In this method, the function f on the interval $[x_j, x_{j+1}]$ is replaced by the polynomial P of degree 2 which interpolates the function f at the points x_j, x_{j+1} and the middle point $m_j = \frac{x_j+x_{j+1}}{2}$.

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_j(x)dx = \frac{x_{j+1}-x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

$$\begin{aligned} P_j(x) &= f(x_j)\frac{(x_{j+1}-x)(x-m_j)}{(x_{j+1}-x_j)(x_j-m_j)} + f(m_j)\frac{(x_{j+1}-x)(x-x_j)}{(x_{j+1}-m_j)(m_j-x_j)} \\ &+ f(x_{j+1})\frac{(x-x_j)(x-m_j)}{(x_{j+1}-x_j)(x_{j+1}-m_j)}. \end{aligned}$$

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_2(x)dx = \frac{x_{j+1} - x_j}{6}(f(x_j)+f(x_{j+1})+4f(m_j)).$$

If the partition is uniform, $x_{j+1} - x_j = \dfrac{b-a}{n}$, then

$$\begin{aligned} S_n(f) &= \frac{b-a}{6n}\sum_{j=0}^{n-1}(f(x_j) + f(x_{j+1}) + 4f(m_j)) \\ &= \frac{b-a}{6n}\Big(f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j) + 4\sum_{j=0}^{n-1} f(m_j)\Big). \end{aligned}$$

This formula is called **The Simpson formula** and it is exact for polynomials of degree at most 3.

If the middle point is not used, taking $n = 2m$ and $P = \{x_0, x_1, \ldots, x_{2m-1}\}$ a partition of the interval $[a, b]$. **The Simpson Formula** has the following form

$$S_n(f) = \frac{b-a}{3n}\Big(f(a) + f(b) + 4\sum_{j=0}^{m-1} f(x_{2j+1}) + 2\sum_{j=1}^{m-1} f(x_{2j})\Big).$$

Example

Let $g(x) = x^2 + 3x - 1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Using Simpson method for $n = 8$, to approximate the integrals $\int_1^3 (x^2 + 3x - 1)dx$ and $\int_1^3 x^3 dx$.

$x_k = 1 + \frac{k}{4}$, $x_{2k} = 1 + \frac{k}{2}$ and $x_{2k+1} = 1 + \frac{2k+1}{4}$,

$g(x_k) = 3 + \frac{5k}{4} + \frac{k^2}{16}$ and $h(x_k) = 1 + \frac{3k}{4} + \frac{3k^2}{16} + \frac{k^3}{64}$.

$$\int_1^3 (x^2 + 3x - 1)dx \approx \frac{1}{12}\left(3 + 17 + 4\sum_{k=0}^{3} g(x_{2k+1}) + 2\sum_{k=1}^{3} g(x_{2k})\right) = 18.666$$

k	x_k	m_k	$m_k g(x_k)$
0	1	1	3
1	$5/4$	4	17.25
2	$3/2$	2	11.5
3	$7/4$	4	29.25
4	2	2	18
5	$9/4$	4	43.25
6	$5/2$	2	25.25
7	$11/4$	4	59.25
8	3	1	17
			224

Example

Approximation of the integral $\int_1^3 \sqrt{1 - x + x^2}\, dx$ using Simpson's rule with $n = 4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1	1	1
1	1.5	1.32287	4	5.2915.3
2	2	1.73205	2	3.4641
3	2.5	2.179449	4	8.717798
4	3	2.645751	1	2.645751
				21.119181

$$\int_1^3 \sqrt{1 - x + x^2}\, dx \approx 3.5198.$$

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function of class C^4 on the interval $[a, b]$. If $n = 2m$ and $P = \{x_0, x_1, \ldots, x_{2m-1}\}$ a partition of the interval $[a, b]$. Then the remainder of the approximation of f by the following sum S_n

$$S_n(f) = \frac{b-a}{3n}\Big(f(a) + f(b) + 4\sum_{j=0}^{m-1} f(x_{2j+1}) + 2\sum_{j=1}^{m-1} f(x_{2j})\Big).$$

is approximated as follows:

$$|R_n| \le \frac{(b-a)^5 M_4}{180n^4}, \qquad M_4 = \sup_{x\in[a,b]} |f^{(4)}(x)|.$$

Example

Let $f(x) = \sqrt{2+x^2}$ defined on the interval $[0, 2]$. Use the Simpson method for $n = 6$ to approximate the integral $\int_0^2 \sqrt{2+x^2}dx$.

$x_k = \frac{k}{3}$,

k	x_k	m_k	$m_k f(x_k)$
0	0	1	$\sqrt{2}$
1	$1/3$	4	5.81186
2	$2/3$	2	3.1269438
3	1	4	6.92820
4	$4/3$	2	3.8873
5	$5/3$	4	8.7432513
6	2	1	2.44948974
			32.361254842

$\int_0^2 \sqrt{2+x^2}dx \approx 3.59569498.$
$f(x) = (2+x^2)^{\frac{1}{2}}$, $f'(x) = x(2+x^2)^{-\frac{1}{2}}$, $f''(x) = 2(2+x^2)^{-\frac{3}{2}}$,
$f^{(3)}(x) = -6x(2+x^2)^{-\frac{5}{2}}$, $f^{(4)}(x) = 12(2x^2-1)(2+x^2)^{-\frac{7}{2}}$ and
$f^{(5)}(x) = 60x(3-2x^2)(2+x^2)^{-\frac{9}{2}}$.
Using the variation of the function $f^{(4)}$ on the interval $[0,2]$, the value of M_4 is 1.07 , then $|R| \leq 10^{-5}$.