

### Chapter 3: Random Time Series Model

The random process  $\{y_t\}$  is called **general liner process** if it's possible to express it in the form:

$$y_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

where,  $\varepsilon_t \sim \text{iid } WN(0, \sigma_\varepsilon^2)$ ,  $\mu$  is a constant and  $\{\psi_t\}$  is a sequence of fixed values.

the process  $\{y_t\}$  is stationary if one of the following conditions is satisfied:

1.  $\psi_i$  are finite.
2.  $\sum_{i=0}^{\infty} \psi_i^2 < \infty$ .

#### Question 1\*:

For the model  $y_t = \varepsilon_t + 0.2y_{t-1}$  find  $\pi$ -weights and  $\psi$ -weights.

The  $\pi$ -weights formula  $y_t = \varepsilon_t + \pi_1y_{t-1} + \pi_2y_{t-2} + \pi_3y_{t-3} + \dots$

$$\pi_1 = 0.2 \quad \pi_2 = 0 \quad \pi_3 = 0$$

The  $\psi$ -weights formula  $y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots$

And for  $\psi$  weight, we find:

$$y_t = \varepsilon_t + 0.2y_{t-1}$$

$$y_{t-1} = \varepsilon_{t-1} + 0.2y_{t-2}$$

$$y_{t-2} = \varepsilon_{t-2} + 0.2y_{t-3}$$

$$y_{t-3} = \varepsilon_{t-3} + 0.2y_{t-4}$$

$$y_t = \varepsilon_t + 0.2[y_{t-1}]$$

$$y_t = \varepsilon_t + 0.2[\varepsilon_{t-1} + 0.2y_{t-2}]$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2[y_{t-2}]$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2[\varepsilon_{t-2} + 0.2y_{t-3}]$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2\varepsilon_{t-2} + 0.2^3y_{t-3}$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2\varepsilon_{t-2} + 0.2^3[y_{t-3}]$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2\varepsilon_{t-2} + 0.2^3[\varepsilon_{t-3} + 0.2y_{t-4}]$$

$$y_t = \varepsilon_t + 0.2\varepsilon_{t-1} + 0.2^2\varepsilon_{t-2} + 0.2^3\varepsilon_{t-3} + 0.2^4y_{t-4}$$

$$\psi_1 = 0.2 \quad \psi_2 = 0.2^2 = 0.04 \quad \psi_3 = 0.2^3 = 0.008$$

**Question 2:**

For the model  $y_t = \varepsilon_t - 0.3\varepsilon_{t-1}$  find the first three  $\pi$  weights and first three  $\psi$  weights.

solution:

The  $\psi$  weights formula  $y_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \psi_3\varepsilon_{t-3} + \dots$

$$\psi_1 = -0.3 \quad \psi_2 = 0 \quad \psi_3 = 0$$

The  $\pi$  weights formula  $y_t = \varepsilon_t + \pi_1y_{t-1} + \pi_2y_{t-2} + \pi_3y_{t-3} + \dots$

And for  $\pi$  weight, we find:

$$y_t = \varepsilon_t - 0.3\varepsilon_{t-1} \Rightarrow \varepsilon_t = y_t + 0.3\varepsilon_{t-1}$$

$$\varepsilon_{t-1} = y_{t-1} + 0.3\varepsilon_{t-2}$$

$$\varepsilon_{t-2} = y_{t-2} + 0.3\varepsilon_{t-3}$$

$$\varepsilon_{t-3} = y_{t-3} + 0.3\varepsilon_{t-4}$$

$$\varepsilon_t = y_t + 0.3[\varepsilon_{t-1}]$$

$$\varepsilon_t = y_t + 0.3[y_{t-1} + 0.3\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2\varepsilon_{t-2}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[\varepsilon_{t-2}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2[y_{t-2} + 0.3\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3\varepsilon_{t-3}$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3[\varepsilon_{t-3}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3[y_{t-3} + 0.3\varepsilon_{t-4}]$$

$$\varepsilon_t = y_t + 0.3y_{t-1} + 0.3^2y_{t-2} + 0.3^3y_{t-3} + 0.3^4\varepsilon_{t-4}$$

$$y_t = \varepsilon_t - 0.3y_{t-1} - 0.3^2y_{t-2} - 0.3^3y_{t-3} - 0.3^4\varepsilon_{t-4}$$

$$\pi_1 = -0.3 \quad \pi_2 = -0.3^2 \quad \pi_3 = -0.3^3$$

- **Finding  $\pi$ -weights and  $\psi$ -weights for mixed:**

The  $\psi$ - weights formula

$$y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots$$

The  $\pi$ - weights formula

$$y_t = \varepsilon_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \dots, \quad t = 0, \pm 1, \pm 2, \dots$$

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \dots$$

To find  $\psi$ - weights  $\phi(B) \psi(B) = \theta(B)$

To find  $\pi$ - weights  $\theta(B) \pi(B) = \phi(B)$

### Question 3:

For the model  $y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$

find  $\psi$ -weights and  $\pi$ -weights.

solution:

to find the  $\psi$ -weight and  $\pi$ -weights:

$$y_t = 0.7y_{t-1} + \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7y_{t-1} = \varepsilon_t + 0.6\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$$

$$y_t - 0.7By_t = \varepsilon_t + 0.6B\varepsilon_t - 0.4B^2\varepsilon_t$$

$$(1 - 0.7B)y_t = (1 + 0.6B - 0.4B^2)\varepsilon_t$$

$$\phi(B) y_t = \theta(B) \varepsilon_t$$

To find  $\psi$ - weights  $\boxed{\phi(B) \psi(B) = \theta(B)}$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^0: \boxed{\psi_0 = 1}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^1: \psi_1 - 0.7\psi_0 = 0.6 \Rightarrow \boxed{\psi_1 = 1.3}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^2: \psi_2 - 0.7\psi_1 = -0.4 \Rightarrow \boxed{\psi_2 = 0.3}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^3: \psi_3 - 0.7\psi_2 = 0 \Rightarrow \boxed{\psi_3 = 0.21}$$

$$(1 - 0.7B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = (1 + 0.6B - 0.4B^2)$$

$$B^4: \psi_4 - 0.7\psi_3 = 0$$

$$B^5: \psi_5 - 0.7\psi_4 = 0$$

$$\psi\text{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 1.3 \\ \psi_2 = 0.3 \\ \psi_j - 0.7\psi_{j-1} = 0 \quad ; \quad j \geq 3 \end{cases}$$

To find  $\pi$ - weights  $\theta(B) \pi(B) = \phi(B)$

$$(1 + 0.6 B - 0.4 B^2) (\pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \dots) = (1 - 0.7 B)$$

$$B^0: \quad \boxed{\pi_0 = 1}$$

$$B^1: \quad \pi_1 + 0.6 \pi_0 = -0.7 \Rightarrow \boxed{\pi_1 = -1.3}$$

$$B^2: \quad \pi_2 + 0.6 \pi_1 - 0.4 \pi_0 = 0$$

$$B^3: \quad \pi_3 + 0.6 \pi_2 - 0.4 \pi_1 = 0$$

$$\pi\text{- weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -1.3 \\ \pi_i + 0.6 \pi_{i-1} - 0.4 \pi_{i-2} = 0 & ; i \geq 2 \end{cases}$$

**Question 4:**

For the model  $y_t = 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1}$

find  $\psi$ -weights and  $\pi$ -weights.

to find the  $\psi$ -weight and  $\pi$ -weights:

$$\begin{aligned}y_t &= 0.1y_{t-1} + \varepsilon_t + 0.2\varepsilon_{t-1} \\y_t - 0.1y_{t-1} &= \varepsilon_t + 0.2\varepsilon_{t-1} \\(1 - 0.1B)y_t &= (1 + 0.2B)\varepsilon_t \\ \phi(B) y_t &= \theta(B) \varepsilon_t\end{aligned}$$

- To find  $\psi$ - weights  $\phi(B) \psi(B) = \theta(B)$

$$(1 - 0.1B) (\psi_0 + \psi_1 B + \psi_2 B^2 + \dots) = (1 + 0.2B)$$

$$B^0: \psi_0 = 1$$

$$B^1: \psi_1 - 0.1\psi_0 = 0.2 \Rightarrow \psi_1 = 0.3$$

$$B^2: \psi_2 - 0.1\psi_1 = 0$$

$$\psi\text{- weights} = \begin{cases} \psi_0 = 1 \\ \psi_1 = 0.3 \\ \psi_j - 0.1\psi_{j-1} = 0 \quad ; \quad j \geq 2 \end{cases}$$

- To find  $\pi$ - weights  $\theta(B) \pi(B) = \phi(B)$

$$(1 + 0.2B) (\pi_0 + \pi_1 B + \pi_2 B^2 + \dots) = (1 - 0.1B)$$

$$B^0: \pi_0 = 1$$

$$B^1: \pi_1 + 0.2\pi_0 = -0.1 \Rightarrow \pi_1 = -0.3$$

$$B^2: \pi_2 + 0.2\pi_1 = 0$$

$$\pi\text{- weights} = \begin{cases} \pi_0 = 1 \\ \pi_1 = -0.3 \\ \pi_i + 0.2\pi_{i-1} = 0 \quad ; \quad i \geq 1 \end{cases}$$

- *Auto-regressive processes:*

$$AR(p): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p}$$

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \dots + \phi_p \rho(k-p) ; k = 1, 2, \dots$$

$$AR(1): y_t = \varepsilon_t + \phi_1 y_{t-1}$$

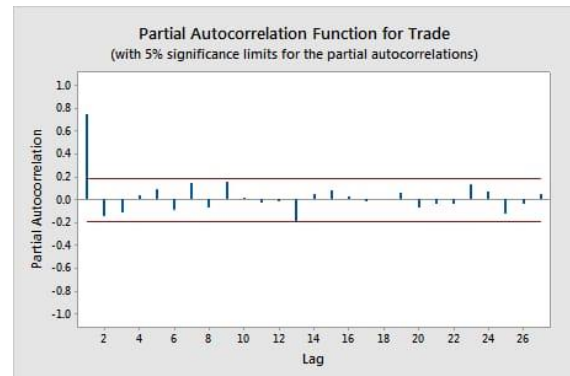
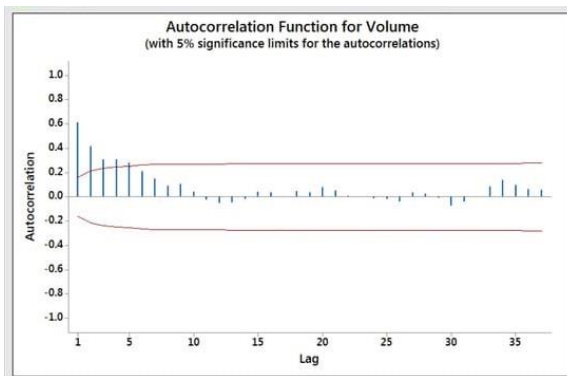
*AR models are always invertible.*

*Stationary condition for AR (1):*

$$|\phi_1| < 1 \quad \text{or} \quad |B| > 1$$

*Auto-correlation function (ACF) for AR (1):*

$$\rho(k) = \phi^k ; k = 1, 2, \dots$$



**Question 5:**

Assume the process  $\{y_t\}$  follow the AR (1) model with parameter  $\phi_1 = 0.4$ , and assuming that  $E(y_t) = 0$

1. write down the mathematical form of this model.

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

where  $\varepsilon_t \sim \text{iid } N(0, \sigma_\varepsilon^2)$  is the white noise process

2. is  $\{y_t\}$  stationary, why?

the stationary condition for the AR (1) is:

$$|\phi_1| < 1 \Rightarrow |0.4| < 1$$

so, this process is stationary.

3. Drive the autocorrelation function  $\rho_k$

$$y_t = \varepsilon_t + 0.4y_{t-1},$$

multiply both sides by  $y_{t-k}$  and taking the expectation.

$$y_t y_{t-k} = \varepsilon_t y_{t-k} + 0.4 y_{t-1} y_{t-k}$$

$$E(y_t y_{t-k}) = \underbrace{E(\varepsilon_t y_{t-k})}_0 + 0.4 E(y_{t-1} y_{t-k})$$

$$\gamma_k = 0.4 \gamma_{k-1} \quad \text{for } k \geq 1$$

dividing both sides by  $\gamma_0$

$$\frac{\gamma_k}{\gamma_0} = 0.4 \frac{\gamma_{k-1}}{\gamma_0}$$

$$\rho_k = 0.4 \rho_{k-1} \quad \text{for } k \geq 1$$

$$\rho_1 = 0.4 \rho_0 = 0.4(1) = 0.4$$

$$\rho_2 = 0.4 \rho_1 = 0.4(0.4) = 0.4^2$$

$$\rho_3 = 0.4 \rho_2 = 0.4(0.4^2) = 0.4^3$$

$$\rho_k = 0.4 \rho_{k-1} = 0.4(0.4^{k-1}) = 0.4^k$$



$$AR(2): y_t = \varepsilon_t + \phi_1 y_{t-1} + \phi_2 y_{t-2}$$

AR models are always invertible.

Stationary condition for AR (2):

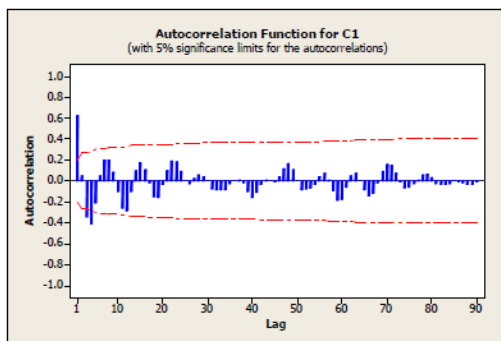
$$|\phi_2| < 1$$

$$\phi_2 + \phi_1 < 1$$

$$\phi_2 - \phi_1 < 1$$

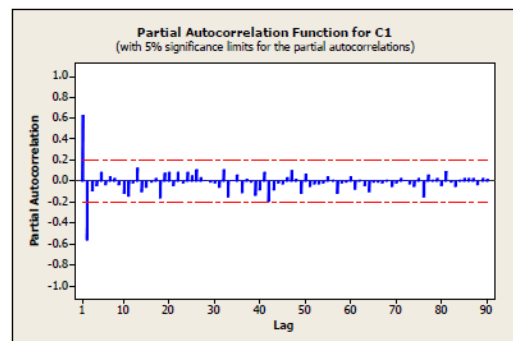
Auto-correlation function (ACF) for AR (2):

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2); \quad k = 1, 2, \dots$$



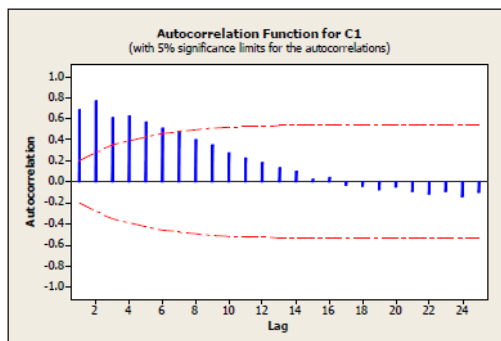
ACF for AR(2) model when

$$\phi_1 = 1, \phi_2 = -0.5$$



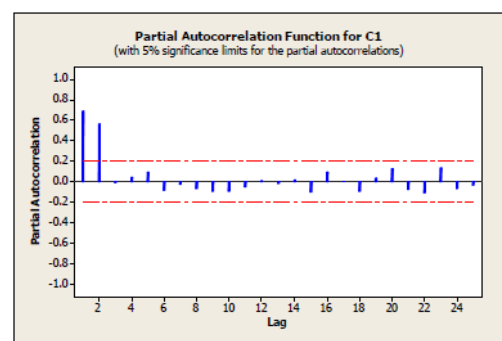
PACF for AR(2) model when

$$\phi_1 = 1, \phi_2 = -0.5$$



ACF for AR(2) model when

$$\phi_1 = 0.4, \phi_2 = 0.5$$



PACF for AR(2) model when

$$\phi_1 = 0.4, \phi_2 = 0.5$$

**Question 6:**

In the following question assume  $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, state its type, and check its stationarity and/or invertibility:

$$y_t = 0.7y_{t-1} + 0.5y_{t-2} + \varepsilon_t$$

model type: *AR (2) or ARMA (2,0) or ARIMA (2,0,0)*

stationarity: *since  $|\phi_2| = |0.5| < 1$*

$$\phi_2 - \phi_1 = 0.5 - 0.7 = -0.2 < 1$$

$$\phi_2 + \phi_1 = 0.5 + 0.7 = 1.2 > 1$$

*then this model is not stationary*

invertibility: *AR models are always invertible*

- Moving average:

$$MA(q) \ y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$\rho(k) = \begin{cases} \frac{-\theta_k + \theta_1 \theta_{k+1} + \dots + \theta_{q-k} \theta_q}{(1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2)} & ; \ k = 1, 2, \dots, q \\ 0 & ; \ k > q \end{cases}$$

$$MA(1): \ y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

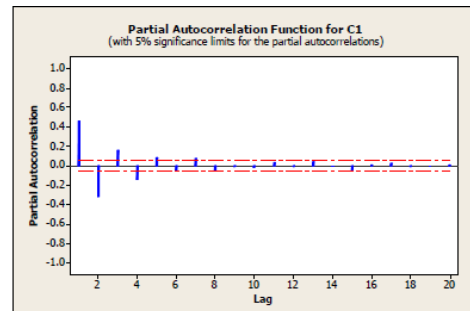
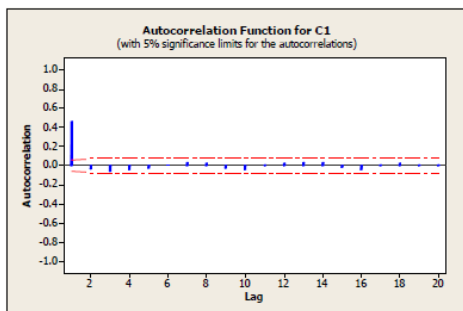
MA models are always stationary.

Invertibility conditions for the MA(1):

$$|\theta_1| < 1$$

Auto-correlation function (ACF) for MA(1):

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta}{(1 + \theta^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

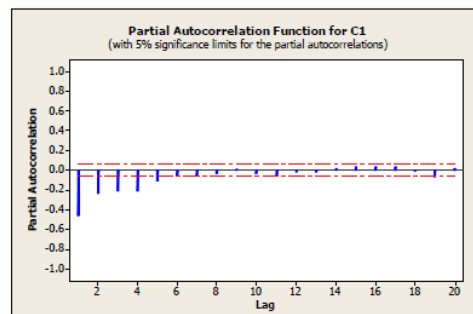
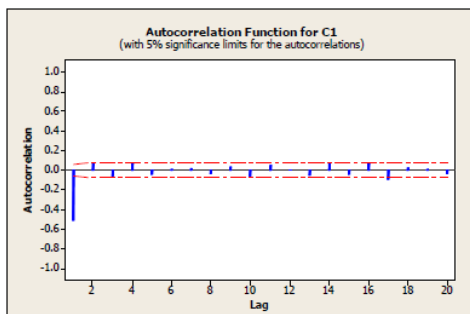


• ACF for MA(1) model when

$$\theta_1 = -0.7$$

• PACF for MA(1) model when

$$\theta_1 = -0.7$$



• ACF for MA(1) model when

$$\theta_1 = 0.7$$

• PACF for MA(1) model when

$$\theta_1 = 0.7$$

$$MA(2): y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

MA models are always stationary.

Invertibility conditions for the MA (2):

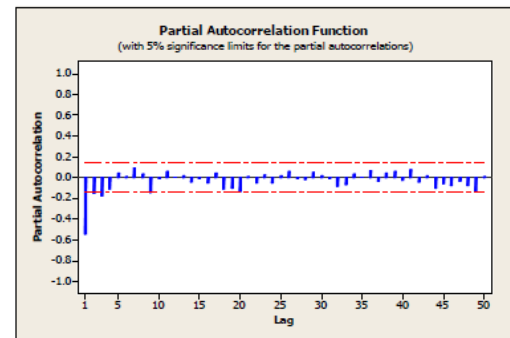
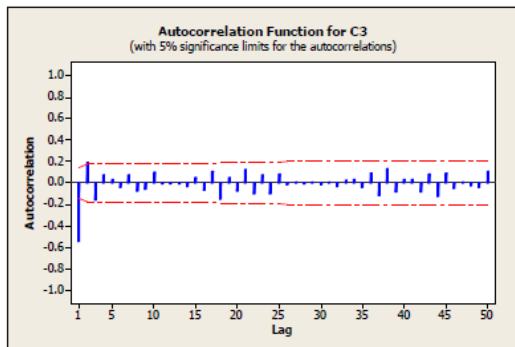
$$|\theta_2| < 1$$

$$\theta_2 + \theta_1 < 1$$

$$\theta_2 - \theta_1 < 1$$

Auto-correlation function (ACF) for MA (2):

$$\rho(k) = \begin{cases} \frac{-\theta_1(1-\theta_2)}{(1+\theta_1^2+\theta_2^2)} & k = 1 \\ \frac{-\theta_2}{(1+\theta_1^2+\theta_2^2)} & k = 2 \\ 0 & k \geq 2 \end{cases}$$

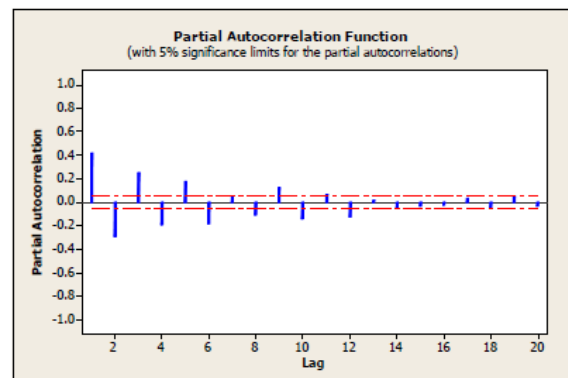
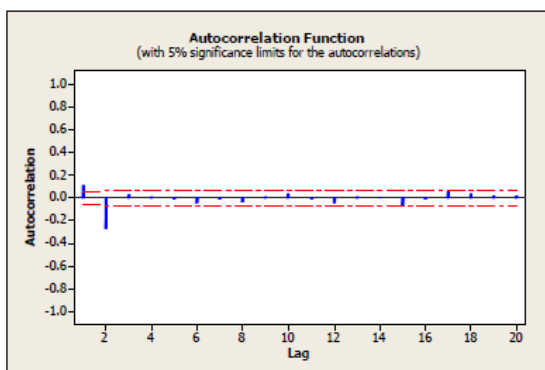


ACF for MA(2) model when

$$\theta_1 = 0.7, \theta_2 = -0.1$$

PACF for MA(2) model when

$$\theta_1 = 0.7, \theta_2 = -0.1$$



ACF for MA(2) model when

$$\theta_1 = 1, \theta_2 = -0.7$$

PACF for MA(2) model when

$$\theta_1 = 1, \theta_2 = -0.7$$

**Question 7:**

For MA (1) process with  $\rho(1) = 0.4$ ,

Find the value of  $\theta$  that make this model is invertible.

$$y_t = \varepsilon_t - \theta \varepsilon_{t-1}$$

$$\text{Var}(y_t) = \text{Var}(\varepsilon_t - \theta \varepsilon_{t-1}) = \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 = (1 + \theta^2) \sigma_\varepsilon^2$$

$$\text{Cov}(y_t, y_{t-k}) = E[(\varepsilon_t - \theta \varepsilon_{t-1})(\varepsilon_{t-k} - \theta \varepsilon_{t-k-1})]$$

$$\text{for } k = 1 \quad E[(\varepsilon_t - \theta \varepsilon_{t-1})(\varepsilon_{t-1} - \theta \varepsilon_{t-2})] = -\theta \sigma_\varepsilon^2$$

$$\rho(k) = \begin{cases} 1 & k = 0 \\ -\frac{\theta}{(1+\theta^2)} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

$$\rho(1) = \frac{\gamma_1}{\gamma_0} = \frac{-\theta \sigma_\varepsilon^2}{(1+\theta^2)\sigma_\varepsilon^2} = \frac{-\theta}{1+\theta^2} = 0.4$$

$$\frac{-\theta}{1+\theta^2} = 0.4 \Rightarrow -\theta = 0.4 + 0.4\theta^2$$

$$\Rightarrow 0.4\theta^2 + \theta + 0.4 = 0$$

or we can use the calculator

The image shows four sequential screenshots of a calculator interface:

- 1:** The calculator is in the 'MODE' menu. The 'EQ' mode is selected.
- 2:** The screen shows '1: Simul Equation' and '2: Polynomial' with a cursor pointing to 'Polynomial'.
- 3:** The screen asks 'Polynomial Degree?' and 'Select 2~4'.
- 4:** The screen shows the quadratic equation form:  $ax^2+bx+c$  with input fields for coefficients. The value '0' is entered in the 'c' field.

$$\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(0.4)(0.4)}}{2(0.4)} = (-0.5) \text{ or } (-2)$$

MA (1) is invertibility  $\Rightarrow |\theta| < 1$  then,  $\theta = -0.5$

**Question 8:**

Assume the process  $\{y_t\}$  follow the MA (2) model with parameters

$$\theta_1 = 0.4 \text{ and } \theta_2 = -0.8$$

1. write down the mathematical form of this model.

$$y_t = \varepsilon_t - 0.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

where  $\varepsilon_t$  is  $WN(0, \sigma_\varepsilon^2)$

2. is  $\{y_t\}$  invertible?

the invertible condition for the MA (2) is:

- $|\theta_2| < 1 \Rightarrow |-0.8| < 1$
- $\theta_2 + \theta_1 < 1 \Rightarrow -0.8 + 0.4 \Rightarrow -0.4 < 1$
- $\theta_2 - \theta_1 < 1 \Rightarrow -0.8 - 0.4 \Rightarrow -1.2 < 1$

the invertibility conditions are satisfied.

**Question 9:**

In the following question assume  $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, **state its type**, and check its **stationarity and/or invertibility**:

$$y_t = \varepsilon_t + 0.5\varepsilon_{t-1} - 1.2\varepsilon_{t-2}$$

model type: **MA (2) or ARMA (0,2) or ARIMA (0,0,2)**

Stationarity: **Moving Average models are always stationary**

Invertibility: **since  $\theta_2 = 1.2 > 1$ , then this model is not invertible.**

Autoregressive-Moving average processes:

$$ARMA(p, q) \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

for ARMA(p, q):

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \dots - \phi_p y_{t-p} = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \dots - \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t - \dots - \phi_p B^p y_t = \varepsilon_t - \theta_1 B \varepsilon_t - \theta_2 B^2 \varepsilon_t - \dots - \theta_q B^q \varepsilon_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) y_t = (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q) \varepsilon_t$$

$$\phi(B) y_t = \theta(B) \varepsilon_t$$

then, ARIMA(p, d, q) is:

$$\phi(B) \nabla^d y_t = \theta(B) \varepsilon_t$$

$$\Rightarrow \phi(B) z_t = \theta(B) \varepsilon_t$$

Model	ARMA(p, q)	ARIMA(p, d, q)
AR(p)	ARMA(p, 0)	ARIMA(p, 0, 0)
AR(2)	ARMA(2, 0)	ARIMA(2, 0, 0)
MA(q)	ARMA(0, q)	ARIMA(0, 0, q)
MA(1)	ARMA(0, 1)	ARIMA(0, 0, 1)

$$\text{ARMA}(1,1) \quad y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

Auto-correlation function (ACF) for ARMA (1,1):

$$\rho_k = \begin{cases} \frac{(\phi_1 - \theta_1)(1 - \phi_1 \theta_1)}{(1 + \theta_1^2 - 2\phi_1 \theta_1)} & ; k = 1 \\ \phi_1 \rho_{k-1} & ; k = 2, 3, \dots \end{cases}$$

**Question 10:**

In the following question assume  $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

For the following model, **state its type**, and check its **stationarity and/or invertibility**:

$$y_t = 1.5y_{t-1} + \varepsilon_t + 0.25\varepsilon_{t-1}$$

model type: **ARMA (1,1) or ARIMA (1,0,1)**

stationarity: **since  $|\phi_1| = |1.5| > 1$ , then this model is not stationary**

invertibility: **since  $|\theta_1| = |-0.25| < 1$ , then this model is invertible.**

**Question 11:**

for the following model  $y_t = 0.2 y_{t-1} + \varepsilon_t - 1.1 \varepsilon_{t-1} + 0.18 \varepsilon_{t-2}$

$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$ , state its type, and check its stationary and / or invertibility.

$$y_t - 0.2 y_{t-1} = \varepsilon_t - 1.1 \varepsilon_{t-1} + 0.18 \varepsilon_{t-2}$$

$$y_t - 0.2B y_t = \varepsilon_t - 1.1 B \varepsilon_t + 0.18 B^2 \varepsilon_t$$

$$(1 - 0.2B) y_t = (1 - 1.1 B + 0.18 B^2) \varepsilon_t$$

$$(1 - 0.2B) y_t = (1 - 0.2 B)(1 - 0.9 B) \varepsilon_t$$

$$y_t = (1 - 0.9 B) \varepsilon_t$$

model type: **MA (1) or ARMA (0,1) or ARIMA (0,0,1)**

stationarity: **MA (1) always stationary**

invertibility: **since  $|\theta_1| = |0.9| < 1$  then this model is invertible.**



**Question 12:**

write the model ARIMA(1,1,1) in the final form in terms of

$$z_t = \nabla y_t, y_t \text{ and } \varepsilon_t$$

for ARMA(1,1)

$$y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 y_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - \phi_1 B y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$(1 - \phi_1 B) y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$


for ARIMA(1,1,1)

$$(1 - \phi_1 B) z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} \quad ; \quad z_t = \nabla y_t = y_t - y_{t-1} = (1 - B)y_t$$

$$z_t - \phi_1 B z_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t - \phi_1 z_{t-1} = \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$z_t = \phi_1 z_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$


$$y_t - y_{t-1} = \phi_1 (y_{t-1} - y_{t-2}) + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t - y_{t-1} = \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$y_t = y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}$$

$$\boxed{y_t = (1 + \phi_1) y_{t-1} - \phi_1 y_{t-2} + \varepsilon_t - \theta_1 \varepsilon_{t-1}}$$