Differential Equations

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2022

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May 26, 2023

Contents

Introduction

This book provides an introduction to ordinary differential equations and systems of differential equations for beginning graduate students. We introduce some basic methods for solving differential equations and study some of the most important of them. As a prerequisite, it is essential the reader has basic knowledge of differential and integral calculus and linear algebra.
This manual is intended primarily for undergraduate students in mathematics, science or engineering, who generally take a course on differential equations.
Note that, differential equations are primarily used to study physical processes and to model them. These differential equations come from physical models.

A differential equation is a relation involving an unknown function, some of its derivatives and known quantities and functions. For example, $\dfrac{dy}{dx} = y + e^{-x}$.
Ordinary differential equations are differential equations whose unknowns are functions of a single variable.
Many physical laws are formulated as differential equations. (The gravity law, the Kepler law,..). For example Newton's second law of motion tells us that the force on an object is equal to the product of its mass, m and its acceleration $F = \dfrac{dv}{dt}$.
Solving the differential equation means finding y in terms of x. This is not usually possible, moreover in general under some assumptions, we have local solutions.

The object of this course is the qualitative study of a differential equations: existence and uniqueness of the solutions, the study of their domain of definition, how to solve differential equations explicitly.

1 First-Order Differential Equations

Definitions and Basic Concepts

Definition 0.1.

1. An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies this equation is called a solution of the differential equation. The most general form of an ordinary differential equation is:

$$F(x, y(x), y^{(1)}(x), \ldots, y^{(n)}(x)) = 0, \tag{1.1}$$

   where $F\colon \Omega \longrightarrow \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^{n+2}$.

2. The highest derivative that appears in the ordinary differential equation is called the order of that ordinary differential equation.

3. (a) If the ordinary differential equation (1.1) is in the form

$$a_0(x)y^{(n)}(x) + \ldots + a_{n-1}(x)y^{(1)}(x) + a_n(x)y(x) = f(x) \tag{1.2}$$

   the differential equation is called linear. If such representation is not possible, the differential equation is called nonlinear.

   (b) If $f = 0$ in (1.2), the equation is called an homogeneous linear ordinary differential equation.

   (c) If the functions $a_0, a_1, \ldots, a_n$ are constants, the equation (1.2) is called a linear ordinary differential equation with constant coefficients.

(d) Similarly, if the functions $a_0, a_1, \dots, a_n$ are constants and $f = 0$, the equation (1.2) is called a linear homogeneous ordinary differential equation with constant coefficients.

Definition 0.2.
A function φ defined on an interval I is called a solution of the differential equation (1.1) provided that the n derivatives of the function exist on the interval I and

$$F(x, \varphi(x), \varphi^{(1)}(x), \dots, \varphi^{(n)}(x)) = 0, \quad \forall x \in I.$$

There is a classification of solutions of an ordinary differential equation:

1. Exact solution, i.e. we obtain a solution in a closed form.
2. A relation $f(x, y) = 0$ is said to be an implicit solution of the differential equation (1.1) on an interval I, if the relation defines implicitly a function $y = \varphi(x)$ which satisfies the differential equation (1.1) on an interval I.
3. We prove under some conditions on the function F the existence, the uniqueness and some others properties of the solution of initial conditions without finding this solution.
4. Using computer techniques and under some conditions, we can give some approximation of the solution of the differential equation.

Remark 1 :

1. An ordinary differential equation may sometimes have solution that can not be obtained from the general solution. Such a solution is called singular solution.
   For example, the differential equation ${y'}^2 - xy' + y = 0$ has the general solution $y = cx - c^2$ but also has a solution $y_s = \frac{1}{4}x^2$ that cannot be obtained from the general solution by choosing specific values of c.
2. In general, there is no simple formula or procedure to find solutions of differential equations.

Example 0.1 :

1. $y = e^{-x}$ is a solution of the ordinary differential equation $y' = -y$.
2. $y = \dfrac{1}{1-x}$ is a solution of the ordinary differential equation $(1-x)y' - y = 0$.

3. The function $y = e^{-2x}$ is a solution of the ordinary differential equation $y^{''} + 5y' - 6y = 0$.
4. $y = \cos(3x)$ is a solution of the ordinary differential equation $y^{''} + 9y = 0$.
5. The relation $2xy + \ln|y| - 1 = 0$ defines an implicit solution of the differential equation $(2xy + 1)y' + 2y^2 + y = 0$.
6. Consider the differential equation $y' = \dfrac{xy}{y^2+1}$. This equation is equivalent to $\left(y + \dfrac{1}{y}\right) dy = xdx.$
   After integration we get $y^2 + 2\ln|y| = x^2 + c$. This is an implicit solution.
7. Consider the following differential equation: $y' = \frac{1}{3y^2+1}$, $y(0) = 1$. After integration we get $y^3 + y = x + 2$. This is an implicit solution.
8. Consider the following differential equation: $y' = \dfrac{\sin(x)}{\cos(y)}$.
   After integration we get $\sin(y) = -\cos(x) + c$. This is an implicit solution.

1 First-Order Differential Equations

1.1 Introduction

Definition 1.1.
Let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function on an open subset Ω of $\mathbb{R}^2$. The following equation is called the reduced form of a first order ordinary differential equation:

$$y' = f(x, y), \quad (x, y) \in \Omega \tag{1.3}$$

Remark 2 :
If f is a continuous function of one variable x, the equation becomes

$$y' = f(x).$$

The solution is a primitive of the function f,

$$y(x) = \int f(x)\, dx + c.$$

1.2 The Cauchy Problem

Definition 1.2.
Let $(x_0, y_0) \in \Omega$, the Cauchy problem of the differential equation (1.3) at (x_0, y_0) is to find a solution $y\colon I \longrightarrow \mathbb{R}$ such that x_0 is an interior point of I and $y(x_0) = y_0$.
The point (x_0, y_0) is called the initial condition of the Cauchy problem.

Remark 3 :

1. An explicit solution of the Cauchy problem is a solution of the form $y = g(x)$, where $g\colon I \longrightarrow \mathbb{R}$ is a differentiable function and fulfills:

   (a) $\forall\, x \in I$, $(x, g(x)) \in \Omega$,

   (b) $\forall\, x \in I$, $y'(x) = f(x, g(x))$.

2. An implicit solution of the Cauchy problem is a solution of the form $G(x, y) = 0$, such that

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases} \Longleftrightarrow G(x,y) = 0.$$

Remark 4 :

1. For any point $M = (x_0, y_0) \in \Omega$, consider the straight line D_M passing through M and of slope $f(x_0, y_0)$. The equation of D_M is

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

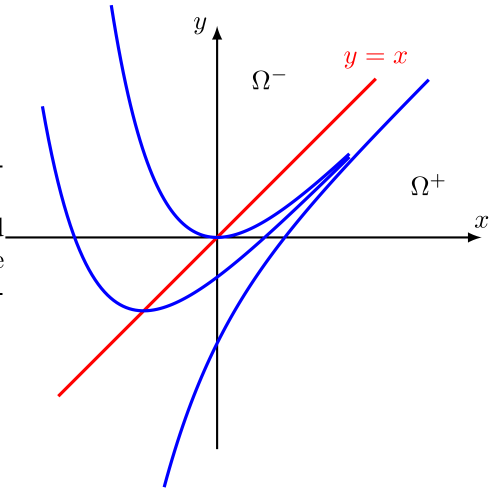
   The straight line D_M is tangent to the solution (if it exists) of the Cauchy problem at (x_0, y_0) of the differential equation (1.3) .

   The map $M \longmapsto D_M$ is called the vector field of the tangents associated to the equation (1.3).

   For example, the function $e^{\alpha x}$ is the solution of the Cauchy problem at $(0, 1)$ of the differential equation $y' = \alpha y$. If $M = (0, 1)$, the equation of the straight line D_M is $y = 1 + \alpha x$.

2. Consider the differential equation (1.3) on the open subset $\Omega^+ = \{(x, y) : f(x, y) > 0\}$, any solution of the differential equation (1.3) is increasing since $y' = f(x, y) > 0$ and on the open set $\Omega^- = \{(x, y) : f(x, y) < 0\}$, any solution of the differential equation (1.3) is decreasing.

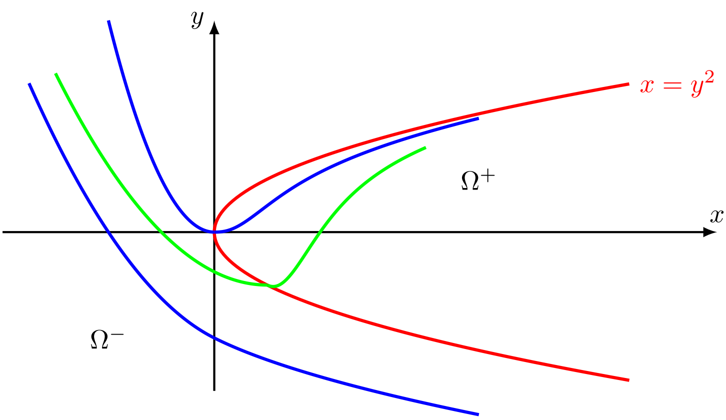
For example, consider the differential equation $y' = x - y$.
$\Omega^+ = \{(x, y) \in \mathbb{R}^2;\ y < x\}$ and $\Omega^- = \{(x, y) \in \mathbb{R}^2;\ y > x\}$. The solutions of this differential equation are $y = (x-1) + ce^{-x}$, $c \in \mathbb{R}$.



Consider also the differential equation

$$y' = f(x, y) = x - y^2$$

$\Omega^+ = \{(x, y) \in \mathbb{R}^2;\ y^2 < x\}$ and $\Omega^- = \{(x, y) \in \mathbb{R}^2;\ y^2 > x\}$.



1.3 Maximal Solutions

Definition 1.3.
Let $y_1 : I \longrightarrow \mathbb{R}$ and $y_2 : J \longrightarrow \mathbb{R}$ be two solutions of the differential equation $y' = f(x, y)$. The function y_2 is called an extension of y_1 if $I \subset J$ and $y_2{\restriction_I} = y_1$.

Definition 1.4.
A solution $y : I \longrightarrow \mathbb{R}$ of the differential equation is called maximal if y can not have an extension $y_1 : J \longrightarrow \mathbb{R}$, with $J \supset I$.

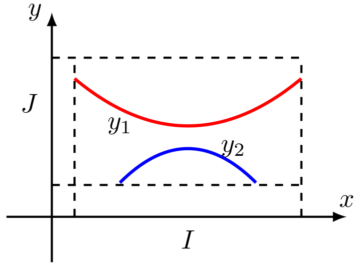
Theorem 1.5.
Any solution y of the differential equation $y' = f(x, y)$ may be extended to a maximal solution (not necessarily unique).

1.4 Global Solutions

If $\Omega = I \times J$, I and J are open intervals of $\mathbb{R}$.

Definition 1.6.
A global solution of (1.3) is a solution defined on the interval I.

Any global solution is maximal but the inverse is false.



Example 1.1 :
Consider the differential equation: $y' = y^2$ on $\mathbb{R} \times \mathbb{R}$.
$y = 0$ is a global solution.
If y is zero free, the differential equation is equivalent to $\frac{y'}{y^2} = 1$, then $\frac{-1}{y(x)} = x + c$ and $y(x) = \frac{-1}{x+c}$.
This solution is defined on $]-\infty, -c[\cup]-c, +\infty[$. It is maximal but non global.

1.5 Regularity of Solutions

Recall that a function of several variables f is called of class $\mathcal{C}^k$ if the partial derivatives of f of order less or equal to k are continuous.

Theorem 1.7.
If $f\colon \Omega \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^k$, any solution of the differential equation $y' = f(x, y)$ is of class $\mathcal{C}^{k+1}$.

Proof .
If $k = 0$, f is continuous. By assumption a solution $g\colon I \longrightarrow \mathbb{R}$ is differentiable and $g' = f(x, g(x))$ is continuous, thus g is of class $\mathcal{C}^1$.
Assume that if f is of class $\mathcal{C}^{k-1}$, any solution of the differential equation $y' = f(x, y)$ is of class $\mathcal{C}^k$.

If f is of class $\mathcal{C}^k$ and g a solution of the differential equation $y' = f(x, y)$. Since $g' = f(x, g(x))$, g is at least of class $\mathcal{C}^k$. Also as f is of class $\mathcal{C}^k$, then g' is of class $\mathcal{C}^k$ and then g is of class $\mathcal{C}^{k+1}$.

□

1.6 Local Existence Theorem

Theorem 1.8. [Local Existence Theorem (The Picard's Theorem)]
If the function $f\colon \Omega \longrightarrow \mathbb{R}$ is continuously differentiable with respect to the variable y and continuous on Ω, then for any $(x_0, y_0) \in \Omega$ there exists a unique solution of the initial value problem $y' = f(x, y)$ and $y(x_0) = y_0$ for x in some open interval containing x_0.

Example 1.2 :
Consider on $\mathbb{R}$ the following differential equation

$$y'(t) = \sqrt{|y|}.$$

$y = 0$ is a solution of the differential equation on $\mathbb{R}$.
If $y > 0$, there is a constant $c \in \mathbb{R}$ such that $2\sqrt{y} = x + c$. This solution is defined only for $x > -c$ and in this case $y = (\dfrac{x+c}{2})^2$.
If $y < 0$, there is a constant $c \in \mathbb{R}$ such that $-2\sqrt{-y} = x + c$. This solution is defined only for $x < -c$ and in this case $y = -(\dfrac{x+c}{2})^2$.
The functions $y_1 = 0$, $y_2(x) = \begin{cases} \frac{1}{4}x^2 & \text{for } x \geq 0, \\ 0 & \text{for } x \leq 0, \end{cases}$, $y_3(x) = \begin{cases} \frac{1}{4}x^2 & \text{for } x \geq 0, \\ -\frac{1}{4}x^2 & \text{for } x \leq 0, \end{cases}$, $y_4(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ -\frac{1}{4}x^2 & \text{for } x \leq 0, \end{cases}$ are solutions of the differential equation with the initial value problem $y(0) = 0$. In spite of the fact that the theorem conditions are not necessary, this gives an example where the theorem conditions are not satisfied and we do not have the uniqueness of the solution.

1.7 Exercises

1-1 Give an example of an initial value problem with multiple solutions.

1-2 Give an example of an initial value problem which has no solution.

1-3 Characterize the continuous functions f on $\mathbb{R}$ such that $\displaystyle\int_0^x f(t)dt + 1 = f(x)$.

1-4 Discuss the uniqueness of solutions of the following differential equation on the interval $[0, +\infty[$

$$y'(t) = -3\sqrt[3]{y^2(x)}, \quad y(0) = 1.$$

2 Exact Differential Equations and Integrating Factor

In general, there is no simple formula or procedure to find solutions of differential equations.

2.1 Exact Differential Equations

By a domain in $\mathbb{R}^2$, we mean a connected open subset.

Definition 2.1.
The differential equation

$$M(x,y)y' + N(x,y) = 0 \tag{1.4}$$

is said to be exact on a domain Ω if there is a function $F(x,y)$ defined on Ω such that

$$\frac{\partial F}{\partial x} = N(x,y), \quad \frac{\partial F}{\partial y} = M(x,y).$$

In this case, if M, N are continuously differentiable on Ω, we get

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}. \tag{1.5}$$

Remark 5 :

1. If the domain Ω is a rectangle, the condition (1.5) is also sufficient for the exactness of the differential equation on Ω.

2. If the differential equation is exact, we get
   $M(x,y(x))y' + N(x,y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' = \frac{d}{dx}F(x,y(x)) = 0$. It follows that $F(x,y(x)) = c$, with $c \in \mathbb{R}$.
   This is an implicit form of the solution y.

Example 2.1 :
Consider the differential equation $2x^2yy' + 2xy^2 + 1 = 0$.
$M = 2x^2y$, $N = 2xy^2 + 1$ and $\Omega = \mathbb{R}^2$. The differential equation is exact on Ω.

$$\frac{\partial N}{\partial y} = 4xy = \frac{\partial M}{\partial x}.$$

To find F we have:

$$\frac{\partial F}{\partial x} = 2xy^2 + 1, \quad \frac{\partial F}{\partial y} = 2x^2y.$$

If we integrate the first differential equation with respect to x holding y fixed, we get

$$F(x, y) = x^2y^2 + x + f(y).$$

Differentiating this equation with respect to y, we have

$$\frac{\partial F}{\partial y} = 2x^2y + f'(y) = 2x^2y$$

using the second equation. Hence $f'(y) = 0$ and $f(y)$ is a constant function. The solutions of the differential equation in implicit form is $x^2y^2 + x = c$.

Example 2.2 :

Consider the differential equation $y' = \dfrac{x - y}{x + y}$. This differential equation can be written in the form $y - x + (x + y)y' = 0$ which is an exact differential equation. In this case, the solution in implicit form is $x(y - x) + y(x + y) = c$, i.e., $y^2 + 2xy - x^2 = c$.

2.2 Integration Factor Method

If the differential equation $My' + N = 0$ is not exact it can sometimes be made exact by multiplying it by a continuously differentiable function $g(x, y)$. Such a function is called an integrating factor.

If g is an integrating factor, then $\dfrac{\partial(gM)}{\partial x} = \dfrac{\partial(gN)}{\partial y}$ and it can be written in the form

$$\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) g = -M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y}.$$

1. If g is a function of x only, then

$$\frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right) = A(x)$$

   is a function only of x and $g' = Ag$.

2. If g is a function of y only, then

$$\frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) = B(y)$$

   is a function only of y and $g' = Bg$.

Example 2.3 :
$(x^2y - x)y' + 2x^2 + y = 0$. In this case

$$\frac{1}{M}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) = \frac{2xy-2}{x^2y-x} = \frac{2}{x}.$$

So $g = \dfrac{1}{x^2}$ is an integrating factor and the general solution of the differential equation is $2x - \dfrac{y}{x} + \dfrac{1}{2}y^2 = c$ or $2x^2 - y + \frac{1}{2}xy^2 = cx$.

Example 2.4 :
Consider the differential equation: $y + (2x - ye^y)y' = 0$. In this case

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y}$$

so there is an integrating factor which is a function of y only and satisfies $g' = \frac{1}{y}g$. Hence y is an integrating factor and $y^2 + (2xy - y^2e^y)y' = 0$ is an exact differential equation with general solution $xy^2 - (y^2 - 2y + 2)e^y = c$.

Remark 6 :
The solutions of the exact differential equation obtained by multiplying by the integrating factor may have solutions which are not solutions of the original differential equation. This is due to the fact that g may be zero and we will have the possibly to exclude those solutions where g vanishes. However, this is not the case for the above examples.

2.3 Exercises

2-1 Solve the following differential equations

1) $(x^2 + 3y^2)y' + 2xy = 0$
2) $(2y + xe^y)y' + e^y = 0$
3) $(2y\sqrt{x^2 - y^2})y' - (1 + 2x\sqrt{x^2 - y^2}) = 0$
4) $(6x^2 - y + 3)y' + (12xy - \sin x) = 0.$
5) $(\dfrac{2x}{y^3})y' + (\dfrac{2}{x} - \dfrac{1}{y^2}) = 0$
6) $(y^2 + x)y' + x^2 + y = 0$
7) $(3x^2y + y^3)y' + (x^2 + 3xy^2) = 0$

2-2 Test the following differential equations for exactness, and find the general solution for those which are exact.

1) $(x^3+y^3)y'+3x^2y=0$,
2) $(y^2-x^2)y'+(x^2-y^2)=0$,
3) $xe^{xy}y'+ye^{xy}=0$,
4) $x^2y'-2xy=0$
5) $xy'+(x^3+y)=0$,
6) $\cos xy'=(y\sin x+e^x)=0$,
7) $y^{'}\tan(x+y)=1-\tan(x+y)$,

2-3 Find an integrating factor and solve the following differential equations:

1) $\dfrac{x^2}{y}y'+2x=0$,
2) $(y^4+3x)y'-y=0$
3) $xy^{'}-y=x^2\sin x$,
4) $[2(\sin(x)+\sin(y))+y\cos(y)]y'+y\cos(x)=0$.

3 Separable Differential Equations

Definition 3.1.
The first order ordinary differential equation $y'=f(x,y)$ is said to be separable if $f(x,y)$ can be expressed as a product of a function of x and a function of y. The differential equation has the form

$$y'=g(x)h(y). \tag{1.6}$$

This differential equation can be rewritten as

$$A(x)dx+B(y)dy=0. \tag{1.7}$$

3.1 Autonomous Differential Equations $y'=f(x)$

where $f\colon I\longrightarrow\mathbb{R}$ is a continuous function.
The solutions of this differential equation are given by: $y(x)=F(x)+c$, where $c\in\mathbb{R}$ and F any primitive of f.

3.2 Logistic Differential Equations $y' = g(y)$

where $g\colon I \longrightarrow \mathbb{R}$ is a continuous function.

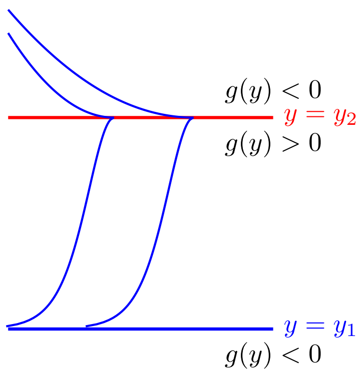
Denote $(y_j)_{j\in M}$ the roots of the equation $g(y) = 0$ in the interval J. The functions $y(x) = y_j$ are singular solutions of the differential equation $y' = g(y)$. On the open set $\Omega = \{(x,y) \in \mathbb{R}\times J;\ g(y) \neq 0\}$, the differential equation $y' = g(y)$ is equivalent to the following differential equation $\frac{dy}{g(y)} = dx$.
The solutions of this differential equation are given by $G(y) = x + c$, where $c \in \mathbb{R}$ and G any primitive of the function $\dfrac{1}{g}$ on each of the open intervals $]y_j, y_{j+1}[$.

As $G' = \dfrac{1}{g}$ and g is positive or negative on the interval $]y_j, y_{j+1}[$, the function G is monotone and bijective $G\colon]y_j, y_{j+1}[\longrightarrow]a_j, b_j[$, with $a_j \in [-\infty, +\infty[$ and $b_j \in]-\infty, +\infty]$. Thus the solution is given by $y = G^{-1}(x+c)$, with $c \in \mathbb{R}$.
Assume now that $g > 0$, G is increasing on the interval $]y_j, y_{j+1}[$.

• If the integral $\displaystyle\int_{y_j}^{y_j+\varepsilon} \frac{dy}{g(y)}$ is divergent, $a_j = -\infty$ since $x = G(y) - c \underset{y\longrightarrow y_j^+}{\longrightarrow} -\infty$. In this case the curve is asymptotic to the straight line of equation $y = y_j$.

• If the integral $\displaystyle\int_{y_j}^{y_j+\varepsilon} \frac{dy}{g(y)}$ converges, $a_j \in \mathbb{R}$ and $x \underset{y\to y_j^+}{\longrightarrow} a_j - c$ and $y' = g(y) \underset{y\longrightarrow y_j^+}{\longrightarrow} 0$.

The solution passes trough the point $(a_j - c, y_j)$ and the line of equation $y = y_j$ is tangent at this point.
In this case, when the integral converges, there is no uniqueness of the Cauchy problem.
The shape of the curve is as follows (we assume that the integral converges at $y_2 - 0$ and diverges at $y_1 \pm 0$ and $y_2 + 0$).



Exercise 3.1 :
Prove that $\displaystyle\int_{y_j}^{y_j+\varepsilon} \frac{dy}{g(y)}$ diverges if g is of class $\mathcal{C}^1$.

Example 3.1 :
Consider a logistic differential equation

$$y' = y(1-y).$$

$y = 0$ and $y = 1$ are singular solutions of the differential equation.
For $y \neq 0$ and $y \neq 1$, the differential equation is equivalent to the following: $\frac{y'}{y(1-y)} = 1$. Integrating both sides with respect to the variable x, we get $\ln\left|\dfrac{y}{1-y}\right| = x + c$ or $\dfrac{y}{1-y} = \lambda e^x$. $\lambda \in \mathbb{R}$ can be determined by the initial condition $y(0) = y_0$. ($\lambda = \dfrac{y_0}{1-y_0}$). Then $y = \dfrac{y_0 e^x}{(1-y_0) + y_0 e^x}$.

- If $0 < y_0 < 1$; $\lambda > 0$. The solution $y(x) \in (0, 1)$, for all $x \in \mathbb{R}$, $\lim\limits_{x \to +\infty} y(x) = 1$ and $\lim\limits_{x \to -\infty} y(x) = 0$.
- If $y_0 > 1$; $\lambda < 0$. $\lim\limits_{x \to +\infty} y(x) = 1$.
- If $y_0 < 0$; $\lambda < 0$. $\lim\limits_{x \to -\infty} y(x) = 0$.

Example 3.2 :
Consider the following differential equation

$$y' = \sqrt{|y|}(1 - y).$$

$y = 0$ and $y = 1$ are singular solutions of this differential equation.
For $y \neq 0$ and $y \neq 1$, the differential equation is equivalent to the following differential equation: $\frac{y'}{\sqrt{|y|}(1-y)} = 1$.

- If $0 < y_0 = y(0) < 1$. Integrating both sides with respect to the variable x, we get $\ln\left|\dfrac{1+\sqrt{y}}{1-\sqrt{y}}\right| = x + c$, then

$$y = \left(\frac{\lambda e^x - 1}{\lambda e^x - 1}\right)^2,$$

where $\lambda = \dfrac{1+\sqrt{y_0}}{1-\sqrt{y_0}} > 0$. $\lim\limits_{x \to +\infty} y(x) = 1$.

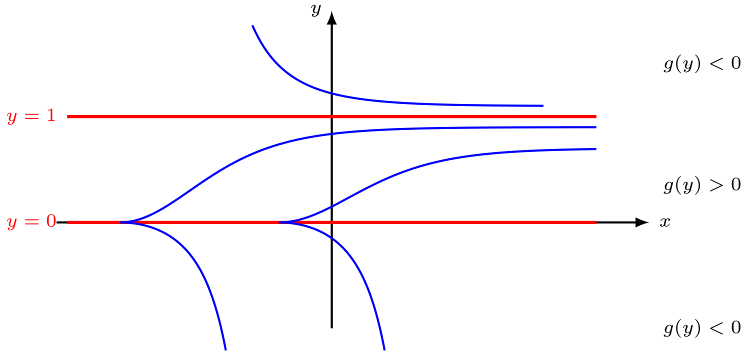
- If $y_0 > 1$:

$$y = \left(\frac{\lambda e^x - 1}{\lambda e^x - 1}\right)^2,$$

$\lambda < 0$. $\lim\limits_{x \to +\infty} y(x) = 1$.

- If $y_0 < 0$:

$$x = 2\tan^{-1}(\sqrt{-y_0}) - 2\tan^{-1}(\sqrt{-y}),$$



From this example, we see that the Cauchy problem has many solutions at $(x_0, 0)$.

Example 3.3 :
Take the differential equation $y' = xy$.
$y = 0$ is a solution. For $y \neq 0$, the differential equation becomes $\dfrac{dy}{dx} = xy$. We get $\ln |y| = \dfrac{x^2}{2} + c$, or

$$y = \lambda e^{\frac{x^2}{2}}.$$

where $\lambda \in \mathbb{R}$.

Example 3.4 :
It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable differential equation

$$y' = \frac{xy}{y^2 + 1}.$$

We separate variables,

$$\frac{y^2 + 1}{y}\ dy = \left(y + \frac{1}{y}\right)\ dy = x\ dx.$$

We integrate to get $y^2 + 2 \ln |y| = x^2 + c$.
It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an implicit solution.

Example 3.5 :
Consider the differential equation: $y' = \frac{2x-1}{y^2}$. Integrating both sides, we get $y^3/3 = x^2 - x + c$. Hence,

$$y = \left(3x^2 - 3x + 3c\right)^{1/3}.$$

Example 3.6 :
Consider the differential equation: $y' = \frac{y-1}{x+1}$, $(x > -1)$.
$y = 1$ is a singular solution. For $y \neq 1$, dividing both sides of the given differential equation by $y - 1$, we get

$$\frac{y'}{y-1} = \frac{1}{x+1}.$$

Integrating both sides we get $\ln|y-1| = \ln(x+1) + c$. Thus $y = 1 + \lambda(x+1)$, $\lambda \in \mathbb{R}$.

Example 3.7 :
Consider the differential equation: $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$, which is equivalent to the following equation:

$$x^2y' = (1-x^2)(1+y^2).$$

Separate variables, integrate, and solve for y.

$$\begin{aligned} \frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\ \tan^{-1}(y) &= \frac{-1}{x} - x + C, \\ y &= \tan\left(\tfrac{-1}{x} - x + C\right). \end{aligned}$$

Now solve for the initial condition,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 3.8 :
Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

First note that $y = 0$ is a solution (a singular solution). So assume that $y \neq 0$ and write $\frac{-3}{y^2}y' = x, \iff \frac{3}{y} = \frac{x^2}{2} + c, \iff y = \frac{3}{\frac{x^2}{2} + c} = \frac{6}{x^2 + 2c}$.

Example 3.9 :
Consider the differential equation: $y' = \frac{x-5}{y^2}$.

To solve it using the above method we multiply both sides of the equation by y^2 to get

$$y^2y' = (x-5).$$

Integrating both sides we get $\frac{1}{3}y^3 = \frac{1}{2}x^2 - 5x + c$. Hence,

$$y = \left[\frac{3}{2}x^2 - 15x + c_1\right]^{\frac{1}{3}}.$$

Example 3.10 :
Consider the differential equation: $y' = \dfrac{y-1}{x+3}$, for $x \in (-3, +\infty)$.
$y = 1$ is a solution. For $y \neq 1$, we have

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

Integrating both sides we get

$$\int \frac{y'}{y-1}\mathrm{dx} = \int \frac{\mathrm{dx}}{\mathrm{x}+3} + \mathrm{c}_1,$$

from which we get $\ln|y-1| = \ln(x+3) + c_1$. Thus the general solution is $y = 1 + c(x+3)$, with $c \in \mathbb{R}$.

Example 3.11 :
Consider the differential equation: $y' = \dfrac{y\cos x}{1+2y^2}$. Transforming in the standard form then integrating both sides we get

$$\int \frac{(1+2y^2)}{y}\mathrm{dy} = \int \cos\mathrm{x}\,\mathrm{dx} + \mathrm{c},$$

from which we get a family of the solutions:

$$\ln|y| + y^2 = \sin x + 1.$$

Example 3.12 :
Consider the differential equation

$$y' = \sqrt{\frac{1-y^2}{1-x^2}}. \tag{1.8}$$

The domain of definition of the differential equation is

$$\Omega = \{|x| < 1,\ |y| \leq 1\} \cup \{|x| > 1,\ |y| \geq 1\}.$$

1. On the open set $\Omega_1 = \{(x,y) \in \mathbb{R}^2;\ |x| < 1,\ |y| < 1\}$ the equation is equivalent to

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}.$$

Then $\sin^{-1} y = \sin^{-1} x + \lambda$, where $\lambda \in \mathbb{R}$. As the function $\sin^{-1} :]-1,1[\longrightarrow]-\frac{\pi}{2},\frac{\pi}{2}[$ is bijective, then $\lambda \in]-\pi,\pi[$. Moreover $\sin^{-1} x \in]-\dfrac{\pi}{2},\dfrac{\pi}{2}[\cap]-\dfrac{\pi}{2}-\lambda,\dfrac{\pi}{2}-\lambda[= \begin{cases}]-\frac{\pi}{2},\frac{\pi}{2}-\lambda[& \text{if } \lambda \geq 0 \\ ]-\frac{\pi}{2}-\lambda,\frac{\pi}{2}[& \text{if } \lambda \leq 0 \end{cases}$, and $\sin^{-1} y \in]-\frac{\pi}{2}+\lambda,\frac{\pi}{2}[$ if $\lambda \geq 0$ and $\sin^{-1} y \in]-\frac{\pi}{2},\frac{\pi}{2}+\lambda[$ if $\lambda \leq 0$.

The integral curves admits the equation $y = \sin(\sin^{-1} x + \lambda) = x\cos\lambda + \sqrt{1-x^2}\sin\lambda$, with $x \in]-1,\cos\lambda[$, $y \in]-\cos\lambda, 1[$ if $\lambda \geq 0$, $x \in]-\cos\lambda, 1[$, $y \in]-1,\cos\lambda[$ if $\lambda \leq 0$. The equation $(y - x\cos\lambda)^2 + x^2\sin^2\lambda = \sin^2\lambda$ is an ellipse.

2. The open set $\{|x| > 1,\ |y| > 1\}$ has 4 connected components.

On the component $\{x > 1,\ y > 1\}$, the differential equation (1.8) is equivalent to

$$\frac{dy}{\sqrt{y^2-1}} = \frac{dx}{\sqrt{x^2-1}}.$$

Then $\cosh^{-1} y = \cosh^{-1} x + \lambda$, $\lambda \in \mathbb{R}$. The function $\cosh^{-1} :]1,+\infty[\longrightarrow]0,+\infty[$ is bijective, then

$$y = x\cosh\lambda + \sqrt{x^2-1}\sinh\lambda,$$

with $x \in]1,+\infty[$, $y \in]\cosh\lambda,+\infty[$ if $\lambda \geq 0$, $x \in]\cosh\lambda,+\infty[$, $y \in]1,+\infty[$ if $\lambda \leq 0$.

3.3 Exercises

3-1 Solve the following differential equations:

1) $y' = 2xy$,
2) $y' = x^2 y$,
3) $(1+x^2)y' = 1$,
4) $y' = \dfrac{y}{x^3-1}$,
5) $y' = \dfrac{y}{x^3-1}$,
6) $x^2y' + (y^2 - 2y) = 0$,
7) $xy' = \sqrt{1-y^2}$,
8) $y' = \left(\dfrac{y-1}{x+1}\right)^2$,
9) $y' = \dfrac{1+y}{4+x^2}$,

10) $y'x\tan y = -1$,

11) $y' = xy + x + y + 1$.

12) $xy' = y + 2x^2y$, $y(1) = 1$.

13) $y' = 3yx^2 - 3x^2$, $y(0) = 2$,

14) $y' = \dfrac{1}{3y^2+1}$, $y(0) = 1$,

15) $xy' = y^2$, $y(1) = 1$,

16) $y' = (y^2 - 1)x$, $y(0) = 0$,

17) $y' = y\sin(x)$, $y(0) = 1$,

18) $y' = \dfrac{x^2+1}{y^2+1}$, $y(0) = 1$,

19) $y' = xe^{-y}$, $y(0) = 1$,

20) $xy' = e^{-y}$, for $y(1) = 1$,

21) $y' = \dfrac{\sin(x)}{\cos(y)}$,

22) $y' = \dfrac{x}{y}$,

3-2 Consider the following differential equation

$$y' = 1 + \cos y \quad y(0) = a. \tag{1.9}$$

1) Solve the differential equation (1.9) for $a = 3\pi$.
2) Solve the differential equation (1.9) for $a = 0$.

3-3 Give an explicit solution (involving a definite integral) to the following differential equations.

1) $y' = \frac{1}{y^2 \ln x}$, $\quad y(2) = 0$,
2) $y' = \frac{y}{x}e^x$, $\quad y(1) = 1$,
3) $yy' = xy + x$, $\quad y(2) = 0$,
4) $y' = \sin x \cos^2 y$, $\quad y(0) = 0$.

3-4 Solve the following differential equations:

1) $y' = \frac{xy+x}{y}$, $\quad y(2) = 0$
2) $y' = \sin x \cos(2y)$, $\quad y(0) = 0$
3) $x^2y' + (y^2 - 2y) = 0$
4) $xy' = \sqrt{1-y^2}$
5) $y' = \left(\dfrac{y-1}{x+1}\right)^2$
6) $y' = \dfrac{\sqrt{1+y}}{x^2+4}$
7) $y' = 1 + y^2$,
8) $y' = y(1-y)$,

9) $xyy' = 1$,

10) $y' = xy$,

11) $y' = 1 - y^2$,

12) $x^2y' + y = 0$.

13) $(1 + 2x)y' + (2 - y) = 0$,

14) $x(\tan y)y^{'} = -1$,

15) $y' = \dfrac{y}{x} + \dfrac{1}{y}$,

16) $y' = e^{2x+3y}$,

17) $\dfrac{y' - 1}{x^2} = 1$,

4 Homogeneous Differential Equations

Definition 4.1.
A differential equation of first order is called homogeneous if it is has the form

$$y' = f(\frac{y}{x}), \tag{1.10}$$

where $g\colon J \longrightarrow \mathbb{R}$ be a continuous function.

A solution g of the equation (1.10) defined on an interval I must fulfills:

1. $0 \notin I$, $\frac{g(x)}{x} \in J$, $\forall\, x \in I$.
2. $g'(x) = f(\frac{g(x)}{x})$, $\forall\, x \in I$.

If we set $z = \frac{y}{x}$ or $y = xz$, we have: $y' = z + xz' = f(z)$. Thus z fulfills the following separated differential equation $z' = \dfrac{f(z) - z}{x}$.
Let $\{z_j\}$ be the set of roots of the equation $f(z) = z$. We have $z = z_j$ and $y(x) = z_j x$ (line passing through 0) are solutions.
On the open set $\{z;\ f(z) \neq z\}$ the equation is equivalent to the following: $\dfrac{dz}{f(z) - z} = \dfrac{dx}{x}$, which is equivalent to: $F(z) = \ln|x| + c = \ln|\lambda x|$, $\lambda \in \mathbb{R}$ and F a primitive of the function $\dfrac{1}{f(z) - z}$ on $]z_j, z_{j+1}[$. We deduce that $z = F^{-1}(\ln \lambda x)$ where the family of integral curves $C_\lambda\colon y = xF^{-1}(\ln \lambda x)$ defines in the sector $z_j < \frac{y}{x} < z_{j+1}$, $\lambda x > 0$.

Example 4.1 :

Consider the differential equation: $xy'(2y-x) = y^2$ or $y' = \dfrac{y^2}{x(2y-x)}$ if $x \neq 0$ and $y \neq \frac{x}{2}$. Thus $y' = \dfrac{(\frac{y}{x})^2}{\frac{2y}{x} - 1}$.
We set $z = \frac{y}{x}$, then $y' = xz' + z = \frac{z^2}{2z-1}$ and $xz' = \dfrac{z(1-z)}{2z-1}$.
$z = 0$, $z = 1$, $y = 0$, $y = x$ are singular solutions.
For $z \neq 0$ and $z \neq 1$, the equation is equivalent to: $\dfrac{2z-1}{z(1-z)} z' = \dfrac{1}{x}$.
Then $\ln|z(1-z)| = -\ln|x| + c$, $z(1-z) = \dfrac{\lambda}{x}$ and $y(x-y) = \lambda x$ or $(y-\lambda)(x-y-\lambda) = \lambda^2$. If we set $X = x - y - \lambda$ and $Y = y - \lambda$, we have

$$XY = \lambda^2.$$

This is the equation of an hyperbola with asymptotes $y = \lambda$, $y = x - \lambda$ (corresponding to the asymptotic directions $y = 0$, $y = x$ the line of singular integrals).

Other Method of Resolution

We set $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, with $r > 0$ and $\theta \in \mathbb{R}$.
The equation becomes $\dfrac{dy}{dx} = \dfrac{\tan\theta dr + r d\theta}{dr - r\tan\theta d\theta}$. The differential equation $y' = f(\frac{y}{x})$ is equivalent to $\tan\theta dr + r d\theta = f(\tan\theta)(dr - r\tan\theta d\theta)$ or

$$\frac{dr}{r} = \frac{1 + \tan\theta f(\tan\theta)}{f(\tan\theta) - \tan\theta} d\theta.$$

Example 4.2 :

Consider the differential equation $y' = \dfrac{x+y}{x-y}$. Let $y = xz$.
The differential equation becomes $xz' + z = \dfrac{1+z}{1-z} \iff xz' = \dfrac{1+z^2}{1-z}$.
Then $\ln|x| + c = \tan^{-1} z - \dfrac{1}{2}\ln(1+z^2)$ or $\tan^{-1}(\dfrac{y}{x}) - \dfrac{1}{2}\ln(x^2+y^2) = \ln|x| + c$.
Using the second method, we get $\dfrac{dr}{r} = d\theta$, then $\ln r = \theta + c$.

Example 4.3 :

Consider the differential equation $y' = \dfrac{x-y}{x+y}$. Let $y = xz$.
The differential equation becomes $xz' + z = \dfrac{1-z}{1+z}$. Then

$$xz' = \frac{1-z}{1+z} - z = \frac{1-2z-z^2}{1+z}.$$

$z=-1\pm\sqrt{2}$ are singular solutions.
For $z\neq -1\pm\sqrt{2}$, we get $\displaystyle\int \frac{(1+z)}{1-2z-z^2}dz=\ln|x|+c$. Then
$-\frac{1}{2}\ln|1-2z-z^2|=\ln|x|+c$ and $\displaystyle\frac{1}{\sqrt{|1-2z-z^2|}}=\lambda|x|$.
We deduce that $z^2+2z-1=\frac{\lambda}{x^2}$. We solve this equation and we get

$$z=-1\pm\sqrt{2+\frac{\lambda}{x^2}}.$$

For example $y=-x+\sqrt{2x^2+7}$ is the solution to the initial value problem

$$y'=\frac{x-y}{x+y},\quad y(1)=2.$$

By the second method, we get $\displaystyle\frac{dr}{r}=\frac{\cos(2\theta)+\sin(2\theta)}{\cos(2\theta)-\sin(2\theta)}d\theta$. Integrate both sides, we get: $r=\displaystyle\frac{\lambda}{\sqrt{\cos(2\theta)+\sin(2\theta)}}$. This is equivalent to $x^2-y^2+2xy=c$.

4.1 Exercises

4-1 Solve the following homogeneous differential equations

1) $y'=\dfrac{2y-x}{y+4x}$,
2) $y'=\dfrac{2xy}{x^2-y^2}$,
3) $xyy'-y^2=\sqrt{x^2-y^2}$.
4) $y^{'}=\dfrac{x^2-y^2}{5xy}$
5) $xy^{'}=y+xe^{\frac{y}{x}}$,
6) $xy'-y=\sqrt{x^2+y^2}$,
7) $y'=\dfrac{3-2y}{2x+2y+1}$

5 Reduction of Differential Equations to Known Types

Sometimes it is possible by change of variable, we transform the differential equation into one of the known types.

5.1 Reduction to Separable Form (Substitution Method)

• Consider the ordinary differential equation $y' = f(ax + by + c)$.
If $b = 0$ the equation is already separable.
Suppose that $b \neq 0$. The substitution $z = ax + by + c$ reduces the equation to a separable form $z' = bf(z) + a$, which is separable.

Example 5.1 :
Consider the differential equation $y' = 1 + \sqrt{y - x}$. Take the substitution $z = y - x$, the equation becomes $z' = \sqrt{z}$. Then $z = \left(\frac{1}{2}x + c\right)^2$ or $y = x + \left(\frac{1}{2}x + c\right)^2$.

Example 5.2 :
Consider the differential equation $y' = (2x + y + 1)^2$. We define a new variable $z = 2x + y + 1$. For this the equation becomes $z' = -2 + z^2$. We solve this by separating variables method.
Note that $z = \sqrt{2}$ and $z = -\sqrt{2}$ are solutions of the differential equation. For $z \neq \pm\sqrt{2}$, $\ln\left|\frac{\sqrt{2}+z}{\sqrt{2}-z}\right| = 2\sqrt{2}x + c$, or $\frac{\sqrt{2}+z}{\sqrt{2}-z} = \lambda e^{2\sqrt{2}x}$, with $\lambda \in \mathbb{R}$. Then $z = \sqrt{2}\frac{\lambda e^{2\sqrt{2}x} - 1}{\lambda e^{2\sqrt{2}x} + 1}$.

Using unsubstitution, we get $y = \sqrt{2}\frac{\lambda e^{2\sqrt{2}x} - 1}{\lambda e^{2\sqrt{2}x} + 1} - 2x - 1$, with $\lambda \in \mathbb{R}$.

Example 5.3 :
Consider also the differential equations of the form

$$y' = \frac{y}{x} + g(x)h(\frac{y}{x}).$$

This equation can be reduced to the separable form by substituting $z = \frac{y}{x}$.

Example 5.4 :
Consider the differential equation $xyy' = y^2 + 2x^2$, $y(1) = 2$.
Let $z = \frac{y}{x}$. We find $xz' = \frac{2}{z} \Rightarrow y^2 = 2x^2(c + \ln x^2)$.
Using $y(1) = 2$, we get $c = 2$. Hence, $y = 2x^2(1 + \ln x^2)$.

5.2 Reduction to Homogeneous Form

Consider the ordinary differential equation

$$y' = \frac{ax + by + c_1}{cx + dy + c_2}, \quad ad - bc \neq 0.$$

With the condition $ad - bc \neq 0$, the lines of equations $ax + by + c_1 = 0$ and $cx + dy + c_2 = 0$ are distinct lines. We assume that they meet at the point (x_0, y_0). The above differential equation can be written in the form

$$y' = f\left(\frac{a(x - x_0) + b(y - y_0)}{c(x - x_0) + d(y - y_0)}\right)$$

which yields the differential equation

$$z' = f\left(\frac{at + bz}{ct + dz}\right) = f\left(\frac{a + b(\frac{z}{t})}{c + d(\frac{z}{t})}\right)$$

after the change of variables $t = x - x_0$, $z = y - y_0$. This equation is homogeneous.

Example 5.5 :
Consider the differential equation $y' = \dfrac{2x + y + 1}{x - y + 2}$.
The equation is equivalent to $y' = \dfrac{2(x + 1) + (y - 1)}{(x + 1) - (y - 1)}$. Take the substitution $z = y - 1$ and $t = x + 1$, the equation becomes $z' = \frac{2t+z}{t-z} = \frac{2+(\frac{z}{t})}{1-(\frac{z}{t})}$. If $w = \dfrac{2 + (\frac{z}{t})}{1 - (\frac{z}{t})}$, $z = t\dfrac{w - 2}{w + 1}$. The differential equation becomes: $\dfrac{w - 2}{w + 1} + \dfrac{3tw'}{(w + 1)^2} = w$. This equation is equivalent to:

$$\frac{3w'}{(w + 1)(2 + w^2)} = w'\left(\frac{1}{w + 1} + \frac{1 - w}{w^2 + 2}\right) = \frac{3}{t}.$$

Then

$$\ln|w + 1| + \frac{1}{\sqrt{2}}\tan^{-1}(\frac{w}{\sqrt{2}}) - \frac{1}{2}\ln(w^2 + 2) - 3\ln|t| = c,$$

where $t = x + 1$, $w = \dfrac{2x + y + 1}{x - y + 2}$.

5.3 Exercises

5-1 Find the solution of the differential equation

$$y' = (y - x)^2 + 1.$$

5-2 Solve the following differential equation

$$\frac{dy}{dx} = \frac{1 - 4x - 4y}{x + y}, \quad x + y \neq 0.$$

5-3 Solve the following differential equation $y' = \dfrac{y}{x}\dfrac{1 + xy}{1 - xy}$, on $\{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ xy \neq 1\}$.

6 Linear Differential Equations of First Order

The general linear first order differential equation has the form

$$a(x)y' + b(x)y = c(x).$$

where $a,\ b,\ c$ are continuous functions on some interval I.
To have the normal form $y' = f(x, y)$ we have to divide both sides of the differential equation by a. This is possible only on the set where the function a is zero free. After possibly shrinking the interval I we assume that $a \neq 0$ on I. The differential equation has now the standard form

$$y' = a(x)y + b(x), \tag{1.11}$$

with a and b, both continuous function on an interval I.

6.1 Resolution of the Homogeneous Differential Equation

The homogeneous differential equation associated to the differential equation (1.11) is the following equation

$$y' = a(x)y. \tag{1.12}$$

$y = 0$ is a solution and any other solution is non zero.
We can write $\dfrac{y'}{y}(x) = \dfrac{\mathrm{d}}{\mathrm{dx}}\ln(y) = a(x)$.
Integrating both sides, we derive $\ln|y(x)| = A(x) + c$, or $y = \lambda e^{A(x)}$, where $\lambda \in \mathbb{R}$ and A any anti-derivative of the continuous function a. ($\lambda e^{A(x)}$ is called the general form of the solution of the homogeneous differential equation (1.12).

6.2 Resolution of the Inhomogeneous Differential Equation

To solve the differential equation (1.11), we propose two methods.

6.2.1 Method of Variation of Constant

Theorem 6.1.
If y_0 is a particular solution of the differential equation (1.11) and z the general solution of the homogeneous differential equation, then the general solution of the differential equation (1.11) has the form $y = y_0 + z$.

We look for a particular solution of the differential equation (1.11) in the form $y = \lambda(x)e^{A(x)}$. We differentiate both sides and we derive that

$$\lambda'(x) = b(x)e^{-A(x)}.$$

Integration both sides we have

$$\lambda(x) = \int_{x_0}^{x} b(t)e^{-A(t)}dt + c \quad x_0 \in I.$$

So $y_0 = e^{A(x)} \int_{x_0}^{x} b(t)e^{-A(t)}dt$ is a particular solution and the general solution is

$$y(x) = \lambda e^{A(x)} + e^{A(x)} \int_{x_0}^{x} b(t)e^{-A(t)}dt.$$

Theorem 6.2.
The set of solutions of the differential equation (1.11) on I is

$$S = \{t \longmapsto \lambda e^{A(x)} + e^{A(x)} \int_{x_0}^{x} b(t)e^{-A(t)}dt, \quad \lambda \in \mathbb{R}\}.$$

where A is any anti-derivative of the continuous function a on I.

Example 6.1 :
Consider the following differential equation: $xy' + 2y = \dfrac{x}{1+x^2}$.
$xy' + 2y = 0 \iff y = \dfrac{\lambda}{x^2}$. The variation of the constant method yields that $\lambda' = \frac{x^2}{1+x^2}$, then $y = \dfrac{c}{x^2} - \dfrac{x - \tan^{-1} x}{x^2}$.
Since $\lim\limits_{x\to 0} \dfrac{x - \tan^{-1} x}{x^2} = 0$, then the only solution of on $\mathbb{R}$ is $y = \dfrac{x - \tan^{-1} x}{x^2}$.

6.2.2 Method of Integration Factor

We multiply both sides of the differential equation (1.11) with a factor $c(x) \neq 0$. The differential equation becomes $c(x)y'(x) = c(x)a(x)y(x) + c(x)b(x)$. This differential equation is equivalent to the differential equation (1.11), (i.e. has the same set of solutions). We choose the function c so that the previous differential equation has the form

$$c(x)y'(x) - c(x)a(x)y(x) = \frac{d}{dx}\big(c(x)y(x)\big).$$

For this, the function c must fulfills $c'(x) = -a(x)c(x)$, and $c(x) \neq 0$ for all $x \in I$.
By solving this linear homogeneous differential equation, we obtain $c(x) = e^{-A(x)}$, with A any anti-derivative of the function a on I. This function is called an integrating factor.
The differential equation (1.11) is reduced to

$$\frac{\mathrm{d}}{\mathrm{dx}}\big(c(x)y(x)\big) = c(x)b(x). \tag{1.13}$$

Integrating both sides, we get

$$c(x)y(x) = \int c(x)b(x)dx + C = \int b(x)e^{-A(x)}dx + C,$$

with $C \in \mathbb{R}$. Solving for y, we get

$$y = e^{A(x)} \int c(x)b(x)dx + Ce^{A(x)} \tag{1.14}$$

as the general solution for the general linear first order

$$y' = a(x)y + b(x).$$

The first part, $y_0(x) = e^{A(x)} \int b(x)e^{-A(x)}dx$ is a particular solution of the inhomogeneous differential equation, while the second part, $y_1(x) = Ce^{-A(x)}$ is the general solution of the associate homogeneous solution.

Example 6.2 :
Consider the differential equation $y' + 2xy = x$. The integrating factor is $c(x) = e^{x^2}$. Hence, after multiplying both sides of our differential equation, we get

$$\frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

which, after integrating both sides, yields

$$e^{x^2}y = \int xe^{x^2}dx + c = \frac{1}{2}e^{x^2} + c.$$

Hence the general solution is $y = \frac{1}{2} + ce^{-x^2}$.

Example 6.3 :
Consider the differential equation $xy' - 2y = x^3 \sin x$, $x \in (0, +\infty)$. The standard form of this differential equation is:

$$y' = \frac{2}{x}y + x^2 \sin x.$$

The integrating factor is $c(x) = e^{-2\ln x} = \frac{1}{x^2}$. The simple form of the differential equation is: $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y}{x^2}\right) = x\sin x$, then $y = -x^3\cos x + x^2\sin x + cx^2$.

Example 6.4 :

Consider the following differential equation: $y' = 3yx^2 + x^5$, $y(0) = 1$. $y = 0$ is a solution of the linear differential equation $y' = 3yx^2$.

If $y \neq 0$, after integration we get $\ln|y| = x^3 + c$. Then $y = \lambda e^{x^3}$.

Using the variation of parameter method, $\lambda' = x^5 e^{-x^3}$. Then $\lambda = -\frac{1}{3}(x^3 + 1)e^{-x^3} + c$ and the general solution of the differential equation is

$$y = -\frac{1}{3}(1 + x^3) + \lambda e^{x^3}.$$

6.3 The Bernoulli Differential Equation

The general Bernoulli's differential equation is

$$y' + p(x)y(x) + q(x)y^{\alpha}(x) = 0, \tag{1.15}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$, $p, q\colon I \longrightarrow \mathbb{R}$ two continuous functions.

The open set where the differential equation is defined is $\Omega = \mathbb{R}\times]0, +\infty[= \{(x, y);\ y > 0\}$. By multiplying by $y^{-\alpha}$, we get

$$y^{-\alpha}\frac{dy}{dx} + p(x)y^{1-\alpha}(x) + q(x) = 0.$$

If $z = y^{1-\alpha}$, the differential equation is equivalent to

$$\frac{1}{1-\alpha}\frac{dz}{dx}(x) + p(x)z(x) + q(x) = 0. \tag{1.16}$$

This differential equation is linear in z.

Example 6.5 :

Consider the differential equation $y' - xy + y^2 = 0$. We set $z = \frac{1}{y}$, we get $z' - z - 1 = 0$. Then $z = \lambda e^x - 1$ and $y = \frac{1}{\lambda e^x - 1}$.

6.4 Riccati Differential Equations

The general Riccati 's differential equation is

$$y' = a(x)y^2 + b(x)y + c(x), \tag{1.17}$$

where $a, b, c\colon I \longrightarrow \mathbb{R}$ three continuous functions.

We assume that we have a particular solution y_0 of the differential equation (1.17).
Set $y = y_0 + z$. y is solution of (1.17) if and only if:

$$\begin{aligned} y_0' + z' &= a(x)(y_0 + z)^2 + b(x)(y_0 + z) + c(x) \\ &= a(x)y_0^2 + a(x)z^2 + 2y_0 a(x)z + b(x)y_0 + b(x)z + c(x). \end{aligned}$$

This differential equation is equivalent to

$$z' = a(x)z^2 + (2y_0(x)a(x) + b(x))z.$$

This is a Bernoulli differential equation with $\alpha = 2$.

Example 6.6 :

$$(1 - x^3)y' + x^2 y + y^2 - 2x = 0.$$

We remark that $y_0(x) = x^2$ is solution. We set $y = x^2 + z$. The differential equation satisfied by z is

$$(1 - x^3)z' + 3x^2 z + z^2 = 0.$$

If $w = \frac{1}{z}$,

$$(1 - x^3)w' + 3x^2 w + 1 = 0,$$

or $w' = \frac{3x^2}{1-x^3}w + \frac{1}{1-x^3}$, if $x \neq 1$.
The linear homogeneous differential equation associated is $\dfrac{w'}{w} = \dfrac{3x^2}{1-x^3}$. Then $\ln|w| = -\ln|1 - x^3| + c$ or $w = \dfrac{\lambda}{1-x^3}$. In use the method of variation of constant, we find $\dfrac{\lambda'}{1-x^3} = \dfrac{1}{1-x^3}$. Then $\lambda' = 1$ and $\lambda(x) = x$. The general solution of the linear differential equation is $w(x) = \dfrac{x+\lambda}{1-x^3}$. Then $y = x^2 + z = x^2 + \frac{1}{w} = x^2 + \frac{1-x^3}{x+\lambda}$ and

$$y(x) = \frac{\lambda x^2 + 1}{x + \lambda}.$$

6.5 Exercises

6-1 Solve the following differential equations:

1) $y' - xy = x,$
2) $y' - y = \cosh x$
3) $y' + 2y = e^x,$
4) $xy' + 2y = \cos x,$
5) $y' = 1 + 2xy,$
6) $y' + y\tan x = \sin(2x),$
7) $xy' + (x^3 + y) = 0,$
8) $y' + 2xy = e^{-x},$
9) $y'\cos x = (y\sin x + e^x),$
10) $y' = e^{2x+3y},$
11) $xy' - y = x^2 \sin x,$

6-2 Consider the following differential equation: $xy' + 2y = \dfrac{x}{1+x^2}$

1) Solve the differential equation on $\mathbb{R}^*$.
2) Prove that there exists only one solution on $\mathbb{R}$. Determine this solution.

6-3 The goal of this exercise is to find the global solutions of the following differential equation:

$$x(1+x^2)y^{'} - (x^2 - 1)y = -2x \tag{1.18}$$

Consider the following differential equation (1.19) on each of the intervals $I_1 = (-\infty, 0)$ and $I_2 = (0, +\infty)$ by:

$$y^{'} - \frac{x^2 - 1}{x(1+x^2)}y = -\frac{2}{1+x^2} \tag{1.19}$$

1) i. Solve the homogeneous equation $y^{'} - \dfrac{x^2-1}{x(1+x^2)}y = 0$ on $I_1 = (-\infty, 0)$ and on $I_2 = (0, +\infty)$
   ii. Determine the solution h_1 of the homogeneous equation on I_1 such that $h_1(-1) = -1$ and the solution h_2 of the homogeneous equation on I_2 such that $h_2(1) = 1$.
2) i. Solve the differential equation (1.19).
   ii. Determine the solutions g_1 of (1.19) on I_1 such that $g_1(-1) = -1$ and the solution g_2 on I_2 such that $g_2(1) = 1$.
3) Define on $\mathbb{R}$ the following function f:

$$f(x) = \begin{cases} g_1(x) + \lambda h_1(x) & \text{si} \quad x < 0 \\ \alpha & \text{si} \quad x = 0 \\ g_2(x) + \mu h_2(x) & \text{si} \quad x > 0 \end{cases}$$

where $\lambda, \mu, \alpha \in \mathbb{R}$.

4) Determine the conditions on α, λ and μ such that the function f is continuous at 0. In what follows, we assume that this condition is satisfied .

5) Deduce the set of solutions of (1.18) on $\mathbb{R}$.

6-4 The goal of this exercise is to solve on $\mathbb{R}$ the following differential equation:

$$x(1+x^2)y^{'} - (x^2-1)y = -2x \tag{1.20}$$

Consider for this the differential equation (1.20) defined on each of the interval $I_1 =]-\infty, 0[$ and $I_2 =]0, +\infty[$ by:

1) i. Solve the homogeneous equation of the equation (1.20) on the intervals I_1 and I_2.

   ii. Determine the solution h_1 of the homogeneous equation of the equation (1.20) on the interval I_1 such that $h_1(-1) = -1$ and the solution h_2 of the homogeneous equation of the equation (1.20) on the interval I_2 such that $h_2(1) = 1$.

2) i. Determine a particular solution g_1 of the differential equation (1.20) on I_1 such that $g_1(-1) = -1$.

   ii. Determine a particular solution g_2 of the differential equation (1.20) on I_2 such that $g_2(1) = 1$.

3) Let λ, μ, α be three reals numbers. Define on $\mathbb{R}$ a function f as follows:

$$f(x) = \begin{cases} g_1(x) + \lambda h_1(x) & \text{if} \quad x < 0 \\ \alpha & \text{if} \quad x = 0 \\ g_2(x) + \mu h_2(x) & \text{if} \quad x > 0 \end{cases}$$

   Compute $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

4) Deduce a condition on α, λ and μ such that the function f is continuous at 0.
   Assume in what follows that this condition is satisfied .

5) Prove that the function f obtained is differentiable on $\mathbb{R}$.

6) Deduce the set of solutions of the differential equation (1.20) on $\mathbb{R}$.

6-5 Solve each of the following, finding the general solution, or the solution satisfying the given initial condition.

1) $xy^{'} + 2y = x$,

2) $y' - y\tan x = \dfrac{x}{\cos x}, \quad y(0) = 0,$

3) $y' + \dfrac{y}{x+1} = x^2 + x,$

4) $y' + \sin(x)y = \cos(x)\sin(x),$

5) $y' - 7y = 3x^2 - 4x, \quad y(0) = 0,$

6) $(x^2 - 1)y' = 1 - 2xy$

7) $3y = x(1 - y'), \quad y(1) = \frac{1}{4}$

8) $y' + xy - e^{-x} = 0.$

6-6 Solve the following differential equations:

1) $y' = \dfrac{y}{x^3 - 1},$

2) $y' - xy = x,$

3) $y' - y = \cosh x,$

4) $y' + 2y = e^x,$

5) $xy' + 2y = \cos x,$

6) $y' = x + 2xy,$

7) $y' + y\tan x = \sin(2x),$

8) $y'\cos x = (y\sin x + e^x),$

6-7 Solve these Bernoulli equations

1) $y' + y = 2xy^2,$

2) $x^2y^{'} - y^3 = xy.$

3) $x^2y' + xy + y^2 = 0,$

4) $y' + \dfrac{2y}{x} - \dfrac{y^2}{x} = 0,$

5) $xy' + 2y - y^2 = 0,$

6) $y' = \dfrac{y}{x} + \dfrac{1}{y}.$

6-8 Solve the Riccati equations

1) $y^{'} = 1 - x^2 + y^2$

2) $xy' - 2y + y^2 = x^4$, ($y = x^2$ is a solution)

2 Higher Order Linear Differential Equations

1 Basic Properties of Linear Differential Equations of High Order

1.1 Introduction

Definition 1.1.
A linear ordinary differential equation of order n is an equation that can be expressed in the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f, \tag{2.1}$$

where $a_0, \ldots, a_n$ and f are continuous functions on an interval (a, b) and that $a_0(x) \neq 0$ for all $x \in (a, b)$. The points where $a_0(x) = 0$ are called singular points. Therefore in this chapter we assume that $a_0(x) \neq 0$ for all $x \in (a, b)$. The right-hand member f of the differential equation (2.1) is called the non-homogeneous term. If f is identically zero, the equation (2.1) is called homogeneous.

1.2 Initial value problem for the homogeneous equation

Theorem 1.2. [Existence of Solutions]
Consider the homogeneous linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0, \tag{2.2}$$

, where $a_0, \ldots, a_n$ continuous functions on an interval (a, b) and $a_0(x) \neq 0$ for any $x \in (a, b)$.
Then for any $x_0 \in (a, b)$ and $c_0, \ldots, c_{n-1}$, n arbitrary real numbers, there exists a unique solution y of the equation (2.2) such that $y(x_0) = c_0, \ldots, y^{(n-1)}(x_0) = c_{n-1}$. This solution is defined on the entire interval (a, b).

Remark 7 :

1. Without the condition $a_0(x) \neq 0$ for any $x \in (a,b)$, the existence of solution may be not true. For example, consider the equation: $xy'+y=0$, with $y(1)=0$. The only solution of this equation is $y=\frac{1}{x}$. This solution id defined on $\mathbb{R}^*$.

2. If $y_1,\dots,y_m$ are solutions of the homogeneous differential equation of (2.2), then for all $a_1,\dots,a_m \in \mathbb{R}$, $a_1y_1+\dots+a_my_m$ is also solution of the homogeneous differential equation.

Definition 1.3.
The functions $f_1,\dots,f_m$ are called linearly independent on (a,b) if the only solution of

$$a_1f_1(x)+\dots+a_mf_m(x)=0, \quad \forall x \in (a,b)$$

is $a_1=\dots=a_m=0$. Otherwise, the functions are called linearly dependent.

Example 1.1 :

1. The functions $\sin x, \cos x$ are linearly independent on the interval $[0,\dfrac{\pi}{2}]$.
   If $a\sin x + b\cos x = 0$ for all $x \in [0,\dfrac{\pi}{2}]$, then for $x=0$, $b=0$ and for $x=\dfrac{\pi}{2}$, $a=0$.
   We can also prove that $\sin x, \cos x$ are linearly independent on any non trivial interval.

2. The functions $e^x, \sin x, \cos(2x)$ are linearly independent on the interval $[0,\dfrac{\pi}{2}]$.
   If $ae^x+b\sin x+c\cos(2x)=0$ for all $x\in[0,\dfrac{\pi}{2}]$, then for $x=0$, $a+c=0$ and for $x=\dfrac{\pi}{2}$, $ae^{\frac{\pi}{2}}+b=0$. Also we can differentiate this function and we get: $ae^x+b\cos x-2c\sin(2x)=0$ for all $x\in[0,\dfrac{\pi}{2}]$. Also for $x=0$ and $x=\dfrac{\pi}{2}$, we have, $a+b=0$ and $a=0$. Then $a=b=c=0$.
   We can also prove that $e^x, \sin x, \cos x$ are linearly independent on any non trivial interval.

3. The functions $\sin x, \cos x, \sin(x+1)$ are linearly dependent on any non trivial interval. Indeed, $\sin(x+1)=\cos 1\sin x+\sin 1\cos x$.

Theorem and Definition 1.4.
The homogeneous linear differential equation (2.2) has n solutions linearly independent. Further, if $f_1,\dots,f_n$ are n linearly independent solutions of (2.2), then every solution f of (2.2) is a linear combination of $f_1,\dots,f_n$:

$$f=c_1f_1+\dots+c_nf_n, \tag{2.3}$$

for some $c_1, \dots, c_n \in \mathbb{R}$.
The expression f in (2.3) is called the general solution of the homogeneous equation (2.2) and $\{f_1, \dots, f_n\}$ is called a fundamental set of solutions of this equation.

Proof .
Let $x_0 \in (a, b)$. According to Theorem (1.2), for all $0 \le k \le n-1$ there exists a solution f_k of (2.2) satisfying

$$f_k^{(j)}(xo) = 0, \forall j \neq k, \; f_k^{(k)}(x_0) = 1.$$

The solutions f_k, $0 \le k \le n-1$, are linearly independent on (a, b). Indeed, suppose that there exist $c_0, \dots c_{n-1}$ such that $\sum_{k=0}^{n-1} c_k f_k(x) = 0$, for all $x \in (a, b)$.

For $x = x_0$, we have $c_0 = 0$. Differentiating we see that $\sum_{k=0}^{n-1} c_k f_k^{(j)}(x) = 0$, for all $x \in (a, b)$ and $1 \le j \le n-1$. For $x = x_0$, we have $c_j = 0$.

Let y be a solution of the equation (2.2) and let $y(x_0) = a_0, \dots y^{(n-1)}(x_0) = a_{n-1}$. The functions y and $z = \sum_{j=0}^{n-1} a_j f_j$.

Since $f_j^{(k)}(x_0) = \delta_{j,k}$ for all $0 \le j, k \le n-1$, then $y^{(j)}(x_0) = z^{(j)}(x_0)$ for all $0 \le j \le n-1$. Then $y = z$.

□

Example 1.2 :

1. Consider the differential equation $x^2 y^{''} - xy' + y = 0$ for $x \in (0, +\infty)$. The functions $y_1 = x$ and $y_2 = \frac{1}{x}$ are solutions of the differential equation. Then the general solution is $y = ax + \dfrac{b}{x}$.

2. $\{\sin x, \cos x\}$ is a fundamental set of solutions of the homogeneous differential equation $y^{''} + y = 0$. Then the general solution of this equation is $y = a \sin x + b \cos x$, with $a, b \in \mathbb{R}$.

3. $\{e^x, xe^x\}$ is a fundamental set of solutions of the homogeneous differential equation $y^{''} - 2y' + y = 0$. Then the general solution of this equation is $y = (ax + b)e^x$, with $a, b \in \mathbb{R}$.

4. $\{e^x, \cos x\}$ is a fundamental set of solutions of the homogeneous differential equation $y^{''}(\cos x + \sin x) - 2y' \cos x + y(\cos x - \sin x) = 0$. Then the general solution of this equation is $y = ae^x + b \cos x$, with $a, b \in \mathbb{R}$.

Remark 8 :
If $y_1, \dots, y_m$ are linearly independent, we will construct a differential equation of order m such that $\{y_1, \dots, y_m\}$ is a fundamental set of solutions of this equation.

We give now a simple criterion for determining whether or not n solutions of (2.2) are linearly independent.

1.3 The Wronskian and Linear Independence

Definition 1.5.
Let $f_1, \dots, f_n$, n functions defined on an interval (a, b) each of which has an $(n-1)$ derivative. The determinant

$$W = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \tag{2.4}$$

is called the Wronskian of the functions $f_1, \dots, f_n$.

Theorem 1.6.
Let $f_1, \dots, f_n$ solutions of the n^{th}-order homogeneous linear differential equation (2.2). These functions are linearly independent on (a, b) if and only if the Wronskian W is not the zero function on the interval (a, b).

We have further:

Theorem 1.7.
The Wronskian of n solutions $f_1, \dots, f_n$ of (2.2) is either identically zero on (a, b) or else is never zero on (a, b).

Proof .
Using the fundamental properties of the determinant, we have

$$\begin{aligned} W'(x) &= \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n)} & f_2^{(n)} & \dots & f_n^{(n)} \end{vmatrix} \\ &= -\frac{a_1}{a_0} \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \\ &= -\frac{a_1}{a_0} W(x). \end{aligned}$$

Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}dt}.$$

□

Example 1.3 :

1. The functions $\sin x, \cos x$ are solutions of the differential equation $y'' + y = 0$ on any non empty open interval (a, b). The Wronskian of $\sin x, \cos x$ is $W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$. Then $\sin x, \cos x$ are linearly independent on the interval (a, b).

2. Let $f(x) = \sin x$, $g(x) = \cos x$, $h(x) = \sin(x + 1)$ on any non empty open interval (a, b). These functions are solutions of the differential equation $y'' + y = 0$. The Wronskian of f, g, h is $W = \begin{vmatrix} \sin x & \cos x & \sin(x+1) \\ \cos x & -\sin x & \cos(x+1) \\ -\sin x & -\cos x & -\sin(x+1) \end{vmatrix} = 0$, since the first and the third row are proportional. Then f, g, h

Theorem 1.8.
Let $f_1, \ldots, f_n$ be n linearly independent functions (a, b) of class C^n. There is a linear differential equation of order n such that $\{f_1, \ldots, f_n\}$ is a fundamental set of solutions of this equation.

Proof .
Consider the linear differential equation defined by:

$$\begin{vmatrix} y & f_1 & \ldots & f_n \\ y' & f_1' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ y^{(n-1)} & f_1^{(n-1)} & \ldots & f_n^{(n-1)} \\ y^{(n)} & f_1^{(n)} & \ldots & f_n^{(n)} \end{vmatrix} = 0.$$

This equation is of order n since $f_1, \ldots, f_n$ are linearly independent.
By definition the set $\{f_1, \ldots, f_n\}$ is a fundamental set of solutions of this equation.

□

Example 1.4 :

1. Consider the functions $f = \sin x$ and $g = \cos x$ on $\mathbb{R}$. The functions f, g are linearly independent. Consider the equation defined by

$$\begin{vmatrix} y & \sin x & \cos x \\ y' & \cos x & -\sin x \\ y'' & -\sin x & -\cos x \end{vmatrix} = y'' + y = 0.$$

Then $\{f, g\}$ is a fundamental set of solutions of the equation $y^{''} + y = 0$.

2. Consider the functions $f = \sin x$, $g = \cos x$ and $h(x) = e^x$ on $\mathbb{R}$. The functions f, g, h are linearly independent.

$$\begin{vmatrix} y & \sin x & \cos x & e^x \\ y' & \cos x & -\sin x & e^x \\ y^{''} & -\sin x & -\cos x & e^x \\ y^{(3)} & -\cos x & \sin x & e^x \end{vmatrix} = -2e^x(y^{(3)} - y^{''} + y^{'} - y).$$

Then $\{f, g, h\}$ is a fundamental set of solutions of the equation

$$y^{(3)} - y^{''} + y^{'} - y = 0.$$

1.4 Reduction of the Order of a Homogeneous Equation

Consider now the homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0,$$

where $a_1, \ldots, a_n$ are continuous functions on an interval (a, b).

Suppose we have a solution $y_1 \neq 0$ of the equation. We look for a solution in the form $y = uy_1$, where u is some function. In use of Leibniz formula: $(uy_1)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} y_1^{(k-j)} u^{(j)}$. Since y_1 is a solution of the differential equation, then uY_1 is also a solution if and only if

$$\sum_{j=1}^{n} \binom{n}{j} y_1^{(n-j)} u^{(j)} + \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{k} \binom{k}{j} y_1^{(k-j)} u^{(j)} \right) = 0.$$

Then u fulfills a linear differential equation of order n in the form $y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-1} y' = 0$. This method is called reduction of order.

Example 1.5 :

1. Consider the differential equation $y^{''} - 3y' + 2y = 0$.
   $y_1 = e^x$ is a solution of the differential equation. If $y_2 = uy_1$ is a solution, we must have $u^{''} - u' = 0$. Then $u = a + be^x$ and $y_2 = e^{2x}$ is a second solution of the equation $y^{''} - 3y' + 2y = 0$ and y_1, y_2 are linearly independent.

2. Consider the differential equation $(1 - x^2)y'' - xy' + y = 0$ on the interval $(1, +\infty)$. $y_1 = x$ is a solution. Consider a solution y in the form $y = xu$, with u not constant. The function u fulfills the following differential equation: $x(x^2 - 1)u^{''} + (3x^2 - 2)u' = 0$. This yields that $u' = \dfrac{\lambda}{x^2\sqrt{x^2 - 1}}$ and $y_2 = \sqrt{x^2 - 1}$ is a solution.

3. Consider the differential equation $(x-1)y^{''} - xy' + y = 0$.
$y_1 = x$ is a solution of the differential equation. If $y_2 = xu$ is a solution, we must have $x(x-1)u^{''} + (-x^2+2x-2)u' = 0$. Then $u' = \left(\frac{e^x}{x}\right)'$ and $y_2 = e^x$ is a solution of the differential equation.

4. Consider the differential equation $x^2y^{''} - 7xy' + 15y = 0$, for $x > 0$. $y_1(x) = x^3$ is a solution of the differential equation. If $y_2 = ux^3$ is a solution, we must have $xu^{''} - u' = 0$. Then $y_2 = x^5$ is also a solution.

1.5 The Non-Homogeneous Equation

Any function y_p that satisfies (2.1) is called a particular solution of the equation. For example, $\sin x$ is a particular solution of the differential equation $xy^{''} + y' + xy = \cos x$.

Remark 9 :
If $y_1, \ldots, y_m$ are solutions of the homogeneous equation (2.2) on an interval I and y_p is any particular solution of the non-homogeneous equation (2.1) on I, then the linear combination

$$c_1y_1 + \ldots + c_my_m + y_p$$

is also a solution of the non-homogeneous equation (2.1).

Theorem 1.9. [General Solution of the Non-Homogeneous Equations]
Let y_p be any particular solution of the non-homogeneous differential equation (2.1) on an interval I, and let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of the associated homogeneous differential equation (2.2)) on I. Then the general solution of the equation (2.1)) on the interval I is

$$y = c_1y_1 + \ldots + c_ny_n + y_p$$

where the $c_1, \ldots, c_n \in \mathbb{R}$.

Proof .
Let y be any solution of the differential equation (2.1)), the function $y - y_p$ is a solution of the homogeneous equation (2.2)). Then there is $c_1, \ldots, c_n \in \mathbb{R}$ such that $y = c_1y_1 + \ldots + c_ny_n + y_p$. □

Example 1.6 :
Consider the differential equation $(\sin x - \cos x)y^{''} + 2y' \sin x + y(\cos x + \sin x) = 2$. The function $\cos x$ is a particular solution and $\{e^x, \sin x\}$ is a fundamental set of solutions of this equation. Then the general solution of this equation is $y = axe^x + b\sin x + \cos x$, $a, b \in \mathbb{R}$.

Theorem 1.10. [Superposition Principle]
Consider the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b_1 + \ldots + b_m, \tag{2.5}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ continuous functions on an interval (a, b).
If y_k is a particular solution of the non-homogeneous differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b_k,$$

for all $1 \leq k \leq m$, then $y_1 + \ldots + y_m$ is a particular solution of the non-homogeneous differential equation (2.5)).

Example 1.7 :
Consider the differential equation $y^{''} + 2y' + y = e^x + 2e^{-x} + \sin x$.
$\frac{1}{4}e^x$ is a particular solution of the differential equation $y^{''} + 2y' + y = e^x$.
$x^2 e^{-x}$ is a particular solution of the differential equation $y^{''} + 2y' + y = 2e^{-x}$.
$-\frac{1}{2}\cos x$ is a particular solution of the differential equation $y^{''} + 2y' + y = \sin x$.
Then $y_p = \frac{1}{4}e^x + x^2 e^{-x} - \frac{1}{2}\cos x$ is a particular solution of the differential equation $y^{''} + 2y' + y = e^x + 2e^{-x} + \sin x$.

1.6 Exercises

1-1 Prove that the set of functions $\{y_1 = e^x, y_2 = e^{2x}, y_3 = x\}$ is linearly independent on any non empty open interval.

1-2
(a) Prove that the set of functions $\{y_1 = e^x, y_2 = \ln x\}$ is linearly independent on any non empty open interval.
(b) Find a differential equation of order 2 such that $\{y_1 = e^x, y_2 = \ln x\}$ is a fundamental set of solutions of this equation.

1-3 Check that e^x is a solution to $y^{''} - 2y' + y = 0$, and then use reduction of order to find a fundamental set of solutions.

1-4 Check that $e^{2x}\cos x$ is a solution to $y^{''} - 4y' + 5y = 0$, and then use reduction of order to find a fundamental set of solutions.

1-5 Check that $y = x$ is a solution to $(x-1)(x-2)y^{''} - xy' + y = 0$. Then use reduction of order to find a second linearly independent solution.

1-6 Without solving, determine the Wronskian of two solutions evaluated at $x = 4$ for the following differential equation: $2x^2 y^{''} + xy' - 3y = 0$. Is the Wronskian defined for all x?

1-7 Verify that 1 and $\sqrt{x}$ are solutions to the differential equation $yy^{''} + (y')^2 = 0$ for $x > 0$. Then show that $a + b\sqrt{x}$ is not in general a solution of this equation. Can you explain why this result doesn't contradict the method of linear superposition?

1-8 If the functions y_1 and y_2 are linearly independent solutions of $y^{''} - a(x)y' + b(x)y = 0$, determine what the necessary and sufficient conditions are such that the functions $y_3 = Ay_1 + By_2$ and also $y_4 = Cy_1 + Dy_2$ form a linearly independent set of solutions.

1-9
1) Show that $y = x$ is a solution of the following differential equation: $x^2y^{''} - (x^2 + 2x)y^{'} + (x + 2)y = 0$.
2) Use reduction of order method to find the general solution of this differential equation.

1-10
1) Show that $y = e^x$ is a solution for $x > 1$ of the following differential equation: $(x - 1)y^{''} - xy^{'} + y = 0$.
2) Find a second linearly independent solution z, and check that the Wronskian of y and z is non-zero for $x > 1$.

2 Linear Differential Equations With Constant Coefficients

2.1 Homogeneous Equations

We consider now the differential equation (2.2)) with $a_0, \ldots, a_n$ constants in $\mathbb{R}$. We seek for solutions in the form $y = e^{rx}$, where r is constant. y is a solution of (2.2)) if and only if $r^n + a_1r^{n-1} + \ldots + a_{n-1}r + a_n = 0$. This equation is called the characteristic equation or the auxiliary equation of (2.2)).

In what follows, we consider the linear homogeneous equations of order 2 and with constant coefficients.

$$y^{''} + ay' + b = 0$$

The characteristic equation is $r^2 + ar + b = 0$. We have three cases

1. If $\Delta = a^2 - 4b > 0$, the characteristic equation has two different solutions r_1 and r_2. $\{e^{r_1x}, e^{r_1x}\}$ is a fundamental set of solutions of the equation.

2. If $\Delta = 0$, the characteristic equation has one solution $r = -\frac{a}{2}$. $\{e^{rx}, xe^{rx}\}$ is a fundamental set of solutions of the equation.

3. If $\Delta < 0$, the characteristic equation has two different complex solutions r_1 and r_2. $r_1 = \alpha + \mathrm{i}\beta$, $r_2 = \alpha - \mathrm{i}\beta$. $\{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$ is a fundamental set of solutions of the equation.

Example 2.1 :

1. $y" - 3y' + 2y = 0$, $\{e^{x}, e^{2x}\}$ is a fundamental set of solutions of the equation.
2. $y" + 4y' = 0$. $\{1, e^{-4x}\}$ is a fundamental set of solutions of the equation.
3. $y" + y' + y = 0$. $\{e^{\frac{-x}{2}}\cos\frac{\sqrt{3}}{2}x, e^{\frac{-x}{2}}\sin\frac{\sqrt{3}}{2}x\}$ is a fundamental set of solutions of the equation.
4. $y" + 2y' + y = 0$. $\{e^{-x}, xe^{-x}\}$ is a fundamental set of solutions of the equation.

2.2 Non-Homogeneous Equations

We consider the differential equation

$$y^{''} + ay' + by = f(x). \tag{2.6}$$

We give two methods to construct a particular solution of the differential equation (2.6).

Theorem 2.1.

1. Let $\{y_1, y_2\}$ be a fundamental set of solutions of the homogeneous equation. For any differentiable function y on I, there exists a unique pair of differentiable functions (U, V) on I such that:

$$\begin{cases} y = Uy_1 + Vy_2 \\ y' = Uy_1' + Vy_2' \end{cases} \tag{2.7}$$

2. If y is a solution of the differential equation (2.6)), there exists a unique pair of differentiable functions (U, V) on I such that:

$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$

This method is called the change of parameters method.

Proof .

1. For any $x \in I$, the determinant of the linear system (2.7)) is $W(x) \neq 0$, thus we have a unique solution
$$U(x) = \frac{\begin{vmatrix} y(x) & y_2(x) \\ y'(x) & y_2'(x) \end{vmatrix}}{W(x)}, \quad V(x) = \frac{\begin{vmatrix} y_1(x) & y(x) \\ y_1'(x) & y'(x) \end{vmatrix}}{W(x)}.$$

2. If y is a is a solution of the differential equation (2.6). There exists a unique pair of differentiable functions (U, V) on I satisfying the system (2.7). If we differentiate the first equation of the system, we get:
$$U'y_1 + V'y_2 = 0. \tag{2.8}$$
y is twice differentiable, then
$$y" = Uy_1" + Vy_2" + U'y_1' + V'y_2'. \tag{2.9}$$
Now y is a solution of the differential equation (2.6) if and only if
$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$
This system is a Crammer system, so it has a unique solution $y = Uy_1 + Vy_2$.
Thus the set of solutions of the differential equation (2.6) is the set $\{y = Uy_1 + Vy_2, \}$ where U, V differentiable functions solutions of the following system:
$$\begin{cases} U'y_1 + V'y_2 = 0 \\ U'y_1' + V'y_2' = f \end{cases}.$$

□

Example 2.2 :

1. Consider the differential equation $y'' + y = \dfrac{1}{3 + \cos(2x)}$.

   The general solution of the homogenous differential equation is $y = a\cos x + b\sin x$. Using the change of parameters method, $y = U\cos x + V\sin x$, we find:
$$\begin{cases} U'\cos x + V'\sin x = 0 \\ -U'\sin x + V'\cos x = \frac{1}{3+\cos(2x)} \end{cases}.$$
   Then
$$U = -\frac{1}{2}\tan^{-1}(\cos x) + a, \qquad V = \frac{1}{2\sqrt{2}}\tan^{-1}(\frac{\sin x}{\sqrt{2}}) + b.$$

2. Consider the differential equation $y''+4y'+5y=\cosh(2x)\cos x$. The characteristic equation is $r^2+4r+5=(r+2+\mathrm{i})(r+2-\mathrm{i})$. $\{e^{-2x}\cos(2x), e^{-2x}\sin(2x)\}$ is a fundamental set of solutions of the equation.
Using the change of parameter method, the general solution of the equation takes the form: $y=Ue^{-2x}\cos(x)+Ve^{-2x}\sin(x)$, with $U'e^{-2x}\cos(x)+V'e^{-2x}\sin(x)=0$ and $U'e^{-2x}(-\sin(x)-2\cos(x))+V'e^{-2x}(\cos(x)-2\sin(x))=\cosh(2x)\cos(x)$. Then $U=-\dfrac{1}{8}\cos(2x)+\dfrac{1}{20}e^{4x}\sin(2x)-\dfrac{1}{40}e^{4x}\cos(2x)+a$ and $V=\frac{1}{40}e^{4x}\sin(2x)+\frac{1}{20}e^{4x}\cos(2x)+\frac{x}{4}+\frac{1}{8}\sin(2x)+\frac{1}{16}e^{4x}+b$.

2.3 Particular Cases of Non-Homogeneous Term

1. If the function f is a polynomial of degree n. We look for a particular solution as polynomial.

   • If $b\neq 0$, there exists a polynomial of degree n as particular solution of the differential equation (2.6).

   • If $b=0$ and $a\neq 0$, there exists a polynomial of degree $n+1$ as particular solution of the differential equation (2.6).

   • If $b=a=0$, there exists a polynomial of degree $(n+2)$ as particular solution of the differential equation (2.6).

2. If $f(x)=P(x)e^{\alpha x}$, with P a polynomial of degree n. We define the function z by: $z=e^{-\alpha x}y$.
$y'=\alpha e^{\alpha x}z+e^{\alpha x}z'$, $y"=\alpha^2 y+2\alpha e^{\alpha x}z'+e^{\alpha x}z"$.

$$\begin{aligned} y"+ay'+by=e^{\alpha x}P(x) &= e^{\alpha x}(\alpha^2 z+2\alpha z'+z"+a\alpha z+az'+bz)\\ &= e^{\alpha x}(z"+z'(a+2\alpha)+z(\alpha^2+a\alpha+b)). \end{aligned}$$

3. Then z verifies the following differential equation
$$z"+z'(a+2\alpha)+z(\alpha^2+a\alpha+b)=P(x).$$

   • If α is not a solution of the characteristic equation, then there exists a polynomial Q of degree n such that $e^{\alpha x}Q$ is a particular solution of the differential equation.

   • If $\alpha^2+a\alpha+b=0$ and $a+2\alpha\neq 0$, (i.e. α is a simple zero of the characteristic equation (algebraic multiplicity 1)). In this case there exists a polynomial Q of degree $n+1$ such that $e^{\alpha x}Q$ is a particular solution of differential equation.

   • If $\alpha^2+a\alpha+b=0$ and $a+2\alpha=0$, (i.e. α is a solution of the characteristic equation with algebraic multiplicity 2). In this case, there exists a polynomial Q of degree $n+2$ such that $e^{\alpha x}Q$ is a particular solution of the differential equation.

2.4 Exercises

2-1 Solve the integral equation for $y(x)$:

$$y(x) + 8\int_0^x y(t)\sin(x-t)dt = 9, \quad x \geq 0.$$

2-2 Let $g\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ be a continuous function and consider the following differential equation

$$y'' + y = g \tag{2.10}$$

1) Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:

$$h(x) = \sin x\int_0^x g(t)\cos t dt - \cos x\int_0^x g(t)\sin t dt.$$

   i. Prove that h is a solution of the differential equation (2.10) .

   ii. Prove that $h(x) = \displaystyle\int_0^x g(t)\sin(x-t)dt$.

   iii. Prove that $h(x)+h(x+\pi) = \displaystyle\int_0^\pi g(x+t)\sin t dt$ and deduce that $h(x) + h(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

2) 1) Prove that any solution f of (2.10) on $\mathbb{R}$ fulfills $f(x)+f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

   2) Deduce that if a function $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ and $F''(x) + F(x) \geq 0$, $\forall x \in \mathbb{R}$, then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

2-3 Solve the differential equation $y'' + y = \dfrac{1}{3+\cos(2x)}$.

2-4 Consider the differential equation $y'' + \lambda y = 0$, with $\lambda \in \mathbb{R}$.

1) Give the general solutions for this differential equation.
2) Determine the values of λ for which there exists a non zero solution y of the differential equation $y'' + \lambda y = 0$ and fulfills $y(0) = y(1) = 0$.

2-5
1) Determine the primitives of the functions $e^{\alpha x}\sin^2 x$ and $e^{\alpha x}\sin x\cos x$, with $\alpha \neq 0$.
2) Find two linearly independent solutions of the differential equation $y'' - 2ky' + (k^2+1)y = 0$, where $k \in \mathbb{R}$.
3) Solve the differential equation $y'' - 2ky' + (k^2+1)y = e^x\sin x$.

2-6 Solve the following differential equations:

1) $y'' - 5y' + 6y = 0$,
2) $4y'' + 4y' + y = 0$,
3) $y'' + y' + y = 0$,
4) $y'' + y' - 2y = 2x^2 - 3x + 1$,
5) $2y'' + 2y' + 3y = x^2 + 2x - 1$,
6) $y'' - 2y' + y = e^{-x}$,
7) $y'' - y' - 2y = x^2 e^{-3x}$,
8) $y'' - 2y' + 2y = e^x + x$,
9) $y'' + 4y = \sin(3x)$,
10) $y'' + 4y = \cos(2x) + \cos(4x)$,
11) $y'' + y = \dfrac{1}{1 + \sin^2 x}$,
12) $y''+4y'+5y = \cosh(2x).\cos x$,
13) $y'' - 6y' + 9y = \sinh^3 x$.

2-7 Consider the following differential equations: : $y''-y = 1$ and $y''+y = 1$.

1) Solve these differential equations.
2) Give the bounded solutions on $\mathbb{R}^+$ of these differential equations.
3) Give the even solutions of these differential equations.
4) Let $a \in \mathbb{R}^*$. Examine if there exist solutions of these differential equations which vanishes at 0 and at a.
   Discuss according to the values of a.

2-8
1) Let α, β be different real numbers.
   Solve the following differential equation $y^{''} - (\alpha+\beta)y' + \alpha\beta y = 0$.
2) Determine the solutions of the differential equation $y^{''} + y = \cos x$.
3) Determine the solutions of the differential equation $y^{''}+y = \dfrac{1}{3+\cos(2x)}$.

2-9 We consider the following differential equations

$$y^{''} - y = 1 \tag{2.11}$$

and

$$y^{''} + y = 1 \tag{2.12}$$

1) Solve the differential equations (5.11) and (5.12).
   Give the solutions of (5.11) and (5.12) which have the same initial conditions $y(0) = \alpha, y'(0) = \beta$, $\alpha, \beta \in \mathbb{R}$.
2) Give if there exists
   1) the bounded solutions on $\mathbb{R}^+$ for the differential equations (5.11) and (5.12),
   2) the even solutions on $\mathbb{R}$ for the differential equations (5.11) and (5.12).

3) Let $a \in \mathbb{R}^*$. Say whether there exist solutions for the differential equations (5.11) and (5.12) vanishing at 0 and at a. Discuss according to the values of a.

4) 1) Let $\lambda \in \mathbb{R}$, f and g two differentiable functions on $\mathbb{R}^+$ such that $f' + \lambda f \leq g$. We set $h(x) = \int_0^x e^{\lambda t} g(t)dt$.
   Compute h' and deduce that

$$f(x) \leq e^{-\lambda x} f(0) + e^{-\lambda x} h(x), \quad \forall\, x \in \mathbb{R}^+.$$

   2) Let φ be function twice differentiable on $\mathbb{R}^+$ such that

$$\forall x \in \mathbb{R}^+, \quad \varphi^{''}(x) - \varphi(x) \leq 1.$$

   Let ψ be the solution of (5.11) such that $\psi(0) = \varphi(0)$, $\psi'(0) = \varphi'(0)$. Prove that $\forall\, x \in \mathbb{R}^+$, $\varphi(x) \leq \psi(x)$. (Hint: we can use the question a) with $f = \varphi' - \varphi$ and $\lambda = 1$).

5) Let $\varphi(x) = 1 - e^{-x}$.

   1) Verify that $\varphi^{''} + \varphi \leq 1$.
   2) Let ψ be the solution of (5.12) such that $\psi(0) = \varphi(0) = 0$ and $\psi'(0) = \varphi'(0) = 1$.
      Is $\varphi(x) \leq \psi(x)$, $\forall\, x \in \mathbb{R}^+$?

2-10 Consider the differential equation $y^{''} + \lambda y = 0$, with $\lambda \in \mathbb{R}$.

1) Give the real general solution of the equation according to the values of λ.
2) Determine the values of λ for which there exists a non zero real solution of the equation such that $y(0) = y(1) = 0$.

2-11

1) Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Determine the anti-derivatives of $e^{\alpha x} \sin^2 x$ and of $e^{\alpha x} \sin x \cos x$.
2) Find two linearly independent solutions of the differential equation: $y'' - 2ky' + (k^2 + 1)y = 0$, with $k \in \mathbb{R}$.
3) Solve the differential equation $y'' - 2ky' + (k^2 + 1)y = e^x \sin x$.

2-12 Solve the following differential equations:

1) $y'' - 5y' + 6y = 0,$
2) $4y'' + 4y' + y = 0,$
3) $y'' + y' + y = 0.$
4) $y'' + y' - 2y = 2x^2 - 3x + 1,$
5) $2y'' + 2y' + 3y = x^2 + 2x - 1,$
6) $y'' - 2y' + y = e^{-x},$
7) $y'' - y' - 2y = x^2 e^{-3x},$
8) $y'' - 2y' + 2y = e^x + x,$
9) $y'' + 4y = \sin(3x),$
10) $y'' + 4y = \cos(2x) + \cos(4x),$
11) $y'' + y = \dfrac{1}{1 + \sin^2 x},$
12) $y'' + 4y' + 5y = \cosh(2x).\cos x,$
13) $y'' - 6y' + 9y = \sinh^3 x.$

2-13 Consider the differential equation

$$y^{''} - 2y' + y = e^x(x + \cos x)$$

2-14 Consider the differential equation

$$y^{''} + y' + y = \cos x$$

2-15 Consider the differential equation

$$y^{''} - 3y' + 2y = e^x + xe^{2x}$$

2-16 Solve the following second-order differential equations

1) $y^{''} = a^2 y$
2) $yy^{''} = (y^{'})^2,$
3) $y^{''} = y^{'}(1 + 3y^2), \quad y(0) = 1,\ y^{'}(0) = 2.$
4) $y^{''} - 2y' = -3x + 7 - \sin(x), \quad y'(2) = 3,\ y(0) = 3$
$y^{''} + 3y' + 4y = 0, \quad y(0) = 1,\ y'(0) = 0.$

2-17 Find the general solution of the following differential equations:

1) $y^{(3)} + 3y^{''} + 3y' + y = 0,$
2) $y^{(4)} + 4y^{(3)} + 6y^{''} + 4y' + y = 0,$
3) $y^{(4)} + 4y^{(3)} + 3y^{''} - 4y' - 4y = 0$

2-18 Let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $g(x) \geq 0$, $\forall x \in \mathbb{R}$ and we consider the following differential equation

$$y^{''} + y = g \tag{2.13}$$

1) Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:

$$h(x) = \sin x \int_0^x g(t)\cos t dt - \cos x \int_0^x g(t)\sin t dt.$$

a) Compute h' and h'' and prove that h fulfills the equation (2.13).

b) Prove that $h(x) = \displaystyle\int_0^x g(t)\sin(x-t)dt.$

c) Prove that $h(x) + h(x+\pi) = \int_0^\pi g(x+t)\sin t dt$ and deduce that $h(x) + h(x+\pi) \geq 0$.

2) a) Prove that any solution f of (2.13) on $\mathbb{R}$ fulfills $f(x)+f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

b) Deduce that if a function $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$ such that $F''(x) + F(x) \geq 0$, then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$. (Hint: we can take the function $g = F'' + F$ and remark that F is solution of the differential equation (2.13)).

2-19 Let E_k be the vector space of complex functions of class $\mathcal{C}^k$ defined on $\mathbb{R}$ and let $\alpha \neq \beta \in \mathbb{C}$. We consider the linear map $D\colon E_2 \longrightarrow E_0$ defined by

$$D(y) = y\text{''} - (\alpha+\beta)y' + \alpha\beta y.$$

1) Compute the kernel of D.
2) Compute y if $D(y) = e^{\mathrm{i}x}$, with $\alpha = \mathrm{i} = -\beta$.
3) Compute y if $D(y) = \dfrac{1}{3+\cos 2x}$, with $\alpha = \mathrm{i} = -\beta$.

2-20 We consider the following differential equations

$$y\text{''} - y = 1 \tag{2.14}$$

and

$$y'' + y = 1 \tag{2.15}$$

1) Solve the differential equations (5.11) and (5.12).
   Give the solutions of (5.11) and (5.12) which have the same initial conditions $y(0) = \alpha, y'(0) = \beta$, $\alpha, \beta \in \mathbb{R}$.
2) Give if there exists
   i. the bounded solutions on $\mathbb{R}^+$ for the differential equations (5.11) and (5.12),

ii. the even solutions on $\mathbb{R}$ for the differential equations (5.11) and (5.12).

3) Let $a \in \mathbb{R}^*$. Is there exist solutions for the differential equations (5.11) and (5.12) vanishing at 0 and at a. Discuss the values of a.

4) i. Let $\lambda \in \mathbb{R}$, f and g two differentiable functions on $\mathbb{R}^+$ such that $f' + \lambda f \leq g$. We set $h(x) = \displaystyle\int_0^x e^{\lambda t} g(t)dt$.
Compute the differential of h in term of the function $x \longmapsto e^{\lambda x} f(x)$.
Deduce that $\forall\, x \in \mathbb{R}^+$, $f(x) \leq e^{-\lambda x} f(0) + e^{-\lambda x} h(x)$.

ii. Let φ be function twice differentiable on $\mathbb{R}^+$ such that

$$\forall t \in \mathbb{R}^+, \quad \varphi"(t) - \varphi(t) \leq 1.$$

Let ψ be the solution of (5.11) such that $\psi(0) = \varphi(0)$, $\psi'(0) = \varphi'(0)$.
Prove that $\forall\ t \in \mathbb{R}^+$, $\varphi(t) \leq \psi(t)$. (Hint: we can use the question a) in the case where $f = \varphi' - \varphi$ and $\lambda = 1$).

5) Let $\varphi(t) = 1 - e^{-t}$. verify that $\varphi" + \varphi \leq 1$.
Let ψ be the solution of (5.12) such that $\psi(0) = \varphi(0) = 0$ and $\psi'(0) = \varphi'(0) = 1$. Do we have $\varphi(t) \leq \psi(t)$, $\forall\ t \in \mathbb{R}^+$?

2-21 We intend to find the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ twice differentiable and fulfills the following equation

$$f(x) + f(-x) = e^x. \tag{2.16}$$

1) Solve the following differential equation

$$y^{''} + y = 2\cosh x. \tag{2.17}$$

2) Prove that the solutions of the equation (2.16) are solutions of the differential equation (2.17).

3) Deduce all solutions of the equation (2.16)..

2-22

1) $y^{''} - 5y' + 6y = 0$,
2) $4y^{''} + 4y' + y = 0$,
3) , $y^{''} + y' + y = 0$.
4) $y^{''} + y' - 2y = 2x^2 - 3x + 1$,
5) $2y^{''} + 2y' + 3y = x^2 + 2x - 1$,

6) $y'' - 2y' + y = e^{-x}$,
7) $y'' - y' - 2y = x^2 e^{-3x}$,
8) $y'' - 2y' + 2y = e^x + x$,
9) $y'' + 4y = \sin(3x)$,
10) $y'' + 4y = \cos 2x + \cos(4x)$,
11) $y'' + y = \dfrac{1}{1+\sin^2 x}$,
12) $y'' + 4y' + 5y = \cosh 2x . \cos x$,
13) $y'' - 6y' + 9y = \sinh^3 x$.

3 The Cauchy-Euler equation

3.1 The Homogeneous Cauchy-Euler equation

In this section, we solve another class of linear second order equations. These equations called the Cauchy Euler equations. The second order homogeneous linear ordinary differential equation of the form

$$ax^2y'' + bxy' + cy = 0, \qquad (2.18)$$

with a, b constants, is called a homogeneous **Euler's equation** or a homogeneous **Cauchy-Euler equation**.
This equation can be reduced to linear homogeneous differential equation with constant coefficients. This conversion can be done in two ways.
The first way is to take the change of variables $x = e^t$. If $z(t) = y(e^t) = y(x)$, we get $z' = e^t y'(e^t) = xy'$ and $z'' = z' + x^2 y''(x)$. Then $ax^2y'' + bxy' + cy = a(z'' - z') + bz' + cz = az'' + (b-a)z' + cz$. The Cauchy-Euler equation (2.18) becomes a linear differential equation

$$az'' + (b-a)z' + cz = 0.$$

The second way is to look for a solutions in the form $y = x^r$. Substituting into the differential equation gives the following: $x^r\left(ar^2 + (b-a)r + c\right) = 0$. For $x \neq 0$, we have

$$ar^2 + (b-a)r + c = 0. \qquad (2.19)$$

As in the second order linear differential equations with constant coefficients, the type of solutions that we obtain using this method depend on whether the roots of the equation (2.20) are real and distinct, repeated or complex.

3.2 Case of Two Real and Distinct Roots

If the equation (2.20) has two roots p, q, then $y_1 = x^p$ and $y_2 = x^q$ are linearly independent solutions. The general solution of Cauchy-Euler equation (2.18) is $y = ax^p + bx^q$.

Example 3.1 :

1. Consider the following Cauchy-Euler equation: $x^2y'' + 3xy' - 3y = 0$. The $z-$differential equation is $z^{''} + 2z' - 3z = 0$. Then $z = ae^t + be^{-3t}$ and $y = ax + bx^{-3}$.

   If we use the second method, we get: $r^2 + 2r - 3 = 0$. Then $r = 1$ or $r = -3$ and $y = ax + bx^{-3}$.

2. Consider the following Cauchy-Euler equation: $x^2y'' - 3xy' + 7y = 0$. The $z-$differential equation is $z^{''} - 4z' + 7z = 0$. Then $z = ae^{2t}\cos(\sqrt{3}t) + be^{2t}\sin(\sqrt{3}t)$ and $y = ax^2\cos(\sqrt{3}\ln x) + bx^2\sin(\sqrt{3}\ln x)$.

   If we use the second method, we get: $r^2 - 4r + 7 = 0$. Then $r = 2 \pm \mathrm{i}\sqrt{3}$ and $y = ax^2\cos(\sqrt{3}\ln x) + bx^2\sin(\sqrt{3}\ln x)$.

3.3 Repeated Root

If the equation (2.20) has a repeated root p, then $y_1 = x^p$ and $y_2 = x^p \ln x$ are linearly independent solutions. The general solution of Cauchy-Euler equation (2.18) is $y = x^p(a + b\ln x)$. We can find the second solution using the variation of constant method.

Example 3.2 :
Consider the following Cauchy-Euler equation: $x^2y'' - 3xy' + 4y = 0$.
If x^r is a solution, then $r^2 - 4r + 4 = (r-2)^2 = 0$. The general solution of the equation on $(0, +\infty)$ is $y = x^2(a + b\ln x)$, $a, b \in \mathbb{R}$.

3.4 Case of Complex Roots

Let $r = s \pm it$ be the complex roots of the equation (2.20), then $y_1 = x^s\cos(t\ln x)$ and $y_2 = x^s\sin(t\ln x)$ are linearly independent solutions of the Cauchy-Euler equation (2.18).

Example 3.3 :
Consider the following Cauchy-Euler equation: $x^2y'' + 3xy' + 5y = 0$.
If x^r is a solution, then $r^2 + 2r + 5 = (r+1)^2 + 4 = 0$. The general solution of the equation on $(0, +\infty)$ is $y = ax\cos(2\ln x) + bx\sin(2\ln x)$, $a, b \in \mathbb{R}$.

3.5 The Non-Homogeneous Cauchy-Euler equation

The non-homogeneous Euler equation is written as

$$ax^2y'' + bxy' + cy = f. \tag{2.20}$$

To solve this equation, we look for a fundamental set of solutions of the homogeneous equation and use the change of parameter method.

Example 3.4 :
Consider the following Cauchy-Euler equation: $x^2y'' + 3xy' - 3y = e^x$.
$\{y_1 = x, y_2 = x^{-3}\}$ is a fundamental set of solutions. In use of the change of parameter method, $y = Ux + Vx^{-3}$, we find $U = \frac{1}{4}e^x$, $V' = -\frac{1}{4}x^4e^x$ and $V = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$. Then $y = ax + bx^{-3} + e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)x^{-3}$.

3.6 Exercises

3-1 Find the general solution to each of the following Cauchy-Euler equations:

1) $x^2y'' - 2xy' + 2y = 0$,
2) $x^2y'' - xy' + y = 0$,
3) $x^2y'' - xy' + 10y = 0$,
(Hint: Use the formula

$$x^{a+bi} = x^a x^{ib} = x^a(e^{\log x})^{ib} = x^a e^{ib\log x} = x^a[\cos(b\ln x) + \mathrm{i}\sin(b\ln x)]$$

to simplify the answer.)
4) $x^2y'' + xy' + y = 0, \quad x > 0$,
5) $2x^2y'' + 5xy' + y = 0, \quad x > 0$,
6) $9x^2y'' + 15xy' + y = 0, \quad x > 0$.

4 Differential Operators and Differential Equations

4.1 Action of Differential Operator on Elementary Functions

1. The exponential function:

$$De^{\lambda x} = \lambda e^{\lambda x}, \quad D^n e^{\lambda x} = \lambda^n e^{\lambda x}$$

In this case the function $e^{\lambda x}$ is called an eigenfunction of the operator D with eigenvalue λ.

2. The sine and cosine functions:

$$D^2 \sin(\lambda x) = -\lambda^2 \sin(\lambda x), \qquad D^2 \cos(\lambda x) = -\lambda^2 \cos(\lambda x).$$

   The functions $\sin(\lambda x)$ and $\cos(\lambda x)$ are called eigenfunctions of the operator D^2 with eigenvalue $-\lambda^2$.

3. The power functions

$$D^n x^k = k(k-1)\cdots(k-n+1)x^{k-n},$$

   where $k \in \mathbb{N}$. In particular $D^n x^k = 0$ for $k < n$.

4.2 Polynomial of the Differential Operator D

Definition 4.1.
Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n, where $a_0, a_1, \cdots, a_n$ are real constants. We define the operator

$$P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0.$$

$P(D)$ is called a polynomial of differential operator D of degree n.

Theorem 4.2.
If $P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0$, then

1. $P(D)e^{\lambda x} = P(\lambda)e^{\lambda x}$.
2. $P(D^2)\sin(\lambda x) = P(-\lambda^2)\sin(\lambda x)$ and $P(D^2)\cos(\lambda x) = P(-\lambda^2)\cos(\lambda x)$.

Theorem 4.3.
If $P(D) = a_n D^n + a_{n-1}D^{n-1} + a_1 D + \cdots + a_0$ and f a function n-times differentiable on $\mathbb{R}$. We have

1. $D^n \left(e^{\lambda x} f(x)\right) = e^{\lambda x} \left(D+\lambda\right)^n f(x)$.
2. $P(D)\left(e^{\lambda x} f(x)\right) = e^{\lambda x} P\left(D+\lambda\right) f(x)$.

Proof .
We prove the Theorem by induction.
For $n = 1$: $D\left(e^{\lambda x} f(x)\right) = \lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) = e^{\lambda x}(Df(x) + \lambda) = e^{\lambda x}(D + \lambda)f(x)$.

Assume the result holds for n, then

$$\begin{aligned} D^{n+1}\left(e^{\lambda x} f(x)\right) &= D\left(D^n\left(e^{\lambda x} f(x)\right)\right) \\ &= D\left(e^{\lambda x}(D+\lambda)^n f(x)\right) \\ &= e^{\lambda x}(D+\lambda)((D+\lambda)^n f(x)) \\ &= e^{\lambda x}(D+\lambda)^{n+1} f(x) \end{aligned}$$

The result of (2) follows from (1):

Definition 4.4.

We interpret the previous theorem as follows:

$$(D+\lambda)^n = e^{-\lambda x} D^n e^{\lambda x}.$$

the function $e^{\lambda x}$ and $e^{-\lambda x}$ are interpreted as operators.

Definition 4.5.

Let $P(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0$ be a polynomial of differential operator D of degree n.
The kernel of the polynomial $P(D)$ is the set of solutions of the linear n-th order ordinary homogeneous differential equation $P(D)y = 0$.

Theorem 4.6.

1. If $P(D) = \prod_{j=1}^{n}(D - r_j)$ where $r_1, \ldots, r_n$ are different, then the kernel of $P(D)$ is the vector space spanned by $\{e^{r_1 x}, \ldots, e^{r_n x}\}$.
   The general solution of the differential equation $P(D)y = 0$ is

   $$y = \sum_{j=1}^{n} \lambda_j e^{r_j x},$$

   where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

2. If $P(D) = \prod_{j=1}^{k}(D - r_j)^{n_j}$, where $r_1, \ldots, r_k$ are different and $\sum_{j=1}^{k} n_j = n$, then the kernel of $P(D)$ is

   $$\oplus_{j=1}^{k} \text{Vect}\left(e^{r_j x}, \ldots, x^{n_j - 1} e^{r_j x}\right),$$

   where $\text{Vect}\left(e^{r_j x}, \ldots, x^{n_j-1} e^{r_j x}\right)$ is the vector space generated by the set of functions $\left(e^{r_j x}, \ldots, x^{n_j-1} e^{r_j x}\right)\}$.
   The general solution of the differential equation $P(D)y = 0$ is

   $$y = \sum_{j=1}^{k} P_j(x) e^{r_j x},$$

   where P_j is a polynomial of degree $\deg P_j \leq n_j - 1$.

3. If $P(D) = \prod_{j=1}^{m}\left((D-r_j)^2+\theta_j^2\right)$, where $n = 2m$ and $r_j \neq r_k$ or $\theta_j \neq \theta_k$ for all $j \neq k$, $j,k = 1,\ldots,m$, then the kernel of $P(D)$ is the vector space generated by the set of functions
$$(e^{r_1 x}\sin(\theta_1 x), e^{r_1 x}\cos(\theta_1 x), \ldots, e^{r_m x}\sin(\theta_m x), e^{r_m x}\cos(\theta_m x)).$$
The general solution of the differential equation $P(D)y = 0$ is
$$y = \sum_{j=1}^{m} e^{r_j x}\left(\alpha_j \cos(\theta_j x) + \beta_j \cos(\theta_j x)\right),$$
where $\alpha_j, \beta_j \in \mathbb{R}$.

4. If $P(D) = \prod_{j=1}^{m}\left((D-r_j)^2+\theta_j^2\right)^{n_j}$, where $n = \sum_{j=1}^{m} n_j$ and $r_j \neq r_k$ or $\theta_j \neq \theta_k$ for all $j \neq k$, $j,k = 1,\ldots,m$, then the kernel of $P(D)$ is the vector space
$$\oplus_{j=1}^{m}\text{Vect}\left(e^{r_j x}\sin(\theta_j x), e^{r_j x}\cos(\theta_j x), \ldots, x^{n_j-1}e^{r_j x}\sin(\theta_j x), x^{n_j-1}e^{r_j x}\cos(\theta_j x)\right).$$
The general solution of the differential equation $P(D)y = 0$ is
$$y = \sum_{j=1}^{r} e^{r_j x}\left(P_j(x)\cos(\theta_j x) + Q_j(x)\cos(\theta_j x)\right),$$
where P_j and Q_j are polynomials of degrees $\leq n_j - 1$.

5. If $P(D) = \prod_{j=1}^{k}(D-r_j)^{n_j}\prod_{j=1}^{m}\left((D-s_j)^2+\theta_j^2\right)^{m_j}$, where $r_1, \ldots, r_k$ are different, $s_j \neq s_k$ or $\theta_j \neq \theta_k$ for all $j \neq k$, $j,k = 1,\ldots,m$ and $\sum_{j=1}^{k} n_j + \sum_{j=1}^{m} m_j = n$, then the kernel of $P(D)$ is the vector space
$$\oplus_{j=1}^{k}\text{Vect}\left(e^{r_j x}, \ldots, x^{n_j-1}e^{r_j x}\right)$$
$$\oplus_{j=1}^{m}\text{Vect}\left(e^{s_j x}\sin(\theta_j x), e^{s_j x}\cos(\theta_j x), \ldots, x^{m_j-1}e^{s_j x}\sin(\theta_j x), x^{m_j-1}e^{s_j x}\cos(\theta_j x)\right).$$
The general solution of the differential equation $P(D)y = 0$ is
$$y = \sum_{j=1}^{k} P_j(x)e^{r_j x} + \sum_{j=1}^{m} e^{s_j x}\left(Q_j(x)\cos(\theta_j x) + R_j(x)\sin(\theta_j x)\right),$$
where P_j, Q_j and R_j are polynomials such that $\deg P_j \leq n_j - 1$, $\deg Q_j \leq m_j - 1$ and $\deg R_j \leq m_j - 1$.

4.3 Non Linear Differential Equations

The general form of a non-homogeneous linear ordinary differential equation with constant coefficients takes the following form:

$$(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)y = P_n(D)y = f.$$

Our purpose is to find a particular solution to the previous differential equation.

Theorem 4.7.
If $f(x) = e^{\lambda x}$:

1. If $P(\lambda) \neq 0$, then $y_p(x) = \frac{1}{P(\lambda)}e^{\lambda x}$ is a particular solution.
2. If $P(D) = (D-\lambda)^m Q(D)$, $1 \leq m \leq n$ and $Q(\lambda) \neq 0$, then

$$y = \frac{1}{Q(\lambda)}\left(\frac{1}{m!}x^m + P_{m-1}(x)\right)e^{\lambda x}$$

   is a particular solution, where P_{m-1} is a polynomial of degree less then $m-1$.

Proof .
We have $P(D)e^{\lambda x} = P(\lambda)e^{\lambda x}$.

1. If $P(\lambda) \neq 0$, then

$$\frac{1}{P(\lambda)}\left[P(D)\left(e^{\lambda x}\right)\right] = P(D)\left(\frac{e^{\lambda x}}{P(\lambda)}\right) = e^{\lambda x}.$$

   Then $y = \frac{1}{P(\lambda)}e^{\lambda x}$ is a particular solution.

2. If $P(D) = (D-\lambda)^m Q(D)$, $1 \leq m \leq n$ and $Q(\lambda) \neq 0$, then the equation becomes $P(D)y = Q(D)(D-\lambda)^m y = e^{\lambda x}$. Then $y = \frac{1}{Q(\lambda)}(\frac{1}{m!}x^m + P_{m-1}(x))e^{\lambda x}$ is a particular solution, with P_{m-1} a polynomial of degree less then $m-1$.

   Because

$$\begin{aligned}(D-\lambda)^m Q(D)\left[(\frac{1}{m!}x^m + P_{m-1})e^{\lambda x}\right] &= Q(\lambda)(D-\lambda)^m\left(e^{\lambda x}\frac{1}{m!}x^m\right) \\ &= Q(\lambda)e^{\lambda x}D^m\left(\frac{1}{m!}x^m\right) = Q(\lambda)e^{\lambda x}\end{aligned}$$

□

Example 4.1 :

1. Particular solution to the differential equation

$$y'' - 2y' + 6y = e^{3x}.$$

$$y_p = \frac{e^{3x}}{3^2 - 2 \cdot 3 + 8} = \frac{1}{11} e^{3x}.$$

2. A particular solution to the differential equation

$$(D-1)^3(D+2)(D-2)y(x) = e^x.$$

is

$$y_p = \frac{e^x}{(1+2)(1-2)} \left(\frac{1}{3!} x^3 \right) = -\frac{x^3}{18} e^x.$$

4.4 Exercises

4-1 Find the general solution of the following differential equations:

(a) $y^{(6)} - 5y^{(4)} - 36y'' = 0$,

(b) $y^{(6)} - 2y^{(4)} + y'' = 0$,

(c) $y^{(6)} - 2y^{(4)} + y'' = e^x + \sin x$

(d) $y^{(6)} - 2y^{(4)} + y'' = e^{2x} + \cos x$.

3 Laplace Transformation and Applications

1 Basic Properties of Laplace Transform

Definition 1.1.

1. A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is said to be piecewise continuous if there is a finite numbers $a_1 = a < \ldots < a_n = b$ such that the function f is continuous on the intervals (a_j, a_{j+1}), for all $j = 1, \ldots n-1$ and $\lim\limits_{x\to a^+} = f(a^+)$, $\lim\limits_{x\to b^-} = f(b^-)$, $\lim\limits_{x\to a_j^-} = f(a_j^-)$ and $\lim\limits_{x\to a_j^+} = f(a_j^+)$ exist and finite for all $j = 2, \ldots n-1$. The set $S = \{a_1 = a, \ldots a_n = b\}$ is called a partition of the interval $[a,b]$.
2. A function $f\colon [0,+\infty) \longrightarrow \mathbb{R}$ is said to be piecewise continuous if f is piecewise continuous on any interval $[a,b] \subset [0,+\infty)$.

Definition 1.2.
Let f be a piecewise continuous function on the interval $[0,+\infty)$. The Laplace transform of f denoted by $F = \mathcal{L}(f)$ is the function defined by

$$F(s) = \int_0^\infty e^{-sx} f(x)dx = \lim_{N\to+\infty} \int_0^N e^{-sx} f(x)dx$$

if the limit exists on $\mathbb{R}$.

Definition 1.3.
A function f is said to be of exponential order if there exist constants $c, M > 0$, and $T \geq 0$ such that $|f(x)| \leq Me^{cx}$ for all $x \geq T$.

Theorem 1.4. (Sufficient Conditions for the Existence of $\mathcal{L}(f)$)
If f is piecewise continuous on $[0,+\infty)$ and of exponential order c, then for all $s > c$, $\mathcal{L}(f)(s)$ is well defined.

Theorem 1.5.
If f is piecewise continuous on $[0,+\infty)$ and of exponential order, then

$$\lim_{s\to+\infty} \mathcal{L}(f)(s) = 0.$$

Proof .
Without loss of generality, we can suppose that $|f(x)| \leq Me^{cx}$ for all $x \geq 0$.

$$\begin{aligned}
\left|\int_0^N e^{-sx} f(x)dx\right| &\leq \int_0^N e^{-sx}|f(x)|dx \\
&\leq M\int_0^N e^{-sx}e^{ax}dx = \frac{M}{s-a} - \frac{M}{s-a}e^{-(s-a)T}.
\end{aligned}$$

Then $|\mathcal{L}(f)|(s) \leq \dfrac{M}{s-a} \underset{s\to+\infty}{\longrightarrow} 0.$ □

Example 1.1 :

1. $\mathcal{L}(\sqrt{x})(s) = \displaystyle\int_0^{+\infty} \sqrt{x}e^{-sx}dx \overset{t^2=sx}{=} \frac{2}{s^{\frac{3}{2}}}\int_0^{+\infty} t^2e^{-t^2}dt = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.$
2. $\mathcal{L}(\frac{1}{\sqrt{x}})(s) = \displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x}}e^{-sx}dx \overset{t^2=sx}{=} \frac{2}{\sqrt{s}}\int_0^{+\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{\sqrt{s}}.$
3. $\mathcal{L}(e^{ax})(s) = \dfrac{1}{s-a}$, for $s > a$, but if a is a complex number, $\mathcal{L}(e^{ax})(s) = \dfrac{1}{s-a}$, for $s > \mathrm{Re}a$. For example

$$\mathcal{L}(e^{(a+\mathrm{i}b)x})(s) = \frac{1}{s-a-\mathrm{i}b}, \quad \forall s > a. \tag{3.1}$$

Theorem 1.6.
Let f and g be piecewise continuous functions on $[0,+\infty)$ and of exponential order, then $\mathcal{L}\left(af+bg\right)(s) = a\mathcal{L}\left(f\right)(s) + b\mathcal{L}\left(g\right)(s), \quad \forall a,b \in \mathbb{C}.$

Proof .
Assume that $|f(x)| \leq Me^{ax}$ for all $x \geq A$ and $|g(x)| \leq Ne^{bx}$ for all $x \geq B$, then $|af+bg|(x) \leq ||a|M+|b|N|e^{cx}$ for all $x \geq \max(A,B)$, where $c = \max(a,b)$. Then

$$\begin{aligned}
\mathcal{L}\left(af+bg\right)(s) &= \int_0^{+\infty}(af(x)+bg(x))e^{-sx}dx \\
&= a\int_0^{+\infty} f(x)e^{-sx}dx + b\int_0^{+\infty} g(x)e^{-sx}dx.
\end{aligned}$$

□

Theorem 1.7. First Shift Theorem
Let f be piecewise continuous function on $[0,+\infty)$ of exponential order. If $|f(x)| \leq Me^{bx}$ for all $x \geq A$, then

$$\mathcal{L}\left(e^{-ax}f\right)(s) = \mathcal{L}(f)(s+a), \quad \forall b \geq a. \tag{3.2}$$

Proof .
$\mathcal{L}\left(e^{-ax}f\right)(s) = \int_0^{+\infty} f(x)e^{-ax}e^{-sx}dx = \int_0^{+\infty} f(x)e^{-(s+a)x}dx.$ □

Theorem 1.8.

$$\mathcal{L}(f(bx))(s) = \frac{1}{b}\mathcal{L}(f(x))(\frac{s}{b}). \tag{3.3}$$

Proof .
$\mathcal{L}(f(bx))(s) = \int_0^{+\infty} f(bx)e^{-sx}dx \overset{t=bx}{=} \frac{1}{b}\int_0^{+\infty} f(t)e^{-\frac{s}{b}t}dt = \frac{1}{b}.$ □

Theorem 1.9. Transforms of Some Basic Functions

1. $\mathcal{L}(1) = \frac{1}{s}$.
2. $\mathcal{L}(x^n) = \frac{n!}{s^{n+1}}$, $n \in \mathbb{N}$.
3. $\mathcal{L}(\sin(ax)) = \frac{a}{s^2+a^2}$, for $s > 0$.
4. $\mathcal{L}(\cos(ax)) = \frac{s}{s^2+a^2}$, for $s > 0$.
5. $\mathcal{L}(\sinh(ax)) = \frac{a}{s^2-a^2}$, for $s > a$.
6. $\mathcal{L}(\cosh(ax)) = \frac{s}{s^2-a^2}$, for $s > a$.

Proof .

1. $\mathcal{L}(1) = \int_0^{+\infty} e^{-sx}dx = \frac{1}{s}$.
2. We can prove the formula $\mathcal{L}(x^n) = \frac{n!}{s^{n+1}}$ by induction on $n \in \mathbb{N}$. Also we can see that the function $F(s) = \int_0^{+\infty} e^{-sx}dx = \frac{1}{s}$ is differentiable and $F'(s) = -\frac{1}{s^2} = -\int_0^{+\infty} xe^{-sx}dx$. Also $F^{(n)}(s) = \frac{(-1)^n n!}{s^{n+1}} = (-1)^n \int_0^{+\infty} x^n e^{-sx}dx$.
   From the formula (3.1),
3. $\mathcal{L}(\sin(bx)) = \frac{1}{2\mathrm{i}}(\frac{1}{s-b\mathrm{i}} - \frac{1}{s+b\mathrm{i}}) = \frac{b}{s^2+b^2}$, for $s > 0$.

4. $\mathcal{L}\left(\cos(bx)\right) = \frac{1}{2}(\frac{1}{s-bi} + \frac{1}{s+bi}) = \frac{s}{s^2+b^2}$, for $s > 0$.

5. $\mathcal{L}\left(\sinh(bx)\right) = \frac{1}{2}(\frac{1}{s-b} - \frac{1}{s+b}) = \frac{b}{s^2-b^2}$, for $s > b$.

6. $\mathcal{L}\left(\cosh(bx)\right) = \frac{1}{2}(\frac{1}{s-b} + \frac{1}{s+b}) = \frac{s}{s^2-b^2}$, for $s > b$.

□

Corollary 1.10.

1. $\mathcal{L}\left(e^{ax}\sin(bx)\right) = \frac{b}{(s-a)^2+b^2}$,

2. $\mathcal{L}\left(e^{ax}\cos(x)\right) = \frac{s-a}{(s-a)^2+b^2}$.

3. $\mathcal{L}\left(x^n e^{ax}\right) = \frac{n!}{(s-a)^{n+1}}$.

Theorem 1.11.

$$\mathcal{L}\left(x^n f(x)\right)(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\left(f(x)\right)(s). \tag{3.4}$$

For $n = 1$ this formula follows from the theorem of derivative inside the integral. The general case follows by induction.

Example 1.2 :

1. If $f = 1$, $\mathcal{L}\left(x^n\right)(s) = -\frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}}$.

2. If $f(x) = \sin(x)$ and $f(x) = \cos(x)$, we get

   $\mathcal{L}\left(x\sin(bx)\right)(s) = -\frac{d}{ds}\frac{b}{s^2+b^2} = \frac{2bs}{(s^2+b^2)^2}$,

   $\mathcal{L}\left(x\cos(bx)\right)(s) = -\frac{d}{ds}\frac{s}{s^2+b^2} = \frac{s^2-b^2}{(s^2+b^2)^2} = \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}$.

Theorem 1.12.

1. If f is continuously differentiable and f, f' are of exponential order, then

$$\mathcal{L}\left(f'(x)\right)(s) = s\mathcal{L}\left(f(x)\right)(s) - f(0).$$

2. If $f \in \mathcal{C}^{n-1}$ on $[0, +\infty)$, $f^{(n)}$ is piecewise continuous on $[0, +\infty)$ and $f^{(k)}$ are of exponential order for all $0 \leq k \leq n$, then

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \tag{3.5}$$

Proof .

1. We get this property by integration by parts:

$$\begin{aligned}\mathcal{L}\left(f'(x)\right)(s) &= \int_0^{\infty} e^{-sx} f'(x)dx = e^{-sx} f(x)\big]_0^{\infty} + s\int_0^{\infty} e^{-sx} f(x)dx \\ &= s\int_0^{\infty} e^{-sx} f(x)dx - f(0).\end{aligned}$$

2. We prove the formula by induction.

Corollary 1.13.
If f is continuously differentiable on $[0, +\infty)$ and of exponential order, then

$$\mathcal{L}\left(\int_0^x f(t)dt\right)(s) = \frac{1}{s}\mathcal{L}(f)(s).$$

1.1 Exercises

1-1 Find the Laplace transforms of the following functions and for each transform give its appropriate domain:

1) $f(x) = xe^x \sin(x)$
2) $f(x) = 2x\cos^2(x)$
3) $f(x) = x^3 e^{-3x}$.
4) $f(x) = 6\sin(2x) - 5\cos(2x)$
5) $f(x) = (\sin x - \cos x)^2$
6) $f(x) = (x^2+1)^2$
7) $f(x) = e^{-4x}\cosh(2x)$.
8) $e^{-5x}\sin(4x)\cos(4x)$;
9) $6\sin(8x)\sin(2x)$;

2 Inverse Laplace Transform

Theorem 2.1.
Suppose f and g are continuous functions. If $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for all $s > a$, then $f = g$.

For the proof, we recall the Weierstrass theorem for approximation of continuous functions by sequence of polynomials.

Theorem 2.2. [Weierstrass Theorem]
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function, then there exists a sequence $(f_n)_n$ of polynomials which converges uniformly on $[a,b]$ to f.

Proof .
We assume in the first case that $a = -\frac{1}{2}, b = \frac{1}{2}$ and $f(-\frac{1}{2}) = f(\frac{1}{2}) = 0$. In this case the function f can be extended to a continuous function on $\mathbb{R}$.
For all $n \in \mathbb{N}$, define $P_n(x) = c_n(1-x^2)^n$ and

$$f_n(x) = \int_{-\infty}^{+\infty} f(y)P_n(x-y)dy = \int_{-\infty}^{+\infty} f(x-y)P_n(y)dy, \tag{3.6}$$

where c_n is chosen such that $\displaystyle\int_{-1}^{1} P_n(x)dx = 1$.

Lemma 2.3.
The functions f_n defined by (3.6) are polynomials and the sequence $(f_n)_n$ converge uniformly to f on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Proof .
From the left side of the formula (3.6), f_n is a polynomial. From the right side of the formula (3.6), we have for $|x| \leq \frac{1}{2}$

$$f(x) - f_n(x) = \int_{-1}^{1} f(x-y)P_n(y)dy \tag{3.7}$$

Let $\varepsilon > 0$, M the maximum of f on $\mathbb{R}$ and $\delta > 0$ such that $|f(x) - f(x-y)| < \varepsilon$ if $|y| < \delta$. It follows from (3.7) that

$$|f(x) - f_n(x)| \leq \int_{|y|<\delta} \varepsilon P_n(y)dy + \int_{\delta \leq |y| \leq 1} M\ P_n(y)dy.$$

We claim that $\displaystyle\lim_{n\to+\infty} \int_{\delta \leq |y| \leq 1} P_n(y)dy = 0$.

If $0 < r < 1$, $\displaystyle\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n dx \geq \int_{-r}^{r} (1-r^2)^n dx = 2r(1-r^2)^n$. Thus $c_n \leq \dfrac{1}{2r(1-r^2)^n}$ and

$$\int_{\delta \leq |y| \leq 1} P_n(y)dy \leq \frac{1}{2r(1-r^2)^n} \int_{-1}^{1} (1-\delta^2)^n dy = \frac{(1-\delta^2)^n}{r(1-r^2)^n}.$$

The result follows by taking $r < \delta$ and tends n to infinity.

Proof of the theorem
If f is zero outside the interval $[-s, s]$, the function $F(x) = f(2sx)$ is zero outside the interval $[-\frac{1}{2}, \frac{1}{2}]$. From the previous lemma there exists a sequence $(f_n)_n$ of polynomials which converges uniformly to F on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The sequence of polynomials $g_n(x) = f_n(\frac{x}{2s})$ converges uniformly to f on the interval $[-s, s]$.

If f is continuous on the interval $[a,b]$. Consider the function $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$, $g(a) = f(a) = g(b)$ and the function $h(x) = \dfrac{b-a}{2s}x + \dfrac{a+b}{2}$ a bijection between $[-s,s]$ and $[a,b]$.
The function $g \circ h$ is a uniform limit of sequence of polynomials. Then f also.
□

Corollary 2.4.

If f is a continuous function on an interval $[a,b]$ and $\int_a^b f(x)x^n = 0$ for all $n \in \mathbb{N}$, then $f = 0$.

Proof .
There exists a sequence of polynomials $(P_n)_n$ which converges uniformly to f on the interval $[a,b]$. Then $\int_a^b f(x)P_n(x)dx = 0$ for all $n \in \mathbb{N}$. This implies that $\int_a^b f^2(x)dx = 0$. As f is continuous, $f = 0$.

□

Proof of theorem 2.1.
By linearity it is enough to prove that if $\mathcal{L}(f) = 0$ then $f = 0$.
For $s = a + n + 1$,

$$\int_0^{+\infty} e^{-t(s-a)}e^{-ta}f(t)dt \overset{x=e^{-t}}{=} \int_0^1 x^n\left(x^a f(-\ln x)\right)dx = 0.$$

Then $f = 0$.

Definition 2.5.
If F is the Laplace transform of a piecewise continuous function f, then f is called the **inverse Laplace transform** of F and denoted by

$$F = \mathcal{L}^{-1}\left(f\right).$$

The inverse Laplace transform is also linear. We have for example

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{1}{2}x\sin(x), \quad \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{1}{2}\sin(x) - \frac{1}{2}x\cos(x).$$

Theorem 2.6.
Some Inverse Transforms

1. $\mathcal{L}^{-1}(\frac{1}{s}) = 1.$
2. $\mathcal{L}^{-1}(\frac{n!}{s^{n+1}}) = t^n,\ n \in \mathbb{N}.$
3. $\mathcal{L}^{-1}(\frac{1}{s-a}) = e^{at},\ n \in \mathbb{N}.$
4. $\mathcal{L}^{-1}(\frac{k}{s^2+n^2}) = \sin(nt).$
5. $\mathcal{L}^{-1}(\frac{s}{s^2+n^2}) = \cos(nt).$
6. $\mathcal{L}^{-1}(\frac{k}{s^2-n^2}) =.$
7. $\mathcal{L}^{-1}(\frac{s}{s^2-n^2}) = \cosh(nt).$

Example 2.1 :
$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \int_0^x \sin t dt = 1 - \cos x.$

2.1 Exercises

2-1 Find the inverse Laplace transform of the following functions.

1) $\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$
2) $f(s) = \frac{s}{(s-3)^3}$
3) $f(s) = \frac{s+1}{s^2+2s+10}$
4) $f(s) = \frac{s^2+4s-15}{(s-1)(s^2+9)}$
5) $f(s) = \frac{4}{(s-3)^3}$
6) $f(s) = \frac{2s-3}{s^2+2s+10}$
7) $f(s) = \frac{9s^2-12s+28}{s(s^2+4)}$
8) $f(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$
9) $f(s) = \frac{s}{(s+1)(s^2+4)}$
10) $f(s) = \frac{3s}{(s^2+9)^2}$
11) $f(s) = \left(\frac{e^{-2s}}{s^2+12s+32}\right)$
12) $f(s) = \left(\frac{s}{s^2+6s+13}\right)$
13) $F(s) = e^{-s}\frac{7-3s}{s^2-8s+20}$
14) $F(s) = \frac{s^2+5}{(s-2)^3}$
15) $F(s) = e^{-3s}\frac{s}{s^4-16}$
16) $F(s) = e^{-\pi s}\frac{1}{s^4+16s^2}$

3 The Heaviside's Unit Step Function

Definition 3.1.
The Unit Step Function is defined to be $H(x-a) = \begin{cases} 0 & 0 \le x \le a \\ 1 & x \ge a \end{cases}.$
For $a = 0$, this function is called also the Heaviside function.

Consider the function $f(x) = \sin x$ for $x \geq \pi$ and 0 otherwise. The expression of f in term of the Heaviside function is: $f(x) = H(x-\pi)\sin x$.
Also if f is the function defined by $f(x) = e^x$ for $x \in [1,2)$. Then $f(x) = e^x\,(H(x-1) - H(x-2))$.

Theorem 3.2. Second Shift Theorem
If f is piecewise continuous on $[0,+\infty)$ and of exponential order, then for $a \geq 0$

$$\mathcal{L}(f(x-a)H(x-a)) = e^{-as}\mathcal{L}(f)(s). \tag{3.8}$$

In particular, $\mathcal{L}\left(H(x-a)\right) = \dfrac{e^{-as}}{s}$.

Proof .

$$\begin{aligned}
\mathcal{L}(f(x-a)H(x-a)) &= \int_0^{+\infty} e^{-sx} f((x-a)H(x-a))dx \\
&= \int_a^{+\infty} e^{-sx} f(x-a)dx \\
&= e^{-as}\int_0^{+\infty} e^{-sx} f(x)dx = e^{-as}F(s).
\end{aligned}$$

□

Example 3.1 :
Using the Heaviside function write down the piecewise function that is 0 for $x < 0$, x^2 for x in $[0,1]$ and x for $x > 1$.
$f(x) = xH(x-1) + x^2(H(x) - H(x-1))$.

Example 3.2 :
$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s-3}\right) = e^{3(x-2)}H(x-3)$.

3.1 Exercises

3-1 Find the Laplace transforms of the following functions:

$$1)\ \ f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ e^{-2x}, & x > 1 \end{cases}$$

$$2)\ \ f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 5-x, & 1 < x \leq 2 \\ 6, & 2 < x \end{cases}$$

$$3)\ \ f(x) = \begin{cases} 0, & x < 3 \\ x^2 - 6x + 18, & x \geq 3 \end{cases}$$

4) $f(x) = x^2 H(x-3)$.
5) $H(x-5)xe^{-6x}$;
6) $H(x-\frac{\pi}{4})\cos(2x)$;
7) $H(x-3)(x^2-x+4)$.
8) $f(x) = (2x-2)(H(x)-H(x-3)) + (10-2x)(H(x-3)-H(x-6))$

4 Solutions of Linear Differential Equations

The previous results will be useful to find the Laplace transform for the functions that are annihilated by a differential operator of constant coefficients.

We use the identity (3.5) to solve linear ordinary differential equations. If

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = g(t)$$

$y(0) = y_0, y^{'}(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}$, where a_j, y_j are constants, for $0 \le j \le n-1$. By the linearity property, the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n \mathcal{L}(y^{(n)}) + a_{n-1}\mathcal{L}(y^{(n-1)}) + \ldots + a_0\mathcal{L}(y) = \mathcal{L}(g(t))$$

Example 4.1 :
We wish to solve for $y(x)$ the following equation $x^2 = \int_0^x e^t y(t)dt$. We apply the Laplace transform, we get

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}(e^t y(x)) = \frac{1}{s}Y(s-1),$$

where $Y(s) = \mathcal{L}(y(x))$. Thus $Y(s-1) = \frac{2}{s^2}$ or $Y(s) = \frac{2}{(s+1)^2}$. We use the shifting property again $y(x) = 2e^{-x}x$.

Example 4.2 :
Given $y^{''} + a^2 y = 1$, with $y(0) = 0, y'(0) = 0,$.

We take the Laplace transform of the equation, we get $s^2Y(s) + a^2Y(s) = F(s) = \frac{1}{s}$. Then $Y(s) = \frac{1}{s^2+a^2}\frac{1}{s}$. Taking the inverse Laplace transform of $Y(s)$ we obtain

$$y(x) = \frac{1-\cos(ax)}{a^2}.$$

Example 4.3 :

1. Let $f(x) = \sin x$, $f^{''} + f = 0$. Then

$$\mathcal{L}\left(f^{''}(x) - f(x)\right)(s) = s^2\mathcal{L}\left(f(x)\right)(s) - 1 - \mathcal{L}\left(f(x)\right)(s).$$

   We get $\mathcal{L}\left(\sin x\right)(s) = \dfrac{1}{1+s^2}$.

2. Let y be a function such that $y^{''} - 3y' + 2y = e^x$, with $y(0) = 1$ and $y'(0) = -1$. If $Y = \mathcal{L}(y)$, then

$$s^2Y - (s-1) - 3(sY-1) + 2Y = \frac{1}{s-1}$$

   and

$$\begin{aligned} Y &= \frac{1}{(s-1)^2(s-2)} + \frac{s-4}{(s-1)(s-2)} = \frac{2}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{s-2} \\ &= \mathcal{L}(2e^x - xe^x - e^{2x}). \end{aligned}$$

Example 4.4 :
Consider the following initial value problem

$$y^{''} + 7y^{'} + 6y = 2H(t - \frac{\pi}{2})\sin^2 t, \quad y(0) = 2, y^{'}(0) = 0.$$

(This equation can be interpreted as follows: If as mass of 1kg attached to a spring with spring constant $k = 6$ such that the system has damping constant $c = 2$. Assume that the mass is displaced $2m$ from equilibrium and released at $t = 0$. Futhermore at time $t = \dfrac{\pi}{2}$, the forcing function $f(t) = 2\sin^2 t$ is applied. $y(t)$ is the displacement of the mass from equilibrium.

Taking Laplace transforms, we get

$$s^2Y(s) - sy(0) - y^{'}(0) + 7(sY(s) - y(0)) + 6Y(s) = e^{-s\frac{\pi}{2}}(1 + \frac{1}{s}).$$

Then $(s^2 + 7s + 6)Y(s) = 2s + 14 + e^{-s\frac{\pi}{2}}(1 + \frac{1}{s})$ and

$$Y(s) = \frac{12}{5(s+1)} - \frac{2}{5(s+6)} + \frac{e^{-s\frac{\pi}{2}}}{6s} - \frac{e^{-s\frac{\pi}{2}}}{6(s+6)}.$$

Hence

$$y = \frac{1}{2}5e^{-t} - \frac{2}{5}e^{-6t} + \frac{1}{6}H(t - \frac{\pi}{2}) - \frac{1}{6}H(t - \frac{\pi}{2}).$$

Example 4.5 :
Consider the differential equation

$$y' + 3y = 13\sin(2t), \quad y(0) = 6.$$

We take the transform of each member of the differential equation: $\mathcal{L}(y') + 3\mathcal{L}(y) = 13\mathcal{L}(\sin(2t))$. Then $sF(s) - 6 + 3F(s) = 6 + \dfrac{26}{s^2+4}$. Then $F(s) = \dfrac{6}{s+3} + \dfrac{26}{(s+3)(s^2+4)} = \dfrac{8}{s+3} + \dfrac{-2s+6}{s^2+4}$ and $y = 6\mathcal{L}^{-1}(\dfrac{1}{s+3}) - 2\mathcal{L}^{-1}(\dfrac{s}{s^2+4}) + 6\mathcal{L}^{-1}(\dfrac{1}{s^2+4}) = 8e^{-3t} - 2\cos(2t) + 3\sin(2t)$.

Example 4.6 :
Consider the differential equation

$$y^{''} - 3y' + 2y = e^{-4t}, \quad y(0) = 1, y'(0) = 5.$$

We take the transform of each member of the differential equation: $\mathcal{L}(y^{''}) - 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^{-4t})$.

Then $F(s) = \dfrac{s^2+6s+9}{(s-1)(s-2)(s+4}$ and $y = -\dfrac{16}{5}e^t + \dfrac{25}{5}e^{2t} + \dfrac{1}{30}e^{-4t}$.

Example 4.7 :
Consider the initial value problem

$$y'' + y' + y = \sin(x), \quad y(0) = 1, \ y'(0) = -1.$$

Let $Y(s) = \mathcal{L}\left(y(x)\right)$, we have

$$\mathcal{L}\left(y'\right) = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\left(y''\right) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Taking Laplace transforms of the differential equation, we get

$$(s^2+s+1)Y(s) - s = \frac{1}{s^2+1}.$$

Then

$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Finding the inverse Laplace transform.

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\left(Y(s)\right) \\ &= \mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s^2+s+1)(s^2+1)}\right). \end{aligned}$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

we have

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) = e^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) - \frac{1}{\sqrt{3}}e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{s+1}{s^2+s+1} - \frac{s}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+s+1)(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right).$$

Since

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right) = \frac{2}{\sqrt{3}}e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x), \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(x)$$

we obtain

$$y(x) = 2e^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) - \cos(x).$$

Example 4.8 :
Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Taking Laplace transforms of the differential equation, we get $(s^2+s+1)Y(s) - s = \frac{1}{s^2+1}$, and $Y(s) = \dfrac{s}{s^2+s+1} + \dfrac{1}{(s^2+s+1)(s^2+1)}$.

$$y(x) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+\frac{1}{2}} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2} - \frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}\ x) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}\ x).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

We get $A = B = 1$, $C = -1$, $D = 0$, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}\ x), \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(x)$$

we obtain

$$y(x) = 2e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}\ x) - \cos(x).$$

Example 4.9 :
Consider the differential equation

$$y^{''} - 6y' + 9y = x^2e^{3x}, \quad y(0) = 2,\ y'(0) = 17.$$

We take the transform of each member of the differential equation:
$F(s) = \dfrac{2s+5}{(s-3)^2} + \dfrac{2}{(s-3)^5}.$

$$\begin{aligned}
\mathcal{L}^{-1}(\frac{2s+5}{(s-3)^3}) &= \mathcal{L}^{-1}(\frac{2(s-3)+11}{(s-3)^3}) \\
&= 2\mathcal{L}^{-1}(\frac{1}{(s-3)^2}) + 11\mathcal{L}^{-1}(\frac{1}{(s-3)^3}) \\
&= 2e^{3x} + 11xe^{3x}.
\end{aligned}$$

$y = 2e^{3x} + 11xe^{3x}\frac{1}{12}x^4e^{3x}.$

Example 4.10 :

Consider the differential equation

$$y''(x) + y(x) = f(x), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f(x) = 1$ if $x \in [1, 2)$ and zero otherwise.

The function $f(x) = H(x-1) - H(x-2)$. Taking the Laplace transform, we get:

$$s^2Y(s) + Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Solving this equation, we obtain

$$Y(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)}.$$

Then

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos x$. Then using Lemma (**??**) and Theorem (3.2), we get $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(1-\cos x)\right) = (1-\cos(x-1))\,H(x-1)$.

Similarly $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(e^{-2s}\mathcal{L}(1-\cos x)\right) = \big(1-\cos(x-2)\big)H(x-2)$. Hence, the solution is

$y(x) = \big(1-\cos(x-1)\big)\,H(t-1) - \big(1-\cos(x-2)\big)\,H(t-2).$

Example 4.11 :

Solve the following differential equation: $y' - 2y = f(x)$, with $y(0) = 3$, $f(x) = 3\cos x$ for $x \geq 1$ and $f(x) = 0$, for $0 \leq x \leq 1$.

$\mathcal{L}\left(f(x)\right) = -\frac{3s}{s^2+1}e^{-s}$. Then $sF(s) - 3 - 2F(s) = -\frac{3s}{s^2+1}e^{-s}$ and

$F(s) = \frac{1}{s-2}\left(3 - \frac{3s}{s^2+1}e^{-s}\right) = \frac{3}{s-2} + \frac{6}{5}\frac{s}{s^2+1}e^{-s} - \frac{3}{5}\frac{1}{s^2+1}e^{-s}.$

$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3e^{2x},$

$\mathcal{L}^{-1}\left(\frac{6}{5}\frac{s}{s^2+1}e^{-s}\right) = \frac{6}{5}\cos(x-1)H(x-1),$

$\mathcal{L}^{-1}\left(\frac{3}{5}\frac{1}{s^2+1}e^{-s}\right) = \frac{3}{5}\sin(x-1)H(x-1).$

$$y(x) = 3e^{2x} + \frac{6}{5}\cos(x-1)H(x-1) - \frac{3}{5}\sin(x-1)H(x-1).$$

4.1 Exercises

4-1 Solve the following differential equations with and without using Laplace transform

1) $y^{'} + 3y = e^{2x}$, $y(0) = -2$
2) $y^{'} + 4y = \sin(3x)$, $y(0) = 5$
3) $y^{'} + y = xe^{x}$, $y(0) = -1$
4) $y^{''} + 9y = 2$, $y(0) = 0$, $y^{'}(0) = 1$
5) $y^{''} + 9y = 5\cos(3x)$, $y(0) = 0$, $y^{'}(0) = 0$
6) $y^{''} + 2y^{'} + y = 3x$, $y(0) = 0$, $y^{'}(0) = 0$

4-2 Using Laplace transforms, solve the following differential equations.

1) $y^{'} + y = e^{-x} + e^{x} + \cos x + \sin x$, $y(0) = 1$
2) $y^{'} - 2y = 5 + \cos x + e^{2x} + e^{-x}$, $y(0) = 4$
3) $y^{'} + y = 5H(x-1) + e^{x}H(x-1) + H(x-1)\cos x$, $y(0) = 2$
4) $y^{'} + 5y = 20, \quad y(0) = 3$
5) $y^{'} + 3y = e^{2x}, \quad y(0) = -2$
6) $y^{'} - y = xH(x-3), \quad y(0) = -4$
7) $y^{''} + 9y = 0, \quad y(0) = 0, \quad y^{'}(0) = 5$
8) $y^{''} - 9y = 0, \quad y(0) = 2, \quad y^{'}(0) = 0$
9) $y^{''} + 9y = 2, \quad y(0) = 0, \quad y^{'}(0) = 1$
10) $y^{''} + 9y = 5\cos x, \quad y(0) = 0, \quad y^{'}(0) = 0$
11) $y^{''} + 4y = \sin(2x), \quad y(0) = 0, \quad y^{'}(0) = 1$
12) $y^{''} + 9y = \cos(2x) + x\cos(2x), \quad y(0) = 0, \quad y^{'}(0) = 1$
13) $y^{''} + 2y^{'} + 5y = e^{-x}\sin(2x), \quad y(0) = 0, \quad y^{'}(0) = 1$
14) $y^{''} + 2y^{'} + 5y = H(x-4), \quad y(0) = 1, \quad y^{'}(0) = 0$
15) $y^{''} - 2y^{'} - 3y = H(x-3), \quad y(0) = 2, \quad y^{'}(0) = 0$
16) $y^{''} + 2y^{'} + 5y = e^{-x}\sin(2x) + H(x-\pi)e^{-x}\cos(2x), \quad y(0) = 0, \quad y^{'}(0) = 1$
17) $y^{''} + y^{'} - 2y = 4e^{x} + H(x-3), \quad y(0) = 1, \quad y^{'}(0) = 0$
18) $y^{''} + 2y^{'} + 5y = H(x-2), \quad y(0) = 1, y^{'}(0) = 0$
19) $y^{''} - 2y^{'} - 3y = 0$, $y(0) = 4$, $y^{'}(0) = 0$
20) $y^{''} + 4y^{'} + 13y = 2H(x-\pi)\sin(3x), \quad y(0) = 1, y^{'}(0) = 0$

4-3 Use the Laplace transform to solve the initial-value problems

1) $y^{'} + y = e^{-x} + e^{x} + \cos x + \sin x, \quad y(0) = 1;$
2) $y^{'} - 2y = 5 + \cos x + e^{2x} + e^{-x}, \quad y(0) = 4;$
3) $y" - 2y^{'} + 2y = \cos x, \quad y(0) = 1, y^{'}(0) = -1;$
4) $y^{'} + y = 5H(x-1) + e^{x}H(x-1) + H(x-1)\cos x, \quad y(0) = 2.$
5) $y" + 2y^{'} + 5y = H(x-2), \quad y(0) = 1, y^{'}(0) = 0;$
6) $y' + 3y = 13\sin(2x), \quad y(0) = 6.$

4-4 Use the Laplace transform to solve the initial-value problems

1) $y^{'} + y = e^{-x} + e^{x} + \cos x + \sin x, \quad y(0) = 1;$
2) $y^{'} - 2y = 5 + \cos x + e^{2x} + e^{-x}, \quad y(0) = 4;$
3) $y^{"} - 2y^{'} + 2y = \cos x, \quad y(0) = 1, y^{'}(0) = -1;$
4) $y^{'} + y = 5H(x-1) + e^{x}H(x-1) + H(x-1)\cos x, \quad y(0) = 2.$
5) $y^{"} + 2y^{'} + 5y = H(x-2), \quad y(0) = 1, y^{'}(0) = 0;$
6) $y' + 3y = 13\sin(2x), \quad y(0) = 6.$
7) $y^{''} - 3y' + 2y = e^{-4x}, \quad y(0) = 1, y'(0) = 5,$
8) $y^{''} - 6y' + 9y = x^2 e^{3x}, \quad y(0) = 2,\ y'(0) = 17.$
9) $y' - 2y = f(x)$, with $y(0) = 3,\ f(x) = \begin{cases} 3\cos x & x \geq 1 \\ 0 & 0 \leq x \leq 1 \end{cases}$
10) $y'' + y' + y = \sin(x), \quad y(0) = 1,\ y'(0) = -1.$

4-5 Using the Laplace transform to solve the following differential equations

(a) $y^{''} + 3y^{'} + 2y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$
(b) $y^{''} + y^{'} + y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$
(c) $y^{''} + 2y^{'} + y = 0, \quad y(0) = a, \quad y^{'}(0) = b,$
(d) $y^{''} + y = H(x-1)$ for initial conditions $y(0) = 0$ and $y^{'}(0) = 0,$
(e) $y^{(3)} + y = x^3 H(x-1)$ for initial conditions $y(0) = 1$ and $y^{'}(0) = 0,$ $y^{''}(0) = 0,$
(f) $y^{''} - y = (x^2 - 1)H(x-1)$ for initial conditions $y(0) = 1,\ y^{'}(0) = 2$
(g) $y'' + 3y' + 2y = -5\sin(x) + 5\cos(x), \quad y(0) = 5, \quad y'(0) = -3.$
(h) $y'' + y = 4H(x-\pi)$, with initial conditions $y(0) = 2,\ y'(0) = 4.$
(i) $y'' - 5y' - 6y = x^2 + 7, \qquad y(0) = 1,\ y'(0) = 0$
(j) $y^{''} - 2y^{'} + 2y = \cos x, \quad y(0) = 1, y^{'}(0) = -1$

(k) $y^{'} + y = 5H(x-1) + e^x H(x-1) + H(x-1)\cos x, \quad y(0) = 2$

(l) $y^{''} + 2y^{'} + 5y = H(x-2), \quad y(0) = 1, y^{'}(0) = 0$

(m) $y' - 2y = f(x)$, with $y(0) = 3$, $f(x) = 3\cos x$ for $x \geq 1$ and $f(x) = 0$, for $0 \leq x \leq 1$.

(n) $y^{''} + 2y^{'} + 5y = H(x-2), \quad y(0) = 1, y^{'}(0) = 0$

4 Systems of Linear First-Order Linear Differential Equations

Introduction

In this chapter we give an introduction to systems of linear first-order differential equations. The general form of such system is

$$\frac{d}{dt}X(t) = A(t)X(t) + F(t), \tag{4.1}$$

where $X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$ and

$F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$.

If $F = 0$, the system is called homogeneous, otherwise it is non-homogeneous.

Theorem 0.1.
The maximal solutions of the linear system (4.1) are global.

In which follows, we consider only the maximal solutions of the system and then we do not precise the interval of the solutions.

Example 0.1 :

Consider the system $X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} t \\ e^t \end{pmatrix}$.
This system is equivalent to

$$\begin{cases} x'(t) &= x(t) + 2y(t) + t \\ y'(t) &= -y(t) + e^t \end{cases}$$

Then $y(t) = ae^{-t} + \frac{1}{2}e^t$, with $a \in \mathbb{R}$. Moreover $x'(t) = x(t) + ae^{-t} + \frac{1}{2}e^t$. Then $x(t) = (\frac{1}{2}t + b)e^t + \frac{a}{2}e^{-t}$, with $b \in \mathbb{R}$.

Theorem 0.2. [Existence and Uniqueness Solution]
If the matrices $A(t)$ and $F(t)$ are continuous on an open interval I. Then for all $t_0 \in I$, there exists a unique solution of the initial value problem $X'(t) = A(t)X(t) + F(t)$ and $X(t_0) = X_0$ on the interval I.

Example 0.2 :
Consider a linear differential equation of second order:

$$y''(t) + 2y'(t) - 3y(t) = e^t + \cos t.$$

This differential equation can be converted into a system of first order differential equations by letting $Y = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then $Y' = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} Y + \begin{pmatrix} 0 \\ e^t + \cos t \end{pmatrix}$.

Example 0.3 :
The equation $y''(t) - 3y'(t) + 2y(t) = \sin t$ can be written as system of first order equations by letting $Y = \begin{pmatrix} y \\ y' \end{pmatrix}$ Then $Y' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} Y + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$.

$y_1 = y$ and $y_2 = y'$. Thus $\begin{cases} y_1' &= y_2 \\ y_2' &= -2y_1 + 3y_2 + \sin t \end{cases}$.

1 Homogeneous Systems of Linear Differential Equations

1.1 Superposition Principle

Theorem 1.1. [Superposition Principle]
If $X_1, \ldots, X_m$ are solutions of the linear differential system $X'(t) = A(t)X(t)$ on an interval I, then $a_1X_1 + \ldots + a_mX_m$ is also solution of the linear differential system on the interval I.

1.2 Linear Dependence and Linear Independence

Definition 1.2.
Let $\{X_1, \ldots, X_m\}$ be a set of vectors solutions of the linear system $X'(t) = A(t)X(t)$ on an interval I. This set is called linearly dependent, if there exist constants $c_1, \ldots, c_m$, not all zero, such that

$$c_1 X_1(t) + \ldots + c_m X_m(t) = 0$$

for all $t \in I$. Otherwise, the set is called linearly independent.

Theorem 1.3.
Let $X_1(t) = \begin{pmatrix} x_{1,1}(t) \\ \vdots \\ x_{1,n}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{n,1}(t) \\ \vdots \\ x_{n,n}(t) \end{pmatrix}$ be n solutions of the linear system $X'(t) = A(t)X(t)$ on an interval I. Then the set $\{X_1, \ldots, X_n\}$ of solution vectors is linearly independent on I if and only if the Wronskian

$$W(X_1, \ldots, X_n) = \begin{vmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{vmatrix} \neq 0,$$

for all $t \in I$.

Proof .

□

Remark 10 :
If $X_1(t) = \begin{pmatrix} x_{1,1}(t) \\ \vdots \\ x_{1,n}(t) \end{pmatrix}, \ldots, X_n(t) = \begin{pmatrix} x_{n,1}(t) \\ \vdots \\ x_{n,n}(t) \end{pmatrix}$ are n solutions of a linear differential system $X'(t) = A(t)X(t)$ on an interval I. Then the set $X_1, \ldots, X_n$ is linearly independent on I if and only if there exists $t_0 \in I$ such that the Wronskian $W(X_1(t_0), \ldots, X_n(t_0)) \neq 0$.

Definition 1.4. [Fundamental Set of Solutions]
Any set of n linearly independent solution vectors of the homogeneous differential system $X'(t) = A(t)X(t)$ on an interval I is called a fundamental set of solutions of the system on the interval I.

Theorem 1.5. [Existence of a Fundamental Set]
Let A be a continuous matrix function of order n on an interval I and consider the linear differential system of differential equations $X'(t) = A(t)X(t)$. Then there exists $S = \{X_1, \ldots, X_n\}$ a fundamental set of solutions of the differential system. Moreover if X is a solution of the system, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $X = c_1 X_1 + \ldots + c_n X_n$.

1.3 Non-Homogeneous Systems of Linear Differential Equations

Theorem 1.6. Existence of Solutions
If the vector-valued functions $A(t)$ and $B(t)$ are continuous over an open interval I that contains t_0, then the initial value problem

$$\begin{cases} X'(t) & = & A(t)X(t) + B(t), \\ X(t_0) & = & X_0 \end{cases}$$

has an unique vector-values solution $X(t)$ that is defined on entire interval I for any given initial value X_0 at t_0.

Corollary 1.7.
If X_p is a particular solution of the non-homogeneous system $X'(t) = A(t)X(t) + F(t)$ on an interval I and $X_1, \ldots, X_n$ is a fundamental set of solutions of the homogeneous differential system, then if X is a solution of the non-homogeneous differential system on the interval I, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $X = c_1X_1 + \ldots + c_nX_n + X_p$.

1.4 Exercises

1-1 Express the system $\begin{cases} x_1' & = & t^2x_1 + 3tx_2 + \sin t \\ x_2' & = & (\sin t)x_1 + t^2x_2 + \cos t \end{cases}$
in the matrix form $X'(t) = A(t)X(t) + F(t)$.

1-2 Transform the following differential equations into a system of differential equations of first order

(a) $y^{(3)} - ty' + y\sin(2t) = e^t$.
(b) $y^{''} + 2y' + 3y = \sin t$.
(c) $y^{''} - 2y' + ty = t$.
(d) $y^{''} - 3y' + ty = 3t^2$, $y(0) = 0, y'(0) = 1$.

1-3 Consider the following system of first-order linear differential equations.
$X' = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} X$.
Find the second-order linear differential equation that y satisfies.

1-4 Determine if the following functions are linearly independent $y_1 = e^t$, $y_2 = \sin t$, $y_3 = \sin(2t)$ on $\mathbb{R}$.

2 Homogeneous Linear Systems with Constant Coefficients

In this section we limit ourselves to systems of constant real square matrices of order 2.

2.1 Eigenvalues and Eigenvectors

Definition 2.1.
A real or complex number λ is called an eigenvalue of a matrix A if there exists a nonzero vector X such that $AX = \lambda X$. The vector X is called an eigenvector of the matrix A with respect to the eigenvalue λ.

If X is an eigenvector of a matrix A with respect to the eigenvalue λ, then $(A - \lambda I)X = 0$. This will occur exactly when the determinant of $(A - \lambda I)$ is zero. The polynomial $q_A(t) = \det(A - tI)$ is called the characteristic polynomial of A.

Theorem 2.2.
The roots of the characteristic polynomial of the matrix A are the eigenvalues of the matrix A.

2.2 Changing Coordinates

We consider a linear system $X'(t) = AX(t)$ and the change of coordinates $X = TY$, where T is an invertible matrix. Then X is a solution of the system $X'(t) = AX(t)$ if and only if Y is a solution of the linear system system $Y'(t) = T^{-1}ATY(t)$. Indeed: $Y' = T^{-1}X' = T^{-1}AX = T^{-1}ATY$.

We have three cases of type of eigenvalues of the matrix:

2.3 Distinct Real Eigenvalues

Theorem 2.3.
If the discriminant $\Delta_{q_A} = (a+d)^2 - 4bc > 0$, there is two different reals numbers λ_1 and λ_2 such that $q_A(\lambda_1) = q_A(\lambda_2) = 0$ and there exist two eigenvectors X_1 and X_2 with respect to λ_1 and λ_2 respectively such that:
X_1 and X_2 are linearly independent and there exists an invertible matrix P such that $P^{-1}AP = D$, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. (We say that the matrix A is diagonalizable.)

Proof .
If $aX_1 + bX_2 = 0$, by applying A, we get $a\lambda_1 X_1 + b\lambda_2 X_2 = 0$. Hence $a(\lambda_1 - \lambda_2)X_1 = 0$. As $X_1 \neq 0$ and $\lambda_1 \neq \lambda_2$, then $a = 0$ and $b = 0$.

Consider the matrix P defined by: X_1 is its first column and X_2 is its second column. The matrix P is invertible and AP is the matrix with first column $\lambda_1 X_1$ and second column $\lambda_2 X_2$. Then $P^{-1}AP = D$. $\square$

Theorem 2.4.
Consider a linear system $X'(t) = AX(t)$, such that the matrix A has two different eigenvalues λ_1 and λ_2. If X_1, X_2 are two corresponding eigenvectors of A with respect to λ_1 and λ_2 respectively. Then The set of solutions of the system $X' = AX$ is

$$\{X = ae^{\lambda_1 t}X_1 + be^{\lambda_2 t}X_2, \quad a, b \in \mathbb{R}\}.$$

Proof .
There is an invertible matrix P such that $P^{-1}AP = D$, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$.
Consider the vector $Y = P^{-1}X$. We have $Y' = P^{-1}X' = P^{-1}APP^{-1}X = DY$. If $Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, then $y_1'(t) = \lambda_1 y_1(t)$ and $y_2'(t) = \lambda_2 y_2(t)$ and $y_1(t) = ae^{\lambda_1 t}$, $y_2(t) = be^{\lambda_2 t}$. Hence $X(t) = PY(t) = ae^{\lambda_1 t}X_1 + be^{\lambda_2 t}X_2$. $\square$

Example 2.1 :
Consider the system $X' = AX$, with $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.
$q_A(\lambda) = (\lambda + 1)(\lambda - 2)$. The vector $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A with respect to $\lambda = -1$ and the vector $X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of A with respect to $\lambda = -2$. The solution of the system is

$$X = ae^{-t}X_1 + be^{2t}X_2, \quad a, b \in \mathbb{R}.$$

Example 2.2 :
Consider the system of linear differential equation $X' = AX$, with $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$. The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} -10 - \lambda & -6 \\ 18 & 11 - \lambda \end{vmatrix} = (\lambda - 2)(1 + \lambda).$$

Then the matrix is diagonalizable. The vector $X_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix A relative to the eigenvalue $\lambda = -1$ and the vector $X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

is an eigenvector of the matrix A relative to the eigenvalue $\lambda = 2$. Then the solution of the linear system is $X = -3e^{-t}X_1 - 5e^{2t}X_2 = \begin{pmatrix} 6e^{-t} - 5e^{2t} \\ -9e^{-t} + 10e^{2t} \end{pmatrix}$.

Example 2.3 :

Consider the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. $\lambda_1 = 1$, $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 3$, $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
The general solution of the system of linear differential equation $X' = AX$ is

$$X = ae^{t}X_1 + be^{3t}X_2 = \begin{pmatrix} ae^{t} + be^{3t} \\ ae^{t} - be^{3t} \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

If $X_0 = X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then $a = 2$, $b = -\frac{1}{2}$ and

$$X = \begin{pmatrix} 2e^{t} - \frac{1}{2}e^{3t} \\ 2e^{t} + \frac{1}{2}e^{3t} \end{pmatrix}$$

2.4 Repeated Eigenvalues

Theorem 2.5.

If the discriminant $\Delta_{q_A} = (a+d)^2 - 4bc = 0$ and $A \neq tI$, there exist $\lambda \in \mathbb{R}$ such that $q_A(t) = (t - \lambda)^2$. We have the following: There exists an invertible matrix P such that $P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Theorem 2.6.

Consider a matrix A which has a unique real eigenvalue λ but $A \neq \lambda I$. Let $X_2 \in \mathbb{R}^2$ such that $(A - \lambda I)X_2 \neq 0$ and $X_1 = (A - \lambda I)X_2$. Then the general solution for the linear first order differential equation $X' = AX$ is given by:

$$X = (at + b)e^{\lambda t}X_1 + ae^{\lambda t}X_2 = be^{\lambda t}X_1 + a(te^{\lambda t}X_1 + e^{\lambda t}X_2), \quad a, b \in \mathbb{R}.$$

Example 2.4 :

Consider the system $X' = AX$, with $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$.

$q_A(\lambda) = (\lambda + 3)^2$. $X_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $AX_1 = -3X_1$ and $AX_2 = X_1 - 3X_2$. The solutions of the system are

$$X = (at + b)e^{-3t}X_1 + ae^{-3t}X_2, \quad a, b \in \mathbb{R}.$$

Example 2.5 :

Consider the system of linear differential equation $X' = AX$, with $X(0) = \begin{pmatrix}1\\2\end{pmatrix}$ and $A = \begin{pmatrix}5 & 4\\-4 & -3\end{pmatrix}$. The characteristic equation of the matrix A is

$$q_A(\lambda) = \begin{vmatrix}5-\lambda & 4\\-4 & -3-\lambda\end{vmatrix} = (1-\lambda)^2.$$

Then the matrix is not diagonalizable. Let $X_2 = \begin{pmatrix}1\\-1\end{pmatrix}$, $(A-I)X_2 = X_1 = \begin{pmatrix}1\\0\end{pmatrix}$. The solution of the system is

$$X = 3e^tX_1 - 2e^t(tX_1 + X_2).$$

Example 2.6 :

Consider the matrix $A = \begin{pmatrix}3 & -2\\2 & -1\end{pmatrix}$. $\lambda = 1$ is the unique eigenvalue of the matrix.

If $X_2 = \begin{pmatrix}1\\0\end{pmatrix}$, then $X_1 = (A-I)X_2 = \begin{pmatrix}2\\2\end{pmatrix}$

The general solution of the system of linear differential equation $X' = AX$ is

$$X = (at+b)e^tX_1 + ae^tX_2 = e^t\begin{pmatrix}2at+2b+a\\2at+2b\end{pmatrix}, \quad a,b\in\mathbb{R}$$

If $X_0 = X(0) = \begin{pmatrix}1\\-1\end{pmatrix}$ then $a = 2$, $b = -\frac{1}{2}$ and

$$X = e^t\begin{pmatrix}4t+1\\4t-1\end{pmatrix}.$$

2.5 Complex Eigenvalues

Theorem 2.7.

If the discriminant $\Delta_{q_A} = (a+d)^2 - 4bc < 0$, there exist $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}^*$ such that $\lambda = \alpha + \mathrm{i}\beta$ and $\overline{\lambda}$ are zeros of the characteristic polynomial q_A. There exist also two eigenvectors X and $\overline{X}$ with respect to λ and $\overline{\lambda}$ respectively and there exists an invertible matrix P such that $P^{-1}AP = \begin{pmatrix}\alpha & \beta\\-\beta & \alpha\end{pmatrix}$.

Proof .

We have $AX = \lambda X$ and $A\bar{X} = \bar{\lambda}\bar{X}$. If $X = X_1 + \mathrm{i}X_2$, we get:

$$\begin{cases}AX_1 & = & \alpha X_1 - \beta X_2\\AX_2 & = & \beta X_1 + \alpha X_2\end{cases}.$$

Consider the matrix P with first column X_1 and second column X_2. P is invertible and $AP = P\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. □

Theorem 2.8.
Consider a matrix A of order 2 and has two non real eigenvalues $\lambda_1 = \alpha + \mathrm{i}\beta$ and $\lambda_1 = \alpha - \mathrm{i}\beta$, $\beta \neq 0$.
If $X = X_1 + \mathrm{i}X_2$ is an eigenvector with respect to the eigenvalue λ_1 then $X_2 = X_1 - \mathrm{i}X_2$ is an eigenvector with respect to the eigenvalue $\overline{\lambda_1}$. Then

$$\{e^{\alpha t}(\cos(\beta t)X_1 - \sin(\beta t)X_2), e^{\alpha t}(\sin(\beta t)X_1 + \cos(\beta t)X_2)\}$$

is a fundamental system of solutions of the system $X' = AX$.

Proof .
Consider the matrix P with first column X_1 and second column X_2. P is invertible and $AP = P\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

If $Y = P^{-1}X$, then $Y' = P^{-1}X' = P^{-1}APP^{-1}X = HY$. If $Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we have

$$\begin{cases} y_1'(t) &= \alpha y_1 + \beta y_2(t) \\ y_2'(t) &= -\beta y_1 + \alpha y_2(t) \end{cases}.$$

Consider the vector $Z = e^{-\alpha t}Y = \begin{pmatrix} z_1(t) = e^{-\alpha t}y_1(t) \\ z_2(t) = e^{-\alpha t}y_2(t) \end{pmatrix}$.

We get $\begin{cases} z_1'(t) &= \beta z_2(t) \\ z_2'(t) &= -\beta z_1(t) \end{cases}$ and $\begin{cases} z_1''(t) &= -\beta^2 z_1(t) \\ z_2''(t) &= -\beta^2 z_2(t) \end{cases}$.
Hence $z_1(t) = a\cos(\beta t) + b\sin(\beta t)$ and $z_2(t) = b\cos(\beta t) - a\sin(\beta t)$. Then $y_1(t) = e^{\alpha t}\left(a\cos(\beta t) + b\sin(\beta t)\right)$ and $y_2(t) = e^{\alpha t}\left(b\cos(\beta t) - a\sin(\beta t)\right)$.

$$\begin{aligned} X &= PY = y_1X_1 + y_2X_2 \\ &= e^{\alpha t}\left(a\cos(\beta t) + b\sin(\beta t)\right)X_1 + e^{\alpha t}\left(b\cos(\beta t) - a\sin(\beta t)\right)X_2 \\ &= ae^{\alpha t}\left(\cos(\beta t)X_1 - \sin(\beta t)X_2\right) + be^{\alpha t}\left(\sin(\beta t)X_1 + \cos(\beta t)X_2\right). \end{aligned}$$

□

Example 2.7 :
Consider the system $X' = AX$ with $A = \begin{pmatrix} -3 & 2 \\ -1 & -5 \end{pmatrix}$.

$q_A(\lambda) = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -5-\lambda \end{vmatrix} = (\lambda+4)^2 + 1$. Then $\lambda = -4 \pm \mathrm{i}$ are the eigenvalues of the matrix A. The vector $X = \begin{pmatrix} -2 \\ 1-\mathrm{i} \end{pmatrix}$ is an eigenvector of the matrix

relative to the eigenvalue $\lambda = -4 + \mathrm{i}$.
The general solution of the system is $X = c_1 e^{-4t}(X_1 \cos(t) - X_2 \sin(t)) + c_2 e^{-4t}(X_1 \sin(t) - X_2 \cos(t))$, where $X_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Example 2.8 :
Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. $\lambda_1 = 2 + \mathrm{i}$, $\lambda_2 = 2 - \mathrm{i}$ are the eigenvalues of the matrix .
If $X_1 = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, then $AX_1 = \lambda_1 X_1$.
In this case $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
The general solution of the system of linear differential equation $X' = AX$ is

$$\begin{aligned} X &= ae^{2t}(\cos t V_1 - \sin t V_2) + be^{2t}(\sin t V_1 + \cos t V_2) \\ &= e^{2t}\begin{pmatrix} a\cos t + b\sin t \\ b\cos t - a\sin t \end{pmatrix}, \quad a, b\mathbb{R} \end{aligned}$$

If $X_0 = X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ then $a = 1$, $b = -1$ and

$$X = e^{2t}\begin{pmatrix} \cos t - \sin t \\ -\cos t - \sin t \end{pmatrix}.$$

2.6 First-Order Non-Homogeneous Systems

Now consider the non-homogeneous system: $X'(t) = AX(t) + B(t)$. Suppose $X_p(t)$ is a particular solution of the non-homogeneous system and $X(t)$ is any other solution. Then $Y(t) = X(t) - X_p(t)$ satisfies the corresponding homogeneous equation $Y'(t) = AY(t)$. We these facts are in the next theorem:

Theorem 2.9.
Consider a non-homogeneous system: $X'(t) = AX(t) + B(t)$. If $\{X_1, X_2\}$ is a fundamental system of solutions of the homogeneous linear system $X'(t) = AX(t)$ and if $X_p(t)$ is a particular solution of the non-homogeneous system, then the set of solutions of the non-homogeneous system is

$$S = \{X_p + aX_1 + bX_2 : a, b \in \mathbb{R}\}.$$

The particular solution of the non-homogeneous system can be found by the variation of constants. If $\{Y_1, Y_2\}$ is a fundamental system of solutions of the homogeneous linear system $X'(t) = AX(t)$, the solutions of the non-homogeneous system can be written in the form $X(t) = U_1(t)Y_1(t) + U_2(t)Y_2(t)$, where U_1 and U_2 are continuously differentiable functions on the interval I. This is because the matrix $W(t)$ with columns Y_1 and Y_2 is invertible for all $t \in I$. (The determinant of the matrix W is the Wronskian of Y_1 and Y_2.

Example 2.9 :
Consider the non-homogeneous linear system of differential equations
$X' = AX + \begin{pmatrix} e^t \\ \cos t \end{pmatrix}$ with $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.
The set $\{Y_1 = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, Y_2 = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$ is a fundamental set of solutions of the homogeneous system. Taking the change of constants method $X = UY_1 + VY_2$, we get: $\begin{cases} U'e^t + V'e^{3t} & = & e^t \\ U'e^t - V'e^{3t} & = & \cos t \end{cases}$. Therefore $U = a + \dfrac{t}{2} + \dfrac{1}{4}e^{-t}(\sin t - \cos t)$,
$V = b - \dfrac{1}{4}e^{-2t} + \dfrac{1}{20}e^{-3t}(3\cos t - \sin t)$.

$$X = \begin{pmatrix} ae^t + be^{3t} + \frac{2t-1}{4}e^t - \frac{1}{10}\cos t + \frac{1}{5}\sin t \\ \\ ae^t - be^{3t} + \frac{2t+1}{4}e^t - \frac{2}{5}\cos t + \frac{3}{10}\sin t \end{pmatrix}, a, b \in \mathbb{R}$$

If $X_0 = X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $a = \frac{1}{4}$, $b = \dfrac{11}{10}$ and

$$X = \begin{pmatrix} \frac{1}{4}e^t + \frac{11}{10}e^{3t} + \frac{2t-1}{4}e^t - \frac{1}{10}\cos t + \frac{1}{5}\sin t \\ \\ \frac{1}{4}e^t - \frac{11}{10}e^{3t} + \frac{2t+1}{4}e^t - \frac{2}{5}\cos t + \frac{3}{10}\sin t \end{pmatrix}.$$

Example 2.10 :
Consider the non-homogeneous linear system of differential equations
$X' = AX + \begin{pmatrix} e^t \\ \cos t \end{pmatrix}$ where $A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$.
Consider $X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X_1 = (A - I)X_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$X = (Ut + V)e^t X_1 + Ue^t X_2 = e^t \begin{pmatrix} 2Ut + 2V + U \\ 2Ut + 2V \end{pmatrix}$

$$\begin{cases} 2U't + 2V' + 2U' & = & 1 \\ 2U't + 2V' & = & e^{-t}\cos t \end{cases}$$

$U = a + \dfrac{1}{2}t + \dfrac{1}{4}e^{-t}(\cos t - \sin t)$, $V = b - \dfrac{1}{4}t^2 + \dfrac{1}{4}te^{-t}(\sin t - \cos t) + \dfrac{1}{4}e^{-t}(2\sin t - \cos t)$

$$X = \begin{pmatrix} (2at + a + 2b + \frac{1}{2}t^2 + \frac{1}{2}t)e^t + \frac{1}{4}(3\sin t - \cos t) \\ \\ (2at + 2b + \frac{1}{2}t^2)e^t + \frac{1}{2}(2\sin t - \cos t) \end{pmatrix}$$

If $X_0 = X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $a = \frac{7}{4}$, $b = -\dfrac{1}{4}$ and

$$X = \begin{pmatrix} (4t + \frac{5}{4} + \frac{1}{2}t^2)e^t + \frac{1}{4}(3\sin t - \cos t) \\ \\ (\frac{7}{2}t - \frac{1}{2} + \frac{1}{2}t^2)e^t + \frac{1}{2}(3\sin t - \cos t) \end{pmatrix}$$

Example 2.11 :
Consider the non-homogeneous linear system of differential equations $X' = AX + \begin{pmatrix} e^t \\ \cos t \end{pmatrix}$ where $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. $\lambda_1 = 2 + \mathrm{i}$, $\lambda_2 = 2 - \mathrm{i}$ are the eigenvalues of the matrix.
The vector $X_1 = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$ is an eigenvector of the matrix A with respect to the eigenvalue λ_1: $AX_1 = \lambda_1 X_1$. In this case $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Let

$$X = e^{2t} \begin{pmatrix} U\cos t + V\sin t \\ V\cos t - U\sin t \end{pmatrix}$$

$$\begin{cases} U'\cos t + V'\sin t &= e^{-t} \\ -U'\sin t + V'\cos t &= e^{-2t}\cos t \end{cases}$$

$$U = \frac{1}{2}e^{-t}(\sin t - \cos t) + \frac{1}{8}e^{-2t}(\cos(2t) + \sin(2t))$$

$$V = \frac{1}{2}e^{-t}(-\cos t + \sin t) - \frac{1}{4}e^{-2t} + \frac{1}{20}e^{-2t}(-2\cos(2t) + \sin(2t))$$

The general solution of the non-homogeneous system is

$$X = e^{2t} \begin{pmatrix} a\cos t + b\sin t - \frac{1}{2}e^{-t} + \frac{1}{8}e^{-2t}(\cos t - \sin t) \\ \\ b\cos t - a\sin t - \frac{1}{2}e^{-t} + \frac{1}{8}e^{-2t}(\sin t - 2\cos t) \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

If $X_0 = X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $a = \frac{11}{8}$, $b = -\dfrac{1}{4}$ and

$$X = e^{2t} \begin{pmatrix} \frac{11}{8}\cos t - \frac{1}{4}\sin t - \frac{1}{2}e^{-t} + \frac{1}{8}e^{-2t}(\cos t - \sin t) \\ \\ -\frac{1}{4}\cos t - \frac{11}{8}\sin t - \frac{1}{2}e^{-t} + \frac{1}{8}e^{-2t}(\sin t - 2\cos t) \end{pmatrix}$$

2.7 Exercises

2-1 Solve the following linear systems of differential equations $X' = AX$, where

(a) $A = \begin{pmatrix} 2 & 7 \\ -5 & -10 \end{pmatrix}$ (b) $A = \begin{pmatrix} -3 & 6 \\ -3 & 3 \end{pmatrix}$ (c) $A = \begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix}$

2-2 Solve the following system of linear differential equations $X' = AX + B$, with $A = \begin{pmatrix} 1 & -1 \\ 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} e^{-3t} \\ e^{2t} \end{pmatrix}$.

2-3 Solve the following system of linear differential equations $X' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} X$, $X(0) = \begin{pmatrix} -2 & 1 \end{pmatrix}$.

2-4 Solve the following system of linear differential equations $X' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X + \begin{pmatrix} -t \\ 2t-1 \end{pmatrix}$, $X(0) = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$.

2-5 Solve the following system of linear differential equations
$\begin{cases} \frac{dx}{dt} &= -2x + y, \\ \frac{dy}{dt} &= x - 2y \end{cases}$ $x(0) = 1$, $y(0) = 2$.

2-6 Use Laplace transforms to solve each non-homogeneous linear system.

(a) $X' = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} X + \begin{pmatrix} 5e^{-3t} \\ -6e^{-3t} \end{pmatrix}$, $X(0) = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$.

(b) $X' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} X + \begin{pmatrix} -1 \\ 8 \end{pmatrix}$, $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) $X' = \begin{pmatrix} 3 & 0 \\ 5 & -2 \end{pmatrix} X + \begin{pmatrix} -4\sin(2t) \\ \cos(2t) \end{pmatrix}$, $X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(d) $X' = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix} X + \begin{pmatrix} 3t+1 \\ -6t-2 \end{pmatrix}$, $X(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

2-7 Solve the following initial linear systems of differential equations

(a) $X' = \begin{pmatrix} 2 & 7 \\ -5 & 10 \end{pmatrix} X$, $X(0) = \begin{pmatrix} -2 & 1 \end{pmatrix}$

(b) $X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X$, $X(0) = \begin{pmatrix} -4 & 2 \end{pmatrix}$

(c) $X' = \begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix} X$, $X(0) = \begin{pmatrix} 5 & -2 \end{pmatrix}$

(d) $X' = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} X$, $X(0) = \begin{pmatrix} 0 & 3 \end{pmatrix}$

(e) $X' = \begin{pmatrix} 6 & 8 \\ 2 & 6 \end{pmatrix} X$, $X(0) = \begin{pmatrix} 8 & 0 \end{pmatrix}$

(f) $X' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X$, $X(0) = \begin{pmatrix} 6 & 4 \end{pmatrix}$

(g) $\mathbf{X}' = \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix} \mathbf{X}, \qquad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(h) $\begin{cases} x'' - y'' + x - 4y = 0 \\ x' + y' = \cos t \end{cases}$ $\quad x(0) = 0, \quad x'(0) = 1, \quad y(0) = 0, \quad y'(0) =$
2

3 The Laplace Transform Method for Solving Systems of Linear Differential Equations

The method of Laplace transforms, in addition to solving individual linear differential equations, can also be used to solve systems of linear differential equations.

Example 3.1 :

Consider the system of linear differential equations: $\begin{cases} x'(t) &= x(t) - y(t) \\ y'(t) &= x(t) + y(t) \end{cases}$

Using Laplace transform, with $X = \mathcal{L}(x)$ and $Y = Y$, we get:

$\begin{cases} sX(s) - 2 &= X(s) - Y(s) \\ sY(s) - 1 &= X(s) + Y(s) \end{cases}$

This system is equivalent to the following: $\begin{cases} (s-1)X(s) + Y(s) &= 2 \\ -X(s) + (s-1)Y(s) &= 1 \end{cases}$

Then $X(s) = \dfrac{2s-3}{(s-1)^1+1} = \dfrac{2(s-1)-1}{(s-1)^1+1}$ and $Y(s) = \dfrac{(s-1)+2}{(s-1)^1+1}$.

Taking the Laplace inverse operator, we get

$$x(t) = e^t(2\cos(t) - \sin(t)), \quad y(t) = e^t(\cos(t) + 2\sin(t)).$$

Example 3.2 :

Consider the system of linear differential equations: $X' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X$ with $X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. This system is rewritten into the explicit form as follows: $\begin{cases} x'(t) &= -4x(t) - 2y(t) \\ y'(t) &= 3x(t) + y(t) \end{cases}$. Using Laplace transform, we get

$$\begin{cases} sX(s) - 2 &= -4X(s) - 2Y(s) \\ sY(s) - 3 &= 3X(s) + Y(s) \end{cases} \Longleftrightarrow \begin{cases} (s+4)X(s) + 2Y(s) &= 2 \\ -3X(s) + (s-1)Y(s) &= 3 \end{cases}$$

Then $X(s) = \dfrac{2s-8}{(s+1))(s+2)} = -\dfrac{10}{s+1} + \dfrac{12}{s+2}$ and
$Y(s) = \dfrac{3s+18}{(s+1)(s+2)} = \dfrac{15}{s+1} - \dfrac{12}{s+2}$.
Taking the Laplace inverse operator, we get

$$x(t) = -10e^{-t} + 12e^{-2t}, \quad y(t) = 15e^{-t} - 12e^{-2t}.$$

3.1 Exercises

3-1 Use Laplace transform to solve the following differential systems

(a) $X' = \begin{pmatrix} -4 & -2 \\ 3 & 1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ 3e^{-2t} \end{pmatrix}$ with $X(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(b) $\begin{cases} x' &= 2x - y + e^t \\ y' &= x + 2y + \cos t \end{cases}, \quad x(0) = 1, y(0) = -1.$

(c) $\begin{cases} x' &= 2x + y + e^t \\ y' &= -x + 2y + \cos t \end{cases}, \ x(0) = 1, \ y(0) = -1$

(d) $\begin{cases} x' &= 2x - y + e^t \\ y' &= -x + 2y + \cos t \end{cases}, \quad x(0) = 1, y(0) = -1.$

(e) $X' = AX + \begin{pmatrix} e^t \\ \cos t \end{pmatrix}$, where $A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ and $x(0) = 1, \ y(0) = -1.$

(f) $\begin{cases} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= x - 2y \end{cases}$, with the initial conditions $x(0) = 1$, $y(0) = 2$.

(g)

$$\begin{cases} \dfrac{dx}{dt} &= -2x + y, \\ \dfrac{dy}{dt} &= x - 2y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$.

(h)

$$\begin{cases} \dfrac{dx}{dt} &= -x + 2y, \\ \dfrac{dy}{dt} &= 2x + y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = -1$.

4 Systems of Linear Differential Equations of Higher Order

4.1 Elimination Method

In this section we intend to solve the systems of linear differential equations by the elimination method. We can treat systems of differential equations of higher order. In which follows, we consider the linear differential operators with constant coefficients $L_k = \sum_{j=0}^{m} a_j D^j$, which is a polynomial in D. These operators commute i.e. $L_k L_j = L_j L_k$. In which follows, we use some properties of the operators L_k to solve the systems of linear differential equations of type:

$$\begin{cases} L_1 x(t) + L_2 y(t) &= f(t) \\ L_3 x(t) + L_4 y(t) &= g(t) \end{cases}, \tag{4.2}$$

To solve this system, we use the following method, called the elimination method. To eliminate x, we apply the operator L_3 to first equation and L_1 to the second equation, we get

$$(L_3 L_2 - L_1 L_4) y = L_3 f - L_1 g.$$

This equation is a linear differential equation with constant coefficients.

Example 4.1 :
Consider the following system of linear differentials equations

$$\begin{cases} x^{'}(t) + x(t) + y(t) &= e^{-3t} \\ x(t) - y'(t) - y(t) &= e^{2t} \end{cases}$$

This system is equivalent to

$$\begin{cases} (D+1)x(t) + y(t) &= e^{-3t} \\ x(t) - (D+1)y(t) &= e^{2t} \end{cases}.$$

Apply $(D+1)$ to the second equation, we get: $(D^2+2D+2)y = e^{-3t} - 3e^{2t}$. Hence $y = e^{-t}(a\cos t + b\sin t) + \frac{1}{5}e^{-3t} - \frac{3}{10}e^{2t}$ and $x = e^{-t}(b\cos t - a\sin t) - \frac{2}{5}e^{-3t} + \frac{1}{10}e^{2t}$.

Example 4.2 :
Consider the following system of linear differentials equations

$$\begin{cases} x^{''}(t) - 3x'(t) + 2x(t) + y'(t) - y(t) &= e^{t} \\ x'(t) - 3x(t) + y'(t) + y(t) &= \cos t \end{cases}$$

This system is equivalent to

$$\begin{cases}(D-1)(D-2)x(t)+(D-1)y(t) &= e^t\\ (D-3)x(t)+(D+1)y(t) &= \cos t\end{cases}.$$

Apply $(D-3)$ to the first equation and $(D-1)(D-2)$ to the second equation, we get:

$$(D-1)^3y(t)=2e^t+\cos t+3\sin t.$$

Then $y=(at^3+bt^2+ct+d)e^t+A\cos t+B\sin t$ and
$x(t)=(a't^3+b't^2+c't+d')e^t+A'\cos t+B'\sin t+Ce^{3t}$, where $a'=a$, $b'=3a+b$, $c'=3a+2b+c$, $2d'=3a+2b+2c$, $10A'=2A+4B-3$, $10B'=-4A+2B+1$.

Example 4.3 :
Consider the following system of linear differentials equations

$$\begin{cases}x''(t)+x(t)-y'(t)+y(t)=e^t\\ x'(t)+x(t)+y''(t)-y(t)=\sin t.\end{cases}.$$

The system has the operator form

$$\begin{cases}(D^2+1)x(t)+(1-D)y(t)=e^t\\ (D+1)x(t)+(D^2-1)y(t)=\sin t.\end{cases}$$

Apply the operator $D+1$ to the first equation and take the sum with the second equation, we get $(D+1)(D^2+1)x(t)+(D+1)x=2e^t+\sin t$, hence $x'''+x''+2x'+2x=2e^t+\sin t$. The general solution of this equation is

$$x(t)=ae^{-t}+b\cos(\sqrt{2}t)+c\sin(\sqrt{2}t)+\frac{1}{3}e^t-\frac{1}{2}(\cos t-\sin t).$$

Hence

$$y'-y=2ae^{-t}-b\cos\sqrt{2}t-c\sin\sqrt{2}t-\frac{5}{3}e^t.$$

We solve this differential equation and find that

$$y(t)=(d-\frac{5}{3}t)e^t-ae^{-t}+(b+\sqrt{2}c)\cos\sqrt{2}t-(\sqrt{2}b+c)\sin\sqrt{2}t.$$

4.2 Exercises

4-1 Solve the following systems of differential equations:

(a) $\begin{cases}y''+x'+x=0,\\ y'-y+x=\sin t.\end{cases}$

(b) $\begin{cases}x''-3x'+y'+2x-y=t,\\ y'+x'-2x+y=\sin t.\end{cases}$ $x(0)=1, y(0)=0, x'(0)=1, y'(0)=1$.

5 Power Series Solutions of Differential Equations

1 Power Series

1.1 Series Product

Definition 1.1.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers. For $n \in \mathbb{N}$, we set

$$c_n = \sum_{k=1}^{n} u_k v_{n-k}. \tag{5.1}$$

The series $\sum\limits_{n\geq 1} c_n$ is called the product of the two given series $\sum\limits_{n\geq 1} u_n$ and $\sum\limits_{n\geq 1} v_n$.

The convergence of the series product depends of the nature of the convergence the series. Consider for example the following series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n+1}}$. This series is convergent and the product of this series with itself is the series $\sum_{n\geq 1} c_n$, where

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}.$$

It is easy to prove that $|c_n| \geq 1$, then the series $\sum_{n\geq 1} c_n$ is not convergent. The following theorem affirms the existence of the series product under some conditions.

Theorem 1.2.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers.

1. Assume that the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ are absolutely convergent. Then the series $\sum_{n\geq 1} c_n$ is absolutely convergent and

$$\sum_{n=1}^{+\infty} c_n = (\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \tag{5.2}$$

2. Assume that the series $\sum_{n\geq 1} u_n$ is absolutely convergent and the series $\sum_{n\geq 1} v_n$ is convergent. Then the series $\sum_{n\geq 1} c_n$ is convergent and we have (5.2).

Proof .
It suffices to proves 2). Define

$$A_n = \sum_{k=1}^{n} u_k, \qquad B_n = \sum_{k=1}^{n} v_k, \qquad C_n = \sum_{k=1}^{n} c_k,$$

$$A = \sum_{n=1}^{+\infty} u_n, \qquad \alpha = \sum_{n=1}^{+\infty} |u_n| \quad \text{and} \quad B = \sum_{n=1}^{+\infty} v_n.$$

Then

$$C_n = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} u_j B_{n-j} = \sum_{j=1}^{n} u_j (B_{n-j} - B) + BA_n.$$

Since $\lim_{n\to+\infty} B.A_n = A.B$, then to show that $\lim_{n\longrightarrow+\infty} C_n = A.B$, it suffices to prove that the sequence $(\Delta_n)_n$ converges to 0, where $\Delta_n = \sum_{j=1}^{n} a_j(B_{n-j} - B)$.

Let $\varepsilon > 0$: $\exists N \in \mathbb{N}$ such that $\forall\ n \geq N$; $|B_n - B| < \frac{\varepsilon}{2\alpha}$ and $\sum_{j=N}^{+\infty} |a_j| \leq \dfrac{\varepsilon}{2M}$.

Then for all $n \geq 2N$,

$$|\Delta_n| \leq \sum_{j=1}^{N} |a_j||B_{n-j} - B| + \sum_{j=N+1}^{n} |a_j||B_{n-j} - B| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The result follows. □

1.2 Power Series

Definition 1.3.
A power series centered at $a \in \mathbb{R}$ is a series of functions $\sum_{n\geq 0} a_n(x-a)^n$, where $(a_n)_n$ a sequence of real numbers.

In the classical analysis course, we prove that for any power series, there exists $R \in [0,+\infty]$ such that the power series $\sum_{n\geq 0} a_n(x-a)^n$ is absolutely convergent on the interval $(a-R, a+R)$ if $R>0$, convergent at $\{a\}$ if $R=0$ and divergent for x such that $|x-a|>R$. This number R is called the radius of convergence of the power series. Moreover, $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$, if this limit exists. Also $R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$, if this limit exists and in general, $R = \frac{1}{\overline{\lim}_{n\to\infty} |a_n|^{1/n}}$.

Theorem 1.4.
Consider a power series $\sum_{n\geq 0}^{+\infty} a_n(x-a)^n$ with radius of convergence $R>0$, then the function $f(x) = \sum_{n=0}^{+\infty} a_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$, $f'(x) = \sum_{n=1}^{+\infty} n a_n (x-a)^{n-1}$ and

$$\int_a^x f(t)dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(x-a)^{n+1} = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n}(x-a)^n.$$

Corollary 1.5.
Consider a power series $\sum_{n\geq 0}^{+\infty} a_n(x-a)^n$ with radius of convergence $R>0$, then the function $f(x) = \sum_{n=0}^{+\infty} a_n(x-a)^n$ is $C^{+\infty}$ on the interval $(a-R, a+R)$, $f(a) = a_0$ and $a_n = \frac{f^{(n)}}{n!}(a)$, for all $n \in \mathbb{N}$.

Example 1.1 :

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \qquad |x| < 1. \text{ By integration, we get:}$$

$$\ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{(n+1)},$$

$$\ln(1-x) = -\sum_{n=0}^{+\infty} \frac{x^{n+1}}{(n+1)},$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)},$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \text{ and}$$

$$\tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$$

$$\frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = (1-x)\sum_{n=0}^{+\infty} x^{3n} \text{ for } |x| < 1.$$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \qquad x \in \mathbb{R}.$$

$$\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R}.$$

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

$$\sinh x = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R}.$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

Theorem 1.6.
Let $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ two power series with radius of convergence R_1 and R_2 respectively. Then for $|x| < \min(R_1, R_2)$:

$$\sum_{n=0}^{+\infty} c_n x^n = \left(\sum_{n=0}^{+\infty} a_n x^n\right)\left(\sum_{n=0}^{+\infty} b_n x^n\right)$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Theorem 1.7.
Let $f(x) = \sum_{n=0}^{+\infty} a_n(x-a)^n$ for $|x-a| < R$, then f is analytic on $(a-R, a+R)$ and for all $c \in (a-R, a+R)$, $f(x) = \sum_{n=0}^{+\infty} b_n(x-c)^n$ for all $|x-c| < R - |a-c|$, where $b_n = \dfrac{f^{(n)}(c)}{n!}$.

Example 1.2 :
The functions $\dfrac{1}{1-x}$, $\tan^{-1}(x)$ are analytic on the interval $(-1,1)$. For $|x| < 1$,
$\dfrac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$, $\dfrac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$ and $\tan^{-1}(x) = \sum_{n=0}^{+\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$.

Definition 1.8.
A function $f\colon I \longrightarrow \mathbb{R}$ is called analytic at $a \in I$ if there is $r > 0$ which depends on a such that $f(x) = \sum_{n=0}^{+\infty} a_n(x-a)^n$, for all $x \in (a-r, a+r) \subset I$.
The point a is also called an **ordinary point**. The largest such r (possibly $+\infty$) is called the **radius of convergence** of the power series. The series converges for every x with $|x-a| < r$ and diverges for every x with $|x-a| > r$.
If f is not analytic at a, a is called a **singular point** for f.
The function f is called **analytic** on I if it is analytic at each point $a \in I$.

1.3 Exercises

1-1 (a) Compute the radius of convergence of the power series $\sum_{n\geq 1} \sin(\dfrac{n\pi}{3})\dfrac{x^n}{n}$

(b) Compute the sum of the series.

1-2 Let $\lambda \in \mathbb{R}\backslash 2\mathbb{Z}$ and $(a_n)_n$ a sequence such that $a_0 \neq 0$ and $\dfrac{a_{n+1}}{a_n} = \dfrac{2n-\lambda}{n+1}$.

(a) Compute the radius of convergence R of the power series $\sum_{n\geq 0} a_n x^n$.

(b) Prove that the function $f\colon x \longmapsto \sum_{n=0}^{+\infty} a_n x^n$ verifies the différential equation and give a simple expression of f.

1-3 Develop the following function in power series

(a) $\dfrac{\ln(1+x)}{1+x}$.

(b) $(\sin^{-1}x)^2$.

(c) $\dfrac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}$,

(d) $\ln(1-2x\cos\alpha+x^2)$,

(e) $e^{2x}\cos x$,

(f) $\dfrac{x}{1-x-x^2}$.

(Hint: Prove that the function $f(x) = (\sin^{-1}x)^2$ verifies a differential equation of order 2.)

2 Series Solutions of Differential Equations

Definition 2.1.
Consider the following linear differential equation

$$y'' + C(x)y' + D(x)y = 0 \tag{5.3}$$

A point a is called an ordinary point of the differential equation (5.3) if both coefficients $C(x)$ and $D(x)$ are analytic at a. A point that is not an ordinary point of (5.3) is called a singular point of the differential equation.

Example 2.1 :
For the following differential equation

$$(x^2-1)y'' + 2xy' + 6y = 0$$

1 and -1 are the unique singular points.

Theorem 2.2. Existence of Power Series Solutions Theorem
If a is an ordinary point of the differential equation

$$y'' + C(x)y' + D(x)y = 0.$$

there exist two linearly independent solutions as power series centered at a

$$y = \sum_{n=0}^{+\infty} a_n(x-a)^n$$

These series converge on the interval $(a-R, a+R)$, where R is the distance from a to the closest singular point of the differential equation. (The singular point can be a complex number).

Remark 11 :

1. The series solutions can be analytic in an interval larger than the interval of convergence of both functions C and D.
   For example, consider the differential equation
   $$y'' - \frac{2}{x-1}y' + \frac{2}{(x-1)^2}y = 0$$
   The functions $y_1 = x-1$ and $y_2 = (1-x)^2$ are solutions of this differential equation which are analytic on $\mathbb{R}$ but the coefficients of the differential equation are analytic only on $(-1,1)$.

2. If the power series of C and D have different radius of convergence, then the power series of the solutions is the smallest one of C and D.

Proof .
Let $y = \sum_{n=0}^{+\infty} a_n x^n$, $y' = \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n$, $y'' = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$, $C = \sum_{n=0}^{+\infty} c_n x^n$ and $D = \sum_{n=0}^{+\infty} d_n x^n$. Then $Dy = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k d_{n-k}\right)x^n$, $Cy' = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}(k+1)a_{k+1}c_{n-k}\right)x^n$. Then $y'' + Cy' + Dy = 0$ if and only if

$$a_{n+2} = -\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}\Big((k+1)a_{k+1}c_{n-k} + a_k d_{n-k}\Big). \qquad (5.4)$$

To find the two linearly independent solutions, take y_1 the unique solution of the initial value problem : $y'' + Cy' + Dy = 0$, $y(0) = 1$, $y'(0) = 0$ and y_2 the unique solution of the initial value problem : $y'' + Cy' + Dy = 0$, $y(0) = 0$, $y'(0) = 1$. The solutions y_1, y_2 are linearly independent since the Wronskian $W = \begin{vmatrix} y_1(0) & y_1'(0) \\ y_2(0) & y_2'(0) \end{vmatrix} = 1$.

□

Remark 12 :
Consider the differential equation

$$y'' + C(x)y' + D(x)y = F(x).$$

If the functions C, D, F are analytic at a with R the smallest radius of convergence of the power series at a of these functions, the solutions of the differential equation are analytic with radius of convergence at least equal to R.

Example 2.2 :

Consider the following differential equation

$$(x^2-4x+3)y^{''}+2xy'+6y=0.$$

There is two linearly independent solutions of this differential equation as power series centered at 1 with radius of convergence $R=2$ and there is two linearly independent solutions of this differential equation as power series centered at 3 with radius of convergence $R=2$.

Example 2.3 :
Let $f(x)=(1+x)^\alpha$ with α a real number, $\alpha\notin\mathbb{N}$. For $x\in]-1,1[$; $f'(x)=\alpha(1+x)^{\alpha-1}$, then f satisfies the following differential equation

$$y'-\frac{\alpha}{1+x}y=0. \tag{5.5}$$

The differential equation has an analytic solution $\sum_{n\geq 0}a_nx^n$ of radius of convergence at least 1. Let $S=\sum_{n=0}^{+\infty}a_nx^n$ be a solution. We have:

$$(1+x)\sum_{n=0}^{+\infty}na_nx^{n-1}-\alpha\sum_{n=0}^{+\infty}a_nx^n=0.$$

Then $(n+1)a_{n+1}+na_n-\alpha a_n=0$, which yields that $a_{n+1}=\dfrac{\alpha-n}{n+1}a_n$, for all $n\geq 0$. Then $a_n=\dfrac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)}a_0$ and

$$S(x)=a_0(1+\sum_{n=1}^{+\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n).$$

By the uniqueness of the solution of the differential equation

$$(1-x)^\alpha=\sum_{n=0}^{+\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!}x^n,\quad \forall|x|<1.$$

For $\alpha=\frac{-1}{2}$,

1. $\dfrac{1}{\sqrt{1-x}}=\displaystyle\sum_{n=0}^{+\infty}\frac{1}{4^n}\binom{2n}{n}x^n,$

2. $\dfrac{1}{\sqrt{1+x}}=\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}x^n,$

3. $\sqrt{1+x}=1+\dfrac{1}{2}\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n}{4^n(n+1)}\binom{2n}{n}x^{n+1},$

4. $\dfrac{1}{\sqrt{1-x^2}}=\displaystyle\sum_{n=0}^{+\infty}\frac{1}{4^n}\binom{2n}{n}x^{2n},$

5. $\dfrac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{4^n}\binom{2n}{n}x^{2n}$,

6. $\sin^{-1} x = \sum_{n=0}^{+\infty} \dfrac{1}{4^n}\binom{2n}{n}\dfrac{x^{2n+1}}{2n+1}$,

7. $\cos^{-1} x = \dfrac{\pi}{2} - \sum_{n=0}^{+\infty} \dfrac{1}{4^n}\binom{2n}{n}\dfrac{x^{2n+1}}{2n+1}$,

8. $\sinh^{-1} x = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{4^n}\binom{2n}{n}\dfrac{x^{2n+1}}{2n+1}$.

Example 2.4 :
Consider the differential equation $(1+x^2)y'' - 6y = 0$. Any solution of this differential equation is analytic on the interval $(-1,1)$. Consider $y = \sum_{n=0}^{+\infty} a_n x^n$ a solution of this differential equation,
$y' = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$ and $x^2 y'' = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$. Then

$$(1+x^2)y'' - 6y = \sum_{n=0}^{+\infty} \left[(n+2)(n-3)a_n + (n+1)(n+2)a_{n+2}\right] x^n = 0.$$

It follows that: $a_{n+2} = -\dfrac{n-3}{n+1}a_n$ for all $n \in \mathbb{N} \cup \{0\}$.
We deduce that $a_5 = 0$, $a_{2n+1} = 0$ for all $n \geq 2$ and $x + x^3$ is a solution of the differential equation. Also $a_{2n} = -\dfrac{2n-5}{2n-1}a_{2(n-1)} = \dfrac{3(-1)^n}{(2n-1)(2n-3)}a_0$.
The general solution of the differential equation is

$$y = a(x+x^3) + b\sum_{n=0}^{+\infty} \frac{3(-1)^n}{(2n-1)(2n-3)}x^{2n}, \quad a, b \in \mathbb{R}.$$

Example 2.5 :
Consider the differential equation $y'' - 2y = 0$. The coefficients of this differential equation are analytic everywhere, in particular at 0. Any solution of this differential equation is analytic everywhere.
If $y = \sum_{n=0}^{+\infty} a_n x^n$ is a solution, $y' = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n$ and $y'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$. Then

$$y'' - 2y = \sum_{n=0}^{+\infty}((n+1)(n+2)a_{n+2} - 2a_n)x^n = 0,$$

for all $x \in \mathbb{R}$. It follows that $(n+1)(n+2)a_{n+2} = 2a_n$ for $n \geq 0$. Thus

$$a_{n+2} = \frac{2a_n}{(n+1)(n+2)}, \quad \forall n \geq 0.$$

We get $a_{2n} = \dfrac{2^n a_0}{(2n)!}, \quad a_{2n+1} = \dfrac{2^n a_1}{(2n+1)!}$ and hence

$$y = a_0 \sum_{n=0}^{+\infty} \frac{2^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{+\infty} \frac{2^n x^{2n+1}}{(2n+1)!} = a_0 \cosh(\sqrt{2}x) + \frac{a_1}{\sqrt{2}} \sinh(\sqrt{2}x).$$

Example 2.6 :
Consider the differential equation $y'' + xy = 0$.
If $y = \sum_{n=0}^{+\infty} a_n x^n$ is a solution of the differential equation, then $R = +\infty$ is the radius of convergence of this series.
$y'' = \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2}x^n$ and $xy = \sum_{n=1}^{+\infty} a_{n-1}x^n$. Then

$$\begin{aligned} y'' + xy &= \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{+\infty} a_{n-1}x^n \\ &= 2a_2 + \sum_{n=1}^{+\infty} \left((n+2)(n+1)a_{n+2} + na_n + a_{n-1}\right) x^n. \end{aligned}$$

Then $a_2 = 0$ and $(n+3)(n+2)a_{n+3} + a_n = 0, \quad \forall n \in \mathbb{N}$. By induction we get $a_{3n+2} = 0$ for $n \geq 0$ and

$$a_{3n} = a_0 \frac{(-1)^n}{3n!} \prod_{k=0}^{n-1} (3(n-k) - 2),$$

$$a_{3n+1} = a_1 \frac{(-1)^n}{(3n+1)!} \prod_{k=0}^{n-1} (3(n-k) - 1).$$

Hence $y = a_0 y_1 + a_1 y_2$ with $y_1 = \sum_{n=0}^{+\infty} a_{3n} x^{3n}$ and $y_2 = \sum_{n=0}^{+\infty} a_{3n+1} x^{3n+1}$.

Example 2.7 :

Consider a power series $y = \sum_{n=0}^{+\infty} a_n x^n$ solution of the differential equation.

$$y'' + xy' + y = 0.$$

$y' = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}$.

$y'' + xy' + y = \sum_{n=0}^{+\infty} \left((n+1)(n+2)a_{n+2} + na_n + a_n\right) x^n$. Then $a_{n+2} = -\frac{1}{n+2} a_n$.

$$a_{2n} = \frac{(-1)^n}{2^n n!} a_0, \quad a_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!} a_1.$$

Then

$$y_1 = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} x^{2n} = e^{-\frac{x^2}{2}}, \qquad y_2 = \sum_{n=0}^{+\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}$$

are independent solutions of the differential equation. $y'' + xy' + y = 0$.

Example 2.8 :

Consider a power series $y = \sum_{n=0}^{+\infty} a_n x^n$ solution of this differential equation

$$(x^2+1)y'' + xy' + y = 0.$$

$xy' = \sum_{n=0}^{+\infty} n a_n x^n$, $x^2 y'' = \sum_{n=0}^{+\infty} n(n-1) a_n x^n$ and $y'' = \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n$.

$(x^2+1)y'' + xy' + y = \sum_{n=0}^{+\infty} \left((n+1)(n+2)a_{n+2} + n(n-1)a_n + na_n + a_n\right) x^n$.

Then $a_{n+2} = \frac{n^2+1}{(n+1)(n+2)} a_n$, $\forall n \geq 0$.

$a_{2n} = \frac{4(n-1)^2+1}{(2n-1)(2n)} a_{2(n-1)} = \frac{a_0}{(2n)!} \prod_{k=0}^{n-1} (4k^2+1)$,

$a_{2n+1} = \frac{(2n-1)^2+1}{(2n)(2n+1)} a_{2n-1} = \frac{a_1}{(2n+1)!} \prod_{k=1}^{n} ((2k-1)^2+1)$

Then

$$y_1 = \sum_{n=0}^{+\infty} \frac{1}{(2n)!} \prod_{k=0}^{n-1} (4k^2+1) x^{2n}, \quad y_2 = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)!} \prod_{k=1}^{n} ((2k-1)^2+1) x^{2n+1}$$

are independent solutions of the differential equation $(x^2+1)y'' + xy' + y = 0$.

Example 2.9 :

Consider $y = \sum_{n=0}^{+\infty} a_n x^n$ a solution of the differential equation $y'' - xy = 0$, $y(0) = a$ and $y'(0) = b$, with $a_0 = a$, $a_1 = b$. We get $a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)}$, for $n \geq 1$ and $a_2 = 0$. Then $a_{3n+2} = 0$ for all $n \geq 0$ and

$$a_{3n} = \frac{a}{3n!} \prod_{k=1}^{n} (3k-2), \quad a_{3n+1} = \frac{b}{3n!} \prod_{k=1}^{n} (3k-1).$$

Define the shifted factorial $(a)_0 = 1$, $(a)_n = \prod_{k=0}^{n-1} (a+k)$, $n \in \mathbb{N}$. Hence the well known property of the Gamma function $\Gamma(x+1) = z\Gamma(x)$, yields that $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$. The shifted factorial is also called the Pochhammer symbol. Then

$$a_{3n} = \frac{a}{3n!} \prod_{k=1}^{n} (3k-2) = \frac{3^n a}{3n!} \prod_{k=0}^{n-1} (\frac{1}{3} + k) = \frac{3^n a}{3n!} \frac{\Gamma(n + \frac{1}{3})}{\Gamma(\frac{1}{3})},$$

$$a_{3n+1} = \frac{b}{3n!} \prod_{k=1}^{n} (3k-1) = \frac{b3^n}{3n!} \prod_{k=0}^{n-1} (k + \frac{2}{3}) = \frac{3^n b}{3n!} \frac{\Gamma(n + \frac{2}{3})}{\Gamma(\frac{2}{3})}.$$

Then

$$y_1 = \sum_{n=0}^{+\infty} \frac{3^n}{3n!} \frac{\Gamma(n + \frac{1}{3})}{\Gamma(\frac{1}{3})} x^{3n}$$

and

$$y_2 = \sum_{n=0}^{+\infty} \frac{3^n}{3n!} \frac{\Gamma(n + \frac{2}{3})}{\Gamma(\frac{2}{3})} x^{3n+1}$$

are independent solutions of the differential equation $y'' - xy = 0$.

Example 2.10 :
Consider the following differential equation $y'' + xy' - xy = 0$.

Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. This series converges on $\mathbb{R}$. $xy' = \sum_{n=0}^{+\infty} n a_n x^n$, $y'' = \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} x^n$ and $xy = \sum_{n=1}^{+\infty} a_{n-1} x^n$.
Then $a_2 = 0$ and $(n+3)(n+2)a_{n+3} + (n+1)a_{n+1} - a_n = 0$, for all $n \in \mathbb{N}$.
Define $b_0 = a_1 + a_0$ and $b_n = (n+1)a_{n+1} + a_n + a_{n-1}$, for all $n \in \mathbb{N}$.
$(n+1)b_{n+1} = (n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n = b_n$. We deduce that $b_n = \dfrac{a_0 + a_1}{n!}$ and $\sum_{n=0}^{+\infty} b_n x^n = (a_0 + a_1)e^x$ on $\mathbb{R}$. Moreover

$\sum_{n=0}^{+\infty} b_n x^n = \sum_{n=0}^{+\infty} ((n+1)a_{n+1} + a_n + a_{n-1})\, x^n = y'(x) + (x+1)y(x)$. Then y is also a solution of the differential equation

$$y'(x) + (x+1)y(x) = (a_0 + a_1)e^x.$$

The solutions of this linear differential equation are $y = e^{\frac{x^2}{2}+2x}(a+b\int_0^x e^{\frac{t^2}{2}+2t}dt)$.

Example 2.11 :
Consider the following differential equation $y^{''} + x^2 y = 0$.
Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of this differential equation. This series converges on $\mathbb{R}$. $y^{''} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$ and $x^2 y = \sum_{n=2}^{+\infty} a_{n-2}x^n$. Then $a_2 = a_3 = 0$ and $(n+1)(n+2)a_{n+2} + a_{n-2} = 0$, for all $n \geq 2$.
We get $a_{4n} = -\frac{a_{4(n-1)}}{4n(4n-1)} = \frac{(-1)^n}{4^n n!}\prod_{k=1}^{n}\frac{1}{4k-1}a_0$, for all $n \geq 1$,
$a_{4n+1} = \frac{(-1)^n}{4^n n!}\prod_{k=1}^{n}\frac{1}{4k+1}a_1$, for all $n \geq 1$ and $a_{4n+2} = a_{4n+3} = 0$, for all $n \geq 0$.
Then

$$y_1 = 1 + \sum_{n=1}^{+\infty}\frac{(-1)^n}{4^n n!}\prod_{k=1}^{n}\frac{1}{4k-1}x^{4n}$$

and

$$y_2 = x + \sum_{n=1}^{+\infty}\frac{(-1)^n}{4^n n!}\prod_{k=1}^{n}\frac{1}{4k+1}x^{4n+1}$$

are independent solutions of the differential equation $y^{''} + x^2 y = 0$.

Example 2.12 : [Hermite's Equation]
Consider the following differential equation

$$y^{''} - 2xy' + 2axy = 0, \qquad a > 0.$$

Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of this differential equation. This series converges on $\mathbb{R}$.

$xy' = \sum_{n=0}^{+\infty} na_n x^n$, $y^{''} = \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n$ and $2axy = \sum_{n=1}^{+\infty} 2aa_{n-1}x^n$.
Then

$$(n+1)(n+2)a_{n+2} = 2(n-a)a_n, \qquad \forall n \geq 0.$$

$$a_{2n} = \frac{2^2(n-1-\frac{a}{2})}{2n(2n-1)}a_{2(n-1)} = \frac{(-1)^n 2^{2n}\Gamma(1+\frac{a}{2})}{2n!\Gamma(\frac{a}{2}-(n-1))}a_0.$$

and

$$a_{2n+1} = \frac{2^2(n-\frac{1}{2}-\frac{a}{2})}{2n(2n+1)}a_{2n-1} = \frac{(-1)^n 2^{2n+1}\Gamma(\frac{1}{2}+\frac{a}{2})}{(2n+1)!\Gamma(\frac{a}{2}-(n-1))}a_1.$$

Then

$$y_1 = \Gamma(1+\frac{a}{2})\sum_{n=0}^{+\infty}\frac{(-1)^n(4)^n}{2n!\Gamma(\frac{a}{2}-(n-1))}x^{2n}$$

and

$$y_2 = \Gamma(\frac{1}{2}+\frac{a}{2})\sum_{n=0}^{+\infty}\frac{(-1)^n(2)^{2n+1}}{(2n+1)!\Gamma(\frac{a}{2}-n+\frac{1}{2})}x^{2n+1}$$

are independent solutions of the Hermite's differential equation.

Example 2.13 : [Chebyshev's Equation]
Consider the following differential equation

$$(1-x^2)y^{''} - xy' + a^2y = 0, \qquad a > 0.$$

Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of this differential equation. This series converges on $(-1,1)$. $xy' = \sum_{n=0}^{+\infty} na_n x^n$, $y^{''} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$ and $x^2y^{''} = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$. Then

$$a_{n+2} = \frac{n^2-a^2}{(n+1)(n+2)}a_n$$

$$a_{2n} = \frac{4(n-1)^2-a^2}{(2n-1)(2n)}a_{2(n-1)} = \frac{1}{2n!}\left(\prod_{k=0}^{n-1}(4k^2-a^2)\right)a_0$$

$$a_{2n+1} = \frac{(2n-1)^2-a^2}{(2n)(2n+1)}a_{2n-1} = \frac{1}{(2n+1)!}\left(\prod_{k=0}^{n-1}((2k+1)^2-a^2)\right)a_1.$$

Then

$$y_1 = 1 + \sum_{n=1}^{+\infty} \left(\prod_{k=0}^{n-1} (4k^2 - a^2) \right) \frac{x^{2n}}{2n!}$$

and

$$y_2 = x + \sum_{n=1}^{+\infty} \left(\prod_{k=0}^{n-1} ((2k+1)^2 - a^2) \right) \frac{x^{2n+1}}{(2n+1)!}$$

are independent solutions of the Chebyshev's differential equation.

2.1 Exercises

2-1 Consider the differential equation

$$x(x-4)y' + (x+2)y = 2$$

1) Find a power series $\sum_{n=0}^{+\infty} a_n x^n$ solution of the differential equation.
2) Find the general solution for $x \in (-4, 4)$ and deduce the value of the following sum: $\sum_{n=0}^{+\infty} \frac{1}{\binom{2n}{n}} = \sum_{n=0}^{+\infty} \frac{(n!)^2}{2n!}$.

2-2 Consider the differential equation

$$x(x-4)y' + (x+2)y = 2$$

1) Find a power series $\sum_{n=0}^{+\infty} a_n x^n$ solution of the differential equation.
2) Find the general solution for $x \in (-4, 4)$ and deduce the value of the following sum: $\sum_{n=0}^{+\infty} \frac{1}{\binom{2n}{n}} = \sum_{n=0}^{+\infty} \frac{(n!)^2}{2n!}$.

2-3 Find a power series $\sum_{n=0}^{+\infty} a_n x^n$ solution of the following differential equations:

(a) $(1+x^2)y'' + 2xy' - 2y = 0$,

(b) $y'' + xy' + 3y = 0$,

(c) $4xy'' + 2y' - y = 0$,

(d) $y'' - xy = \frac{1}{1-x}$

(e) $y'' + xy' + y = \cos x$,

(f) $4xy'' - 2y' + 9x^2y = \cos x$,

(g) $y'' - 2xy' + y = 0$.

3 Series Solutions Near a Regular Singular Point (Frobenius Method)

Definition 3.1.
A singular point a is called a regular singular point of the differential equation

$$y'' + C(x)y' + D(x)y = 0 \tag{5.6}$$

if the functions $(x-a)C(x)$ and $(x-a)^2D(x)$ are both analytic at a. A singular point that is not regular is said to be an irregular point of the equation. (i.e. one or both of the functions $(x-a)C(x)$ and $(x-a)^2D(x)$ fail to be analytic at a.)

Example 3.1 :
Consider the following differential equation

$$(x^2-4)^2y'' + 3(x-2)y' + 5y = 0.$$

For $x \neq \pm 2$, this equation becomes $y'' + \dfrac{3}{(x-2)(x+3)^2}y' + \dfrac{5}{(x^2-4)^2}y = 0.$
In this case 2 is a regular singular point and -2 is an irregular singular point.

Theorem 3.2. Frobenius' Theorem
If a is a regular singular point of the differential equation (5.6), then there exist two linearly independent solutions y_1 and y_2 such that:

1. either $y_1(x) = (x-a)^{r_1}z_1(x)$ and $y_2(x) = (x-a)^{r_2}z_2(x)$, where z_j are analytic on the interval $(-R, R)$ (and maybe in a larger interval) and $z_j(a) \neq 0$. The functions y_1 and y_2 are defined on the interval (a, R).
2. or $r_1 = r_2 + N$, where N is a non-negative integer, and $y_1(x) = (x-a)^{r_1}z_1(x)$ and $z_2(x) = (x-a)^{r_2}z_2(x) + y_1(x)\ln(x-a)$ where z_j are analytic on the interval $(-R, R)$ (and maybe in a larger interval) and $z_j(a) \neq 0$. The functions y_1 and y_2 are defined on the interval (a, R).

Proof .
Without loss of generality we can, after possibly a change of variable $x - a = t$, assume that $a = 0$. The normalized differential equation

$$y'' + C(x)y' + D(x)y = 0, \quad (x > 0),$$

such that $xC(x)$ and $x^2D(x)$ are analytic at $x = 0$. A necessary and sufficient condition for this is that

$$\lim_{x\to 0} xC(x) = c_0, \quad \lim_{x\to 0} x^2D(x) = d_0$$

exist and are finite. In this case

$$xC(x) = \sum_{n=0}^{+\infty} c_n x^n, \quad x^2 D(x) = \sum_{n=0}^{+\infty} d_n x^n$$

and the given differential equation has the same solutions as the differential equation

$$x^2 y'' + x(xC(x))y' + x^2 D(x) y = 0.$$

This differential equation is an Euler differential equation if $xC(x) = c_0$, $x^2 D(x) = d_0$.

If $y = x^r \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+r}$, with $a_0 \neq 0$ is a solution, we get

$$x^2 y''(x) = \sum_{n=0}^{+\infty} (n+r)(n+r-1) a_n x^{n+r}$$

$$x^2 C(x) y'(x) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} c_k (n-k+r) a_{n-k} \right) x^{n+r}$$

$$x^2 D(x) y(x) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} d_k a_{n-k} \right) x^{n+r}.$$

Then $a_0(r(r-1) + c_0 r + d_0) = 0$ and

$$\left[(n+r)(n+r-1) + c_0(n+r) + d_0\right] a_n + \sum_{k=1}^{n} \left(c_k(n-k+r) + d_k\right) a_{n-k} = 0$$

for all $n \in \mathbb{N}$. Since $a_0 \neq 0$, y is a solution if and only if:

$$F(r) = r(r-1) + c_0 r + d_0 = 0 \tag{5.7}$$

and

$$F(n+r) a_n = -\sum_{k=1}^{n} \left(c_k(n-k+r) + d_k\right) a_{n-k}. \tag{5.8}$$

The polynomial F is called the indicial polynomial of the differential equation and the equation (5.7) is called the indicial equation of the differential equation.

Remark 13 :

1. The roots r_1 and r_2 of the indicial polynomial can be complex numbers.

2. If the roots r_1 and r_2 of the indicial polynomial F verifies $\mathrm{Re}(r_1 - r_2) \geq 0$, then $F(n + r_1) \neq 0$ for all $n \in \mathbb{N}$. Indeed $F(n + r_1) = (n + r_1)^2 + (c_0 - 1)(n + r_1) + d_0 = n(n + 2r_1 + c_0 - 1) = n(n + (r_1 - r_2)) \neq 0$. Therefore we can solve the recurrence relation

$$F(n + r_1)a_n + \sum_{k=1}^{n} (c_k(n - k + r_1) + d_k)\, a_{n-k} = 0$$

for a_n in terms of $a_1, \dots, a_{n-1}, c_k$ and d_k, $k = 1, \dots, n$. Hence there exists a solution

$$y_1 = x^{r_1} \sum_{n=0}^{+\infty} a_n x^n$$

of the differential equation (5.6). A second linearly independent solution can then be found by reduction of order.

3. If $F(n + r) \neq 0$ for all $n \in \mathbb{N}$, the recursion equation (5.8) can be solved. Let $a_n(r)$ be the solution with $a_0(r) = 1$ and define

$$y = y(x, r) = x^r \left(\sum_{n=0}^{\infty} a_n(r) x^n \right).$$

Then, we have the following equality with two variables (x, r):

$$x^2 y'' + x^2 C(x) y' + x^2 D(x) y = x^r (r - r_1)(r - r_2). \tag{5.9}$$

3.1 First case 1: $r_1 - r_2 \notin (\mathbb{N} \cup \{0\})$

In this case, as mentioned in the previous remark, the recursive equation (5.8) determines a_n uniquely for $r = r_1$ and $r = r_2$. For $a_0 = 1$, we obtain two linearly independent solutions

$$y_1 = x^{r_1} \left(\sum_{n=0}^{\infty} a_n(r_1) x^n \right), \quad y_2 = x^{r_2} \left(\sum_{n=0}^{\infty} a_n(r_2) x^n \right).$$

3.2 Second case 2: $r_1 = r_2$

From the equality (5.9), we get

$$x^2 y'' + x^2 C(x) y' + x^2 D(x) y = x^r (r - r_1)^2.$$

Differentiating this equation with respect to r at $r = r_1$, we get

$$x^2\left(\frac{\partial y}{\partial r}\right)^{''}+x^2C(x)\left(\frac{\partial y}{\partial r}\right)^{'}+x^2D(x)\frac{\partial y}{\partial r}=0.$$

We find that the second linearly independent solution

$$\begin{aligned} y_2=\frac{\partial y}{\partial r}(x,r_1) &= x^{r_1}\ln(x)\sum_{n=0}^{\infty}a_n(r_1)x^n+x^{r_1}\sum_{n=0}^{\infty}a_n'(r_1)x^n \\ &= y_1\ln(x)+x^{r_1}\sum_{n=0}^{\infty}a_n'(r_1)x^n, \end{aligned}$$

where $a_n'(r)$ is the derivative of $a_n(r)$ with respect to r.

3.3 Third case 3: $r_1 - r_2 = N \in \mathbb{N}$

Let $z(x,r)=(r-r_2)y(x,r)$. From the equality (5.9), we have

$$x^2z^{''}+x^2C(x)z'+x^2D(x)z=(r-r_1)(r-r_2)^2x^r.$$

Differentiating this equation with respect to r at $r=r_2$, we get

$$x^2\left(\frac{\partial z}{\partial r}\right)^{''}+x^2C(x)\left(\frac{\partial z}{\partial r}\right)^{'}+x^2D(x)\frac{\partial z}{\partial r}=0.$$

Then the function $y_2=\frac{\partial z}{\partial r}(x,r_2)=\lim\limits_{r\to r_2}\frac{z(x,r)}{r-r_2}$ is a solution of the given differential equation. If $b_n(r)=(r-r_2)a_n(r)$, we have

$$F(n+r)b_n(r)+\sum_{k=0}^{n-1}\big[(k+r)c_{n-k}+d_{n-k}\big]b_k(r)$$

and

$$y_2=\lim_{r\to r_2}\left(x^r\ln(x)\sum_{n=0}^{\infty}b_n(r)x^n+x^r\sum_{n=0}^{\infty}b_n'(r)x^n\right). \tag{5.10}$$

This method is due to Frobenius and is called the **Frobenius method**.

Example 3.2 :
Consider the differential equation:

$$y^{''}+\frac{1}{x}(x-\frac{1}{2})y'+\frac{1}{2x^2}y=0$$

around $x = 0$. $xC(x) = x - \dfrac{1}{2}$ and $x^2 D(x) = \frac{1}{2} = d_0$.
The indicial equation of the differential equation is $2r^2 - 3r + 1 = 0$. Then $r_1 = 1, r_2 = \frac{1}{2}$. Moreover, $(n + r - \dfrac{1}{2})(n + r - 1)a_n = -(n - 1 + r)a_{n-1}$. Then $a_n(r) = a_0(-1)^n \displaystyle\prod_{k=1}^{n} \frac{1}{k + r - \frac{1}{2}}$.

$$y_1 = \sqrt{x}\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n = \sqrt{x}e^{-x}, \quad x > 0$$

and

$$y_2 = \sum_{n=0}^{+\infty} \frac{(-1)^n 4^n n!}{(2n+1)!} x^{n+1}, \quad x > 0$$

are two linearly independent solutions of the differential equation.

Example 3.3 :
Consider the differential equation:

$$x(1-x)y^{''} + (1-x)y' - y = 0$$

around $x = 0$. $xC(x) = 1 = c_0$ and $x^2 D(x) = -\frac{x}{(1-x)}$.
$c_0 = 1$, $c_n = 0$ for all $n \geq 1$. $x^2 D(x) = -\frac{x}{1-x} = -\sum_{j=1}^{+\infty} x^n$.
The indicial equation of the differential equation is $r^2 = 0$. Then $r_1 = r_2 = 0$.
The recursive equation is: $(n + r)^2 a_n = \displaystyle\sum_{k=1}^{n} a_{n-k}$. Then $a_1 = \dfrac{1}{(1+r)^2} a_0$ and for $n \geq 2$, $(n + r - 1)^2 a_{n-1} = \sum_{k=2}^{n} a_{n-k}$. Then $a_n = \dfrac{1 + (n + r - 1)^2}{(n+r)^2} a_{n-1}$.
If we choose $a_0 = 1 + r^2$, we have for all $n \in \mathbb{N}$, $a_n = \displaystyle\prod_{k=1}^{n} \frac{1 + (k + r - 1)^2}{(k + r)^2}$.

$$y = x^r(1 + r^2) + x^r \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \frac{1 + (k + r - 1)^2}{(k + r)^2} x^n.$$

For $r = 0$, we have the first solution

$$y_1 = 1 + \sum_{n=1}^{+\infty} \left(\prod_{k=1}^{n} \frac{1 + (k-1)^2}{(k)^2} \right) x^n = 1 + \sum_{n=1}^{+\infty} \left(\prod_{k=0}^{n-1} (1 + k^2) \right) \frac{x^n}{(n!)^2}.$$

$$\frac{\partial a_n}{\partial r} = \prod_{k=1}^{n} \frac{1 + (k + r - 1)^2}{(k + r)^2} \left(\sum_{k=1}^{n} \frac{2(k + r - 1)}{1 + (k + r - 1)^2} - \frac{2(k + r)}{(k + r)^2} \right)$$

The second solution

$$y_2 = y_1 \ln x + \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} \left(\prod_{k=1}^{n-1} (1+k^2) \right) \left(\sum_{k=1}^{n} \frac{2(k-2)}{k(k^2-2k-2)} \right) x^n.$$

Example 3.4 :
Consider the differential equation:

$$xy^{''} + 2y' - y = 0$$

around $x = 0$.
$xC(x) = 2 = c_0$ and $x^2 D(x) = -x$.
The indicial equation of the differential equation is $r(r+1) = 0$. Then $r_1 = 0, r_2 = -1$. Moreover, $(n+r)(n+r+1)a_n = a_{n-1}$. Then $a_n(r) = \prod_{k=1}^{n} \frac{1}{(r+k)(r+k+1)} a_0$.

$$y = x^r \left(1 + \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \frac{1}{(r+k)(r+k+1)} x^n \right).$$

The first solution is:

$$y_1 = \sum_{n=0}^{+\infty} \frac{1}{(n!)^2 (n+1)} x^n, \quad x > 0.$$

Consider the function

$$z = (r+1)y(x,r) = (r+1)x^r \left(1 + \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \frac{1}{(r+k)(r+k+1)} x^n \right).$$

$$\begin{aligned} y_2 &= \frac{\partial z}{\partial r}(-1) \\ &= y_1 \ln x + \frac{1}{x} \left(1 - \sum_{n=1}^{+\infty} \frac{1}{n!(n-1)!} \left(-\frac{1}{n} + 2 \sum_{k=1}^{n} \frac{1}{k} \right) x^n \right), \quad x > 0. \end{aligned}$$

Example 3.5 :
The differential equation $2xy'' + y' + 2xy = 0$ has a regular singular point at $x = 0$ since $xC(x) = \frac{1}{2}$ and $x^2 D(x) = x^2$. The indicial equation is

$$r(r-1) + \frac{1}{2} r = r(r - \frac{1}{2}).$$

The roots are $r_1 = \frac{1}{2}$, $r_2 = 0$ which do not differ by an integer. We have

$$\begin{cases} (r+1)(r+\frac{1}{2})a_1 & = & 0, \\ (n+r)(n+r-\frac{1}{2})a_n & = & -a_{n-2} \quad \text{for } n \geq 2, \end{cases}$$

so that $a_n = -2\dfrac{a_{n-2}}{(n+r)(2n+2r-1)}$ for $n \geq 2$. Hence $a_{2n+1} = 0$ for $n \geq 0$ and $a_{2n} = -\dfrac{2}{(2n+r)(4n+2r-1)}a_{2(n-1)}$.

Setting, $r = \frac{1}{2}$ and $r = 0$, $a_0 = 1$, we get

$$y_1 = \sqrt{x}\sum_{n=0}^{+\infty} \frac{x^{2n}}{n!\prod_{k=1}^{n}(4k+1)}, \quad y_2 = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!\prod_{k=1}^{n}(4k-1)}.$$

The infinite series have an infinite radius of convergence since $x = 0$ is the only singular point of the differential equation.

Example 3.6 :

The differential equation $xy'' + y' + y = 0$ has a regular singular point at $x = 0$ with $xC(x) = 1$, $x^2D(x) = x$. The indicial equation is

$$r(r-1) + r = r^2 = 0.$$

This equation has only one root $x = 0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is $a_n(r) = \dfrac{(-1)^n}{(r+1)^2(r+2)^2\cdots(r+n)^2}$. For $r = 0$, we have the first solution

$$y_1 = \sum_{n=0}^{+\infty}(-1)^n \frac{x^n}{(n!)^2}.$$

Taking the logarithmic derivative of $a_n(r)$ with respect to r we get:
$\dfrac{d}{dr}\ln(a_n(r)) = \dfrac{a_n'(r)}{a_n(r)}$. Then

$$a_n'(r) = \left(\frac{2}{r+1} + \frac{2}{r+2} + \cdots + \frac{2}{r+n}\right) a_n(r).$$

Then $a_n'(0) = 2(-1)^n \dfrac{c_n}{(n!)^2}$, where $c_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Therefore a second linearly independent solution is

$$y_2 = y_1 \ln(x) + 2\sum_{n=1}^{+\infty} \frac{(-1)^n c_n}{(n!)^2} x^n.$$

The above series converge for all x. Any bounded solution of the given differential equation must be a scalar multiple of y_1.

Example 3.7 : [Bessel's Equation]
Consider the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν a non negative real number. This differential equation is known as the **Bessel's equation of order ν.**
$x = 0$ is the unique regular singular point.
The Bessel's equation can also be written $y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$. The indicial equation is $r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0$ whose roots are $r_1 = \nu$ and $r_2 = -\nu$. The recursion equations are $\left[(1+r)^2 - \nu^2\right]a_1 = 0$, $\left[(n+r)^2 - \nu^2\right]a_n = -a_{n-2}$, for $n \geq 2$. The general solution of these equations is $a_{2n+1} = 0$ for $n \geq 0$ and

$$a_{2n}(r) = \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}.$$

• If ν is not an integer and $\nu \neq \frac{1}{2}$, $(\nu - (-\nu) \neq 0)$. There is two linearly independent solutions of the Bessel's equation $J_\nu(x)$, $J_{-\nu}(x)$ can be obtained by taking $r = \pm\nu$ and $a_0 = \frac{1}{2^\nu}\Gamma(\nu + 1)$. In this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(r+1)(r+2)\cdots(r+n)},$$

For $r = \pm\nu$,

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+r}.$$

These functions are called Bessel functions of first kind of order ν.
• If $\nu = \frac{1}{2}$, $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$, $\quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\cos(x)$.
• For $m = 0$, the first solution is

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

In this case, the indicial equation has a repeated root, the second solution is of the form

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} a'_{2n}(0)x^{2n},$$

where $a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\cdots(r+2n)^2}$. It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n}\right)$$

so that $a'_{2n}(0) = (c_n)\, a_{2n}(0) = h_n a_{2n}(0)$, where $c_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Hence

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} \frac{(-1)^n h_n}{2^{2n}(n!)^2} x^{2n}.$$

- If $\nu = -m$, with $m \in \mathbb{N}$, the first solution is

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left(\frac{x}{2}\right)^{2n+m}.$$

The second solution has the form

$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^{\infty} b'_{2n}(-m)x^{2n+m}$$

where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = b_{2m}(-m)$.

3.4 Exercises

3-1

Solutions of Exercises on Chapter 1

1-1 $y' = \sqrt{|y|}$, $y(0) = 0$.

1-2 $(y')^2 + (y)^2 = -1$.

1-3 Since f is continuous, then f is $\mathcal{C}^1$ and by iteration f is C^∞ on $\mathbb{R}$. Moreover by differentiation of the condition on f, we have $f'(x) = f(x)$, then $f(x) = \lambda e^x$. The condition $\displaystyle\int_0^x f(t)dt + 1 = f(x)$ yields that $f(x) = e^x$.

1-4 $y(x) = (1-x)^3$ is a solution of the differential equation. However, this solution is not unique. Consider for any $c \geq 1$ the continuously differentiable function

$$y(x) = \begin{cases} (1-x)^3 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } 1 \leq x \leq c, \\ (c-x)^3 & \text{for } x \geq c \end{cases}$$

is also a solution of the initial value problem. We can also take $y = 0$ for all $x \geq 1$.

2-1

1) Let $M = x^2 + 3y^2$ and $N = 2xy$. $\dfrac{\partial M}{\partial x} = 2x = \dfrac{\partial N}{\partial y}$. The differential equation $(x^2 + 3y^2)y' + 2xy = 0$ is exact.
$\dfrac{\partial F}{\partial x} = 2xy$, then $F = xy^2 + f(y)$ and $\dfrac{\partial F}{\partial y} = x^2 + 3y^2 = 2xy + f'(y)$. Then $f'(y) = 3y^2$ and $F(x, y) = xy^2 + y^3$. The set of solutions of the differential equation $(x^2 + 3y^2)y' + 2xy = 0$ is $\{(x, y) \in \mathbb{R}^2 : xy^2 + y^3 = c;\ c \in \mathbb{R}\}$.

2) Let $M = 2y + xe^y$ and $N = e^y$. $\dfrac{\partial M}{\partial x} = e^y = \dfrac{\partial N}{\partial y}$. The differential equation $(2y + xe^y)y' + e^y = 0$ is exact.
$\dfrac{\partial F}{\partial x} = e^y$, then $F = xe^y + f(y)$ and $\dfrac{\partial F}{\partial y} = xe^y = xe^y + f'(y)$. Then $f'(y) = 0$ and $F(x, y) = xe^y$. The set of solutions of the differential equation $(2y + xe^y)y' + e^y = 0$ is $\{(x, y) \in \mathbb{R}^2 : xe^y = c;\ c \in \mathbb{R}\}$.

3) Let $M = 2y\sqrt{x^2 - y^2}$ and $N = -(1 + 2x\sqrt{x^2 - y^2})$. $\dfrac{\partial M}{\partial x} = \dfrac{2xy}{\sqrt{x^2 - y^2}} = \dfrac{\partial N}{\partial y}$. The differential equation $(2y\sqrt{x^2 - y^2})y' - (1 + 2x\sqrt{x^2 - y^2}) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = -(1 + 2x\sqrt{x^2 - y^2})$, then $F = -x - \dfrac{2}{3}(x^2 - y^2)^{\frac{3}{2}} + f(y)$

and $\dfrac{\partial F}{\partial y} = 2y\sqrt{x^2-y^2} = 2y\sqrt{x^2-y^2} + f'(y)$. Then $f'(y) = 0$ and $F(x,y) = -x - \dfrac{2}{3}(x^2-y^2)^{\frac{3}{2}} + f(y)$. The set of solutions of the differential equation $(2y\sqrt{x^2-y^2})y' - (1+2x\sqrt{x^2-y^2}) = 0$ is $\{(x,y) \in \mathbb{R}^2 : x + \dfrac{2}{3}(x^2-y^2)^{\frac{3}{2}} = c;\ c \in \mathbb{R}\}$.

4) Let $M = 6x^2 - y + 3$ and $N = 12xy - \sin x$. $\dfrac{\partial M}{\partial x} = 12x = \dfrac{\partial N}{\partial y}$. The differential equation $(6x^2 - y + 3)y' + (12xy - \sin x) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = 12xy - \sin x$, then $F = 6x^2y + \cos x + f(y)$ and $\dfrac{\partial F}{\partial y} = 6x^2 - y + 3 = 6x^2 + f'(y)$. Then $f'(y) = 3 - y$ and $F(x,y) = 6x^2y + \cos x + 3y - \frac{1}{2}y^2$. The set of solutions of the differential equation $(6x^2 - y + 3)y' + (12xy - \sin x) = 0$ is $\{(x,y) \in \mathbb{R}^2 : 6x^2y + \cos x + 3y - \frac{1}{2}y^2 = c;\ c \in \mathbb{R}\}$.

5) Let $M = \dfrac{2x}{y^3}$ and $N = \dfrac{2}{x} - \dfrac{1}{y^2}$. $\dfrac{\partial M}{\partial x} = \dfrac{2}{y^3} = \dfrac{\partial N}{\partial y}$. The differential equation $(\dfrac{2x}{y^3})y' + (\dfrac{2}{x} - \dfrac{1}{y^2}) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = \dfrac{2}{x} - \dfrac{1}{y^2}$, then $F = \ln x^2 - \frac{x}{y^2} + f(y)$ and $\dfrac{\partial F}{\partial y} = \dfrac{2x}{y^3} = \dfrac{2x}{y^3} + f'(y)$. Then $f'(y) = 0$ and $F(x,y) = \ln x^2 - \frac{x}{y^2}$. The set of solutions of the differential equation $(\dfrac{2x}{y^3})y' + (\dfrac{2}{x} - \dfrac{1}{y^2}) = 0$ is $\{(x,y) \in \mathbb{R}^2 : \ln x^2 - \frac{x}{y^2} = c;\ c \in \mathbb{R}\}$.

6) Let $M = y^2 + x$ and $N = x^2 + y$. $\dfrac{\partial M}{\partial x} = 1 = \dfrac{\partial N}{\partial y}$. The differential equation $(y^2 + x)y' + x^2 + y = 0$ is exact.
$\dfrac{\partial F}{\partial x} = x^2 + y$, then $F = \frac{1}{3}x^3 + xy + f(y)$ and $\dfrac{\partial F}{\partial y} = x + y^2 = x + f'(y)$. Then $f'(y) = y^2$ and $F(x,y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3$. The set of solutions of the differential equation $(y^2 + x)y' + x^2 + y = 0$ is $\{(x,y) \in \mathbb{R}^2 : x^3 + 3xy + y^3 = c;\ c \in \mathbb{R}\}$.

7) Let $M = 3x^2y + y^3$ and $N = x^2 + 3xy^2$. $\dfrac{\partial M}{\partial x} = 6xy = \dfrac{\partial N}{\partial y}$. The differential equation $(3x^2y + y^3)y' + (x^2 + 3xy^2) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = x^2 + 3xy^2$, then $F = \frac{1}{3}x^3 + \frac{3}{2}x^2y^2 + f(y)$ and $\dfrac{\partial F}{\partial y} = 3x^2y + y^3 = 3x^2y + f'(y)$. Then $f'(y) = y^3$ and $F(x,y) = \frac{1}{3}x^3 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4$. The set of solutions of the differential equation $(3x^2y + y^3)y' + (x^2 + 3xy^2) = 0$ is $\{(x,y) \in \mathbb{R}^2 : 4x^3 + 18x^2y^2 + 3y^4 = c;\ c \in \mathbb{R}\}$.

2-2 1) Let $M = x^3 + y^3$ and $N = 3x^2y$. $\dfrac{\partial M}{\partial x} = 3x^2 = \dfrac{\partial N}{\partial y}$. The differential equation $(x^3 + y^3)y' + 3x^2y = 0$ is exact.
$\dfrac{\partial F}{\partial x} = 3x^2y$, then $F = x^3y + f(y)$ and $\dfrac{\partial F}{\partial y} = x^3 + y^3 = x^3 + f'(y)$. Then $f'(y) = y^3$ and $F(x, y) = x^3y + \frac{1}{4}y^4$. The set of solutions of the differential equation $(x^3 + y^3)y' + 3x^2y = 0$ is $\{(x, y) \in \mathbb{R}^2 : 4x^3y + \frac{1}{4}y^4 = c;\ c \in \mathbb{R}\}$.

2) Let $M = y^2 - x^2$ and $N = x^2 - y^2$. $\dfrac{\partial M}{\partial x} = -2x$, $\dfrac{\partial N}{\partial y} = -2y$. The differential equation $(y^2 - x^2)y' + (x^2 - y^2) = 0$ is not exact.

3) Let $M = xe^{xy}$ and $N = ye^{xy}$. $\dfrac{\partial M}{\partial x} = (1 + xy)e^{xy} = \dfrac{\partial N}{\partial y} = -2y$. The differential equation $xe^{xy}y' + ye^{xy} = 0$ is exact.
$\dfrac{\partial F}{\partial x} = ye^{xy}$, then $F = e^{xy} + f(y)$ and $\dfrac{\partial F}{\partial y} = xe^{xy} = xe^{xy} + f'(y)$. Then $f'(y) = 0$ and $F(x, y) = e^{xy}$. The set of solutions of the differential equation $xe^{xy}y' + ye^{xy} = 0$ is $\{(x, y) \in \mathbb{R}^2 : xy = c;\ c \in \mathbb{R}\}$.

4) Let $M = x^2$ and $N = -2xy$. $\dfrac{\partial M}{\partial x} = 2x$, $\dfrac{\partial N}{\partial y} = -2x$. The differential equation $x^2y' - 2xy = 0$ is not exact.

5) Let $M = x$ and $N = x^3 + y$. $\dfrac{\partial M}{\partial x} = 1 = \dfrac{\partial N}{\partial y} = -2y$. The differential equation $xy' + (x^3 + y) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = x^3 + y$, then $F = \frac{1}{4}x^4 + xy + f(y)$ and $\dfrac{\partial F}{\partial y} = x = x + f'(y)$. Then $f'(y) = 0$ and $F(x, y) = \frac{1}{4}x^4 + xy$. The set of solutions of the differential equation $xy' + (x^3 + y) = 0$ is $\{(x, y) \in \mathbb{R}^2 : x^4 + 4xy = c;\ c \in \mathbb{R}\}$.

6) Let $M = \cos x$ and $N = -y \sin x - e^x$. $\dfrac{\partial M}{\partial x} = -\sin x = \dfrac{\partial N}{\partial y}$. The differential equation $\cos xy' = (y \sin x + e^x) = 0$ is exact.
$\dfrac{\partial F}{\partial x} = -(y \sin x + e^x)$, then $F = y \cos x - e^x + f(y)$ and $\dfrac{\partial F}{\partial y} = \cos x = \cos x + f'(y)$. Then $f'(y) = 0$ and $F(x, y) = y \cos x - e^x$. The set of solutions of the differential equation $\cos xy' = (y \sin x + e^x) = 0$ is $\{(x, y) \in \mathbb{R}^2 : y \cos x - e^x = c;\ c \in \mathbb{R}\}$.

7) Let $M = \tan(x + y)$ and $N = \tan(x + y) - 1$. $\dfrac{\partial M}{\partial x} = \sec^2(x + y) = \dfrac{\partial N}{\partial y}$. The differential equation $y' \tan(x + y) = 1 - \tan(x + y)$ is

exact.
$\dfrac{\partial F}{\partial x} = \tan(x+y) - 1$, then $F = \ln|\sec(x+y)| - x + f(y)$ and $\dfrac{\partial F}{\partial y} = \tan(x+y) - 1 = \tan(x+y) - 1 + f'(y)$. Then $f'(y) = 0$ and $F(x,y) = \ln|\sec(x+y)| - x$. The set of solutions of the differential equation $y^{'}\tan(x+y) = 1 - \tan(x+y)$ is $\{(x,y) \in \mathbb{R}^2 : \ln|\sec(x+y)| - x = c;\ c \in \mathbb{R}\}$.

2-3 1) Let $M = \dfrac{x^2}{y}$ and $N = 2x$. $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial y}\right) = \dfrac{1}{y}$.
$g(y) = y$ is an integrating factor. The differential equation becomes, $x^2y' + 2xy = 0$.
Let $M_1 = x^2$ and $N_1 = 2xy$. $\dfrac{\partial M_1}{\partial x} = frm{-}ex = \dfrac{\partial N_1}{\partial y}$. The differential equation $x^2y' + 2xy = 0$ is exact.
$\dfrac{\partial F}{\partial x} = 2xy$, then $F = x^2y + f(y)$ and $\dfrac{\partial F}{\partial y} = x^2 = x^2 + f'(y)$. Then $f'(y) = 0$ and $F(x,y) = x^2y$. The set of solutions of the differential equation $\dfrac{x^2}{y}y' + 2x = 0$ is $\{(x,y) \in \mathbb{R}^2 : x^2y = c;\ c \in \mathbb{R}\}$.

2) $y = 0$ is a solution of the differential equation. For $y \neq 0$, let $M = y^4 + 3x$ and $N = -y$. $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial x} - \dfrac{\partial N}{\partial y}\right) = -\dfrac{4}{y}$.
$g(y) = \dfrac{1}{y^4}$ is an integrating factor. The differential equation becomes for $y \neq 0$, $(1 + 3\dfrac{x}{y^4})y' - \dfrac{1}{y^3} = 0$. This differential equation is exact.
$\dfrac{\partial F}{\partial x} = -\dfrac{1}{y^3}$, then $F = -\dfrac{x}{y^3} + f(y)$ and $\dfrac{\partial F}{\partial y} = 1 + 3\dfrac{x}{y^4} = 3\dfrac{x}{y^4} + f'(y)$. Then $f'(y) = 1$ and $F(x,y) = y - \frac{x}{y^3}$. The set of solutions of the differential equation $(y^4 + 3x)y' - y = 0$ for $y \neq 0$ is $\{(x,y) \in \mathbb{R}^2 : y - \frac{x}{y^3} = c;\ c \in \mathbb{R}\}$.

3) Let $M = x$ and $N = -y - x^2\sin x$. $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial x}\right) = -\dfrac{2}{x}$.
$g(x) = \dfrac{1}{x^2}$ is an integrating factor. The differential equation becomes for $x \neq 0$, $\frac{1}{x}y^{'} - \frac{y}{x^2} - \sin x$.
$\dfrac{\partial F}{\partial x} = -\dfrac{y}{x^2} - \sin x$, then $F = \dfrac{y}{x} + \cos x + f(y)$ and $\dfrac{\partial F}{\partial y} = \dfrac{1}{x} = \dfrac{1}{x} + f'(y)$. Then $f'(y) = 0$ and $F(x,y) = \frac{y}{x} + \cos x$. The set of

solutions of the differential equation $xy^{'} - y = x^2 \sin x$ for $x \neq 0$ is $\{(x,y) \in \mathbb{R}^2 : \frac{y}{x} + \cos x = c;\ c \in \mathbb{R}\}$.

4) Let $M = 2(\sin(x)+\sin(y))+y\cos(y)$ and $N = y\cos(x)$. $\frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) = \frac{1}{y}$. $g(y) = y$ is an integrating factor. The differential equation becomes, $[2(\sin(x) + \sin(y)) + y\cos(y)]yy' + y^2\cos(x) = 0$, $\frac{\partial F}{\partial x} = y^2 \cos x$, then $F = y^2 \sin x + f(y)$ and $\frac{\partial F}{\partial y} = [2(\sin(x) + \sin(y)) + y\cos(y)]y = 2y\sin x + f'(y)$. Then $f'(y) = 2y\sin y + y^2\cos y$ and $F(x,y) = y^2 \sin x + y^2 \sin y$. The set of solutions of the differential equation $[2(\sin(x) + \sin(y)) + y\cos(y)]y' + y\cos(x) = 0$, is $\{(x,y) \in \mathbb{R}^2 : y^2\sin x + y^2 \sin y = c;\ c \in \mathbb{R}\}$.

5) Let $M = 3x^2+y-y^2$ and $N = x^3-3xy^2+2$. $\frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) = \frac{1}{y}$. $g(y) = y$ is an integrating factor. The differential equation becomes, $[2(\sin(x) + \sin(y)) + y\cos(y)]yy' + y^2\cos(x) = 0$, $\frac{\partial F}{\partial x} = y^2\cos x$, then $F = y^2 \sin x + f(y)$ and $\frac{\partial F}{\partial y} = [2(\sin(x)+\sin(y))+y\cos(y)]y = 2y\sin x + f'(y)$. Then $f'(y) = 2y\sin y + y^2\cos y$ and $F(x,y) = y^2 \sin x + y^2 \sin y$. The set of solutions of the differential equation $[2(\sin(x) + \sin(y)) + y\cos(y)]y' + y\cos(x) = 0$, is $\{(x,y) \in \mathbb{R}^2 : y^2 \sin x + y^2 \sin y = c;\ c \in \mathbb{R}\}$.

3-1

1) $y = 0$ is a solution of the differential equation. If $y \neq 0$, $\frac{dy}{y} = 2xdx$. Then $\ln|y| = x^2 + c$. Therefore, the general solution of the differential equation $y' = 2xy$ is $y = \lambda e^{x^2}$, with $\lambda \in \mathbb{R}$.

2) $y = 0$ is a solution of the differential equation. If $y \neq 0$, $\frac{dy}{y} = x^2dx$. Then $\ln|y| = \frac{1}{3}x^3 + c$. Therefore, the general solution of the differential equation $y' = x^2y$ is $y = \lambda e^{\frac{1}{3}x^3}$, with $\lambda \in \mathbb{R}$.

3) The differential equation $(1+x^2)y' = 1$ is equivalent to $y' = \frac{1}{1+x^2}$, then $y = \tan^{-1}(x) + c$ is the general solution of the differential equation.

4) $y = 0$ is a solution of the differential equation.
$$\frac{1}{x^3-1} = \frac{1}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right).$$
For $y \neq 0$, $\ln|y| = \frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$

5) $\dfrac{1}{x^3-1} = \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{1}{3}\dfrac{1}{x-1} - \dfrac{1}{3}\dfrac{x+2}{x^2+x+1}$.
$\displaystyle\int \frac{dx}{x^3-1} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{3\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c.$
$y = 0$ is a solution of the differential equation. For $y \neq 0$,
$\ln|y| = \dfrac{1}{3}\ln|x-1| - \dfrac{1}{6}\ln(x^2+x+1) - \dfrac{1}{3\sqrt{3}}\tan^{-1}(\dfrac{2x+1}{\sqrt{3}}) + c.$

6) $y = 0$ and $y = 2$ are a singular solutions. For $x \neq 0$, $y \neq 0$ and $y \neq 2$, the equation becomes: $\dfrac{y'}{y(y-2)} = \dfrac{1}{2}\left(\dfrac{y'}{y-2} - \dfrac{y'}{y}\right) = \dfrac{1}{x^2}$.
Then $\ln\left|\dfrac{y}{y-2}\right| = \dfrac{1}{x} + c$. We get $\dfrac{2}{|}y| = 1 + \lambda e^{-\frac{1}{x}}$.

7) $y = \pm 1$ are a singular solutions. For $x \neq 0$ and $|y| < 1$, the equation becomes: $\dfrac{y'}{\sqrt{1-y^2}} = \dfrac{1}{x}$. Then $\sin^{-1}(y) = \ln|x| + c$ and $y = \sin(\ln|x| + c)$.

8) $y = 1$ is a singular solution. For $x \neq -1$ and $y \neq 1$, the equation becomes: $\dfrac{y'}{(y-1)^2} = \dfrac{1}{(x+1)^2}$. Then $\dfrac{1}{y-1} = \dfrac{1}{x+1} + c$.

9) $y = -1$ is a singular solution. For $y \neq -1$, the equation becomes: $\dfrac{y'}{1+y} = \dfrac{1}{4+x^2}$. Then $\ln|y+1| = \dfrac{1}{2}\tan^{-1}(\dfrac{x}{2}) + c$ and $y = \lambda e^{\frac{1}{2}\tan^{-1}(\frac{x}{2})}$, $\quad \lambda \in \mathbb{R}$.

10) The equation $y'x\tan y = -1$, is a separable equation. For $x \neq 0$, the equation fulfills the following implicit equation $\dfrac{1}{\cos y} = \ln|x| + c$.

11) $y' = xy + x + y + 1 \iff y' = (x+1)(y+1)$. Then $y = -1 + \lambda e^{\frac{1}{2}(x+1)^2}$, with $\lambda \in \mathbb{R}$.

12) $y' = 3yx^2 - 3x^2 \iff y' = 3x^2(y-1)$. The function $z = y - 1$ fulfills the differential equation $z' = 3x^2 z$. Then $z = \lambda e^{x^3}$ and $y = \lambda e^{x^3} + 1$, with $\lambda \in \mathbb{R}$. Since $y(0) = 2$, $y = e^{x^3} + 1$.

13) $y' = \dfrac{1}{3y^2+1} \iff (3y^2+1)y' = 1$. Then $y^3 + y = x + c$. Since $y(0) = 1$, then $y^3 + y = x + 2$.

14) In this case $y' = \dfrac{y^2}{x}$. The domain of the function $f(x,y) = \dfrac{y^2}{x}$ is $\Omega = \mathbb{R}^2 \setminus \{(0,y);\ y \in \mathbb{R}\}$.
$y = 0$ is a solution of the differential equation $xy' = y^2$ and defined on $\mathbb{R}$.

For $y \neq 0$, $\dfrac{1}{y} = \ln|x| + c$. Then $y = \dfrac{-1}{\ln|x|+c}$. Since $y(1) = 1$, $y = \frac{1}{1-\ln|x|}$. (This solution is defined in a neighborhood of 1.)

15) $\dfrac{dy}{y^2-1} = xdx$. Then $\dfrac{1}{2}x^2 + c = \dfrac{1}{2}\ln\left|\dfrac{1-y}{1+y}\right|$, which is equivalent to $\dfrac{1-y}{1+y} = \lambda e^{x^2}$ or $y = \dfrac{\lambda e^{x^2}-1}{\lambda e^{x^2}+1}$. Since $y(0) = 0$, $y = \dfrac{e^{x^2}-1}{e^{x^2}+1}$. (This solution is defined in a neighborhood of 0.)

16) $\ln|y| = -\cos(x) + c$, then $y = \lambda e^{-\cos x}$. Since $y(0) = 1$, then $y = e^{1-\cos x}$.

17) $xy' = y(1+2x^2)$.
$y = 0$ is a solution. For $x \neq 0$ and $y \neq 0$, $\ln|y| = \ln|x| + x^2 + c$. Then $y = \lambda x e^{x^2}$, with $\lambda \in \mathbb{R}$. Since $y(1) = 1$, $y = xe^{x^2-1}$.

18) $\tan^{-1}(y) = \tan^{-1}(x) + c$. Since $y(0) = 1$, $\tan^{-1}(y) = \tan^{-1}(x) + \frac{\pi}{4}$.

19) $y' = xe^{-y} \iff y'e^y = x$, then $e^y = \frac{1}{2}x^2 + c$. Since $y(0) = 1$, $e^y = \frac{1}{2}x^2 + e$.

20) For $x \neq 0$, $xy' = e^{-y}$ is equivalent to $y'e^y = \dfrac{1}{x}$. Since $y(1) = 1$, $e^y = \ln|x| + e$.

21) The solutions of the differential equation $y' = \dfrac{\sin(x)}{\cos(y)}$ fulfills $\sin(y) = -\cos(x) + c$.

22) The solutions of the differential equation $y' = \dfrac{x}{y}$ fulfills $y^2 = x^2 + c$.

3-2 The function $f(x,y) = 1+\cos(y)$ is $\mathcal{C}^\infty$, then there is a unique solution for the Cauchy problem. Moreover, the equation is equivalent to: $y' = 2\cos^2(\dfrac{y}{2})$. For $y \neq (2k+1)\pi$, for $k \in \mathbb{Z}$, the equation is also equivalent to $\dfrac{1}{2}y'\sec^2(\dfrac{y}{2}) = 1$ or $(\tan(\frac{y}{2}))' = 1$.

1) The unique solution of the differential equation (1.9) for $a = 3\pi$ is $y = 3\pi$

2) The solution of the differential equation (1.9) for $a = 0$ is $y = 2\tan^{-1}(x)$.

3-3 1) $y' = \frac{1}{y^2 \ln x}$, $\quad y(2) = 0 \iff y^2 dy = \frac{dx}{\ln x}$, $\quad y(2) = 0 \iff y^3 = 3\displaystyle\int_2^x \frac{dt}{\ln t}$.

2) $y' = \frac{y}{x}e^x, \quad y(1) = 1 \iff y\,dy = \frac{e^x dx}{x}, \quad y(1) = 1 \iff y^2 = 2\int_1^x \frac{e^t dt}{t}$.

3) $y = -1$ is a solution. For $y \neq -1$, the equation becomes $y' - \dfrac{y'}{y+1} = x$. Then $y - \ln|y+1| = \frac{x^2}{2} - 2$.

4) For all $k \in \mathbb{Z}$, $y = \dfrac{\pi}{2} + k\pi$ is a solution of the equation.
For $y \neq \dfrac{\pi}{2} + k\pi$, the equation becomes $\dfrac{y'}{\cos^2 y} = \sin x$. Then $\tan y = 1 - \cos x$.
If $y \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi) = ((k+1)\pi - \frac{\pi}{2}, \frac{\pi}{2} + (k+1)\pi)$, $y = (k+1)\pi + \tan^{-1}(1 - \cos x)$.

3-4

1) $y' = \frac{1}{y^2 \ln x}, \quad y(2) = 0 \iff y^2 dy = \frac{dx}{\ln x}, \quad y(2) = 0 \iff y^3 = 3\int_2^x \frac{dt}{\ln t}$.

2) $y' = \frac{y}{x}e^x, \quad y(1) = 1 \iff \frac{1}{y}dy = \frac{e^x dx}{x}, \quad y(1) = 1 \iff y^2 = 2\int_1^x \frac{e^t dt}{t}$.

4-1

1) For the equation $y' = \dfrac{2y - x}{y + 4x}$, we set $y = xz$. The equation becomes: $xz' = \dfrac{2z-1}{z+4} - z = -\dfrac{(z+1)^2}{z+4}$. After integration, we get: $\dfrac{3}{z+1} - \ln|z+1| = \ln|x| + c$ or $\dfrac{3x}{y+x} - \ln|y+x| = c$.

2) This equation is homogeneous. We set $y = xz$. The equation is equivalent to $xz' + z = \dfrac{2z}{1-z^2}$. Then $z'\dfrac{1-z^2}{z(1+z^2)} = \dfrac{1}{x}$. After integration we get $\dfrac{z}{1+z^2} = \lambda x$ or $x^2 + y^2 = \alpha y$, $\alpha \in \mathbb{R}$.
We can also take $x = r\cos\theta$ and $y = r\sin\theta$. The equation is equivalent to $\dfrac{dr}{r} = \dfrac{d\theta}{\tan\theta} \Rightarrow \ln r = \ln\sin\theta + c$ or $r = \lambda\sin\theta$.

3) For $y = xz$, the equation $xyy' - y^2 = \sqrt{x^2 - y^2}$ becomes: $x^3 zz' = \sqrt{x^2(1-z^2)}$. For $x > 0$, we get $\dfrac{zz'}{\sqrt{1-z^2}} = \dfrac{1}{x^2}$. Then $\sqrt{1-z^2} = \dfrac{1}{x} + c$.

4) For $y = xz$, the equation $y^{'} = \dfrac{x^2 - y^2}{5xy}$ becomes: $-\frac{5}{12}\frac{-12zz'}{1-6z^2} = \frac{1}{x}$. Then $\ln|1 - 6z^2| = -\dfrac{12}{5}\ln|x| + c$.

5) For $y = xz$, the equation $xy^{'} = y + xe^{\frac{y}{x}}$ becomes: $e^{-z}z' = \dfrac{1}{x}$ for $x \neq 0$. Then $e^{-z} = -\ln|x| + c$

6) For $y = xz$, the equation $xy' - y = \sqrt{x^2 + y^2}$ becomes: $\dfrac{z'}{\sqrt{1+z^2}} = \dfrac{1}{x}$, for $x > 0$, then $\ln(z + \sqrt{1+z^2}) = \cosh^{-1}(z) = \ln x + c$, for $x > 0$.

7) Let $z = 2y - 3$ and $t = 2x + 4$, the equation $y' = \dfrac{3 - 2y}{2x + 2y + 1}$ becomes: $z' = \dfrac{-2z}{z + t}$. Let $z = tw$, the equation becomes: $tw' + w = -2\frac{w}{1+w}$, which is equivalent to $\frac{2w'}{1+3w} - \frac{w'}{w} = \frac{1}{t}$, hence $\frac{2}{3}\ln|1 + 3w| - \ln|w| = \ln|t|$.

5-1 If $z = y - 2x$, then $z' = y' - 2 = z^2 + 1 - 2$. Hence $z' = z^2 - 1$, $\ln\left|\dfrac{1-z}{1+z}\right| = x + c$ and $z = \dfrac{\lambda e^x - 1}{\lambda e^x + 1}$, with $\lambda \in \mathbb{R}$. The solutions of the differential equation are given by

$$y = 2x + \frac{\lambda e^x - 1}{\lambda e^x + 1},$$

where $\lambda \in \mathbb{R}$.

5-2 The straight lines $1 - 4x - 4y = 0$, and $x + y = 0$ are parallel, hence for $z = x + y$, we get $z' - 1 = \dfrac{1 - 4z}{z}$ or $z' = \dfrac{1 - 3z}{z}$. We deduce that the solutions of the differential equation are given by

$$\frac{x + y}{3} + \frac{1}{9}\ln|1 - 3x - 3y| + x = c,$$

where $c \in \mathbb{R}$.

5-3 Let $z = xy$ or $y = \frac{z}{x}$ then $xy' + y = z'$ and $z' = \dfrac{z}{x}\left(\dfrac{1+z}{1-z} + 1\right)$. The function z fulfills the differential equation $z' = \frac{2z}{x(1-z)}$. Then $y - \lambda x e^{xy} = 0$, where $\lambda \in \mathbb{R}^*$.

6-1 1) The general solution of the homogeneous equation $y' - xy = 0$ is $y = \lambda e^{\frac{1}{2}x^2}$, $\lambda \in \mathbb{R}$. $y = -1$ is a particular solution. Then the general solution of the equation is $y = \lambda e^{\frac{1}{2}x^2} - 1$, $\lambda \in \mathbb{R}$.

2) The general solution of the homogeneous equation $y' - y = 0$ is $y = \lambda e^x$, $\lambda \in \mathbb{R}$. $y = \dfrac{1}{2}(x+1)e^x - \dfrac{1}{4}e^{-x}$ is a particular solution. Then the general solution of the equation is $y = \lambda e^x + \dfrac{1}{2}(x+1)e^x - \dfrac{1}{4}e^{-x}$, $\lambda \in \mathbb{R}$.

3) The general solution of the homogeneous equation $y' + 2y = 0$ is $y = \lambda e^{-2x}$, $\lambda \in \mathbb{R}$. $y = \dfrac{1}{3}e^x$ is a particular solution. Then the general solution of the equation is $y = \lambda e^{-2x} + \dfrac{1}{3}e^x$, $\lambda \in \mathbb{R}$.

4) The general solution of the homogeneous equation $xy' + 2y = 0$ is $y = \dfrac{\lambda}{x^2}$, $\lambda \in \mathbb{R}$.
Using the variation of the constant method, $y = \frac{\lambda}{x^2}$, with $\lambda' = x\cos x = x\sin x + \cos x + c$. Then the general solution of the equation is $y = \frac{\lambda}{}x^2 + \frac{x\sin x + \cos x}{x^2}$, with $\lambda \in \mathbb{R}$.

5) The general solution of the homogeneous equation $y' = 2xy$ is $y = \lambda e^{x^2}$, $\lambda \in \mathbb{R}$.
Using the variation of the constant method, $y = \lambda e^{x^2}$, with $\lambda' = e^{-x^2}$. Then the general solution of the equation is $y = \lambda e^{x^2} + e^{x^2}\displaystyle\int_0^x e^{-t^2}dt$, with $\lambda \in \mathbb{R}$.

6) The general solution of the homogeneous equation $y' + y\tan x = 0$ is $y = \lambda\cos x$, $\lambda \in \mathbb{R}$.
Using the variation of the constant method, $y = \lambda\cos x$, with $\lambda' = 2\sin x$. Then the general solution of the equation is $y = \lambda\cos x - 2\cos^2 x$, with $\lambda \in \mathbb{R}$.

7) The general solution of the homogeneous equation $xy' + y = 0$ is $y = \dfrac{\lambda}{x}$. $y = -\dfrac{1}{4}x^3$ is a particular solution. Then $y = -\dfrac{1}{4}x^3 + \dfrac{\lambda}{x}$ is the general solution of the equation $xy' + y + x^3 = 0$.

8) The general solution of the homogeneous equation $y' + 2xy = 0$ is $y = \lambda e^{-x^2}$. By the variation of parameter method, the general solution of the equation $y' + 2xy = e^{-x}$ is $y = \lambda e^{-x^2} + e^{-x^2}\int_0^x e^{t^2-t}dt$.

9) The equation $y'\cos x = (y\sin x + e^x)$ is equivalent to $\frac{d}{dx}(y\cos x) = e^x$. Then the general solution is: $y = \dfrac{\lambda}{\cos x} + \dfrac{e^x}{\cos x}$.

10) The equation $y' = e^{2x+3y}$ is equivalent to: $y'e^{-3y} = e^{2x}$, then the general solution fulfills $-\dfrac{1}{3}e^{-3y} = \dfrac{1}{2}e^{2x} + c$.

11) The general solution of the homogeneous equation is: $y = \lambda x$. The general solution is: $y = \lambda x - x\cos x$.

6-2

1) $xy' + 2y = 0 \iff y = \dfrac{\lambda}{x^2}$. The variation of the constant method yields that $\lambda' = \frac{x^2}{1+x^2}$, then $y = +\dfrac{c}{x^2} - \dfrac{x - \tan^{-1} x}{x^2}$.

2) Since $\lim\limits_{x\to 0} \dfrac{x - \tan^{-1} x}{x^2} = 0$, then the only solution of the differential equation on $\mathbb{R}$ is $y = \dfrac{x - \tan^{-1} x}{x^2}$.

6-3

1) i. $\dfrac{x^2 - 1}{x(1+x^2)} = -\dfrac{1}{x} + \dfrac{2x}{1+x^2}$.
The solutions of the homogeneous equation are:

$$y = \lambda\frac{1+x^2}{x} = \lambda x + \frac{\lambda}{x}, \qquad \lambda \in \mathbb{R}.$$

ii. $h_1(x) = \dfrac{1+x^2}{2x}$ and $h_2(x) = \dfrac{1+x^2}{2x}$.

2) i. By the variation of constant method, we look for a particular solution in the form $\lambda(x)\dfrac{1+x^2}{x}$.
$\lambda'(x) = -\dfrac{2x}{(1+x^2)^2}$, then $\lambda(x) = \dfrac{1}{(1+x^2)}$.

ii. The function $f(x) = \dfrac{1}{x}$ is a particular solution of the equation, on I_1 or on I_2, and the functions $g_1(x) = \frac{1}{x}$ and $g_2(x) = \frac{1}{x}$.

3) f is continuous at 0 if and only if $\lambda = \mu = -2$ and $\alpha = 0$. Then $f(x) = -x$ for all $x \in \mathbb{R}$.

4) f is the unique solution of (1.18) on $\mathbb{R}$.

6-4 1) i. We have $-\dfrac{1}{x}+\dfrac{2x}{1+x^2}=\dfrac{x^2-1}{x(1+x^2)}$.
On the interval I_1 or I_2, the homogenous equation is equivalent to

$$\frac{dy}{y}=-\frac{dx}{x}+\frac{2xdx}{1+x^2}.$$

Then the general solution of the homogeneous equation on I_1 or on I_2 is $y=\dfrac{\lambda(1+x^2)}{|x|}$.

ii. $h_1(x)=\dfrac{1+x^2}{2x}$ and $h_2(x)=\dfrac{1+x^2}{2x}$.

2) i. By the variation of the constant method, we look for a particular solution in the form $\lambda(x)\dfrac{(1+x^2)}{x}$ with λ a differentiable function. We give $\lambda'(x)=-\dfrac{2x}{(1+x^2)^2}$, then $\lambda(x)=\dfrac{1}{1+x^2}$.
$g_1(x)=\frac{1}{x}$ on I_1.

ii. $g_2(x)=\frac{1}{x}$ on I_2.

3) We have:

$$\begin{cases} f(x) & =\dfrac{1}{x}+\dfrac{\lambda(1+x^2)}{2x}=\dfrac{2+\lambda}{2x}+\dfrac{\lambda x}{2} \quad \text{if } x \quad <0 \\ f(x) & =\dfrac{1}{x}+\dfrac{\mu(1+x^2)}{x}=\dfrac{2+\mu}{2x}+\dfrac{\mu x}{2} \quad \text{if } x \quad >0 \end{cases}$$

4) If $\lambda\neq-2$, $\lim\limits_{x\to 0^-}|f(x)|=+\infty$ diverges at 0^- and if $\mu\neq-2$, $\lim\limits_{x\to 0^+}|f(x)|=+\infty$ diverges at 0^-.

5) f is continuous at 0 if and only if $\lambda=\mu=-2$.

6) In this case $f(x)=-x$ for all $x\in\mathbb{R}$ and it is easy to prove that this function is the unique solution of the equation (1.20) on $\mathbb{R}$.

6-5 1) The general solution of the homogeneous equation is $y=\dfrac{\lambda}{x^2}$. $y=\frac{1}{3}x$ is a particular solution. Then the general solution of the differential equation $xy'+2y=x$ is $y=\frac{1}{3}x+\dfrac{\lambda}{x^2}$, $\lambda\in\mathbb{R}$.

2) The general solution of the homogeneous equation is $y=\lambda\sec x$. Using the variation of the constant method, we get $\lambda'=x$. Then the general solution of the differential equation $y'-y\tan x=\dfrac{x}{\cos x}$, is $y=\dfrac{x^2}{2}+\lambda\sec x$, $\lambda\in\mathbb{R}$. If $y(0)=0$, then $y=\dfrac{x^2}{2\cos x}$.

3) The general solution of the homogeneous equation is $y = \dfrac{\lambda}{x+1}$. Using the variation of the constant method, we get $\lambda' = x(x+1)^2$. Then the general solution of the differential equation $y' + \dfrac{y}{x+1} = x^2 + x$ is $y = \dfrac{\lambda}{x+1} + \dfrac{x}{3}(x+1)^2 - \dfrac{1}{12}(x+1)^3$.

4) The general solution of the homogeneous equation is $y = \lambda(\csc x - \cot x)$. Using the variation of the constant method, we get $\lambda' = \cos x + \cos^2(x)$. Then the general solution of the differential equation $y' + \sin(x)y = \cos(x)\sin(x)$ is $y = \lambda(\csc x - \cot x) + (1 + \dfrac{x}{2}\cos x + \dfrac{1}{2}\sin x \cos x)(1 - \cos x)$.

5) The general solution of the homogeneous equation is $y = \lambda e^{7x}$. $-\dfrac{3}{7}x^2 + \dfrac{22}{7^2}x + \dfrac{22}{7^3}$ is a particular solution. Then the solution of the differential equation $y' - 7y = 3x^2 - 4x, \quad y(0) = 0$ is $y = \dfrac{22}{7^3}(e^{7x} - 1) - \dfrac{3}{7}x^2 + \dfrac{22}{7^2}x$.

6) The general solution of the homogeneous equation is $y = \dfrac{\lambda}{x^2-1}$. Using the variation of the constant method, we get $\lambda' = 1$. Then the general solution of the differential equation $(x^2-1)y' = 1 - 2xy$ is $y = \dfrac{\lambda}{x^2-1} + \dfrac{1}{x-1}$.

7) The general solution of the homogeneous equation is $y = \dfrac{\lambda}{x^3}$. Using the variation of the constant method, we get $\lambda' = x^3$. Then the solution of the differential equation $3y = x(1-y'), \quad y(1) = \frac{1}{4}$ is $y = \dfrac{x}{4}$.

8) The general solution of the homogeneous equation is $y = \lambda e^{-x}$. Using the variation of the constant method, we get $\lambda' = 1$. Then the general solution of the differential equation $y' + xy - e^{-x} = 0$ is $y = (x+\lambda)e^{-x}$.

6-6 1) $y = 0$ is a solution of the differential equation. For $y \neq 0$, $\dfrac{y'}{y} = \dfrac{1}{x^3-1} = \dfrac{1}{3}\dfrac{1}{x-1} + \dfrac{1}{3}\dfrac{2-x}{x^2+x+1}$. Then

$$\ln|y| = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{5}{3\sqrt{3}}\tan^{-1}(1 + (\frac{2x+1}{\sqrt{3}}) + c.$$

2) The solution of the homogeneous equation is $y = \lambda e^{\frac{x^2}{2}}$. $y = -1$ is a particular solution. Then the general solution of the differential equation $y' - xy = x$ is $y = -1 + \lambda e^{\frac{x^2}{2}}$, $\lambda \in \mathbb{R}$.
3) The solution of the homogeneous equation is $y = \lambda e^x$. Using the variation of the constant method, we get $\lambda' e^x = \cosh x$. Then $\lambda' = \frac{1}{2}(1 + e^{-2x})$. Hence the general solution of the differential equation $y' - y = \cosh x$ is $y = \lambda e^x + (\dfrac{x}{2}e^x - \dfrac{1}{4}e^{-x})$, $\lambda \in \mathbb{R}$.
4) The solution of the homogeneous equation is $y = \lambda e^{-2x}$. Using the variation of the constant method, we get $\lambda' e^{-2x} = e^x$. Then $\lambda' = e^{3x}$. Hence the general solution of the differential equation $y' + 2y = e^x$ is $y = \lambda e^x + \dfrac{1}{3}e^x$, $\lambda \in \mathbb{R}$.
5) The solution of the homogeneous equation is $y = \dfrac{\lambda}{x^2}$. Using the variation of the constant method, we get $\lambda' = x\cos x$. Then the general solution of the differential equation $xy' + 2y = \cos x$ is $y = \dfrac{\lambda}{x^2} + \dfrac{x\sin x + \cos x}{x^2}$, $\lambda \in \mathbb{R}$.
6) The solution of the homogeneous equation is $y = \lambda e^{x^2}$. Using the variation of the constant method, we get $\lambda' = xe^{-x^2}$. Then the general solution of the differential equation $y' = x + 2xy$ is $y = \lambda e^{x^2} - \dfrac{1}{2}$, $\lambda \in \mathbb{R}$.
7) The solution of the homogeneous equation is $y = \lambda \cos x$. Using the variation of the constant method, we get $\lambda' \cos x = 2\sin x \cos x$. Then the general solution of the differential equation $y' + y\tan x = \sin(2x)$ is $y = \lambda \cos x - 2\cos^2 x$, $\lambda \in \mathbb{R}$.
8) The solution of the homogeneous equation is $y = \dfrac{\lambda}{\cos x}$. Using the variation of the constant method, we get $\lambda' = e^x$. Then the general solution of the differential equation $y'\cos x = (y\sin x + e^x)$ is $y = \dfrac{\lambda + e^x}{\cos x}$, $\lambda \in \mathbb{R}$.

6-7

1) $y' + y = 2xy^2$ is a Bernoulli equation. $y = 0$ is a solution. For $y \neq 0$, we set $z = \dfrac{1}{y}$, we get $z' - z = -2x$. $z = 2x + 2$ is a particular solution. Then $y = \dfrac{1}{\lambda e^x + 2x + 2}$ is the general solution of the equation.
2) $y = 0$ is a solution of the differential equation $x^2y' - y^3 = xy$. For $xy \neq 0$, we set $z = \dfrac{1}{y}$. The equation becomes, $x^2z' + 2xz + 2 = 0$.

The general solution of the homogeneous equation $x^2z' + 2xz = 0$ is $z = \dfrac{\lambda}{x^2}$ and by the variation of parameter method, the general solution of the equation $x^2z' + 2xz + 2 = 0$ is $z = \dfrac{\lambda}{x^2} - \dfrac{2}{x}$.

3) If $z = \frac{1}{y}$, then $-xz' + 2z - 1 = 0$.
$z = \frac{1}{2}$ is a particular solution and the general solution of the homogeneous equation is: $z = \lambda x^2$. Then $\dfrac{1}{y} = \lambda x^2 + \dfrac{1}{2}$.

4) For $y \neq 0$, the equation is equivalent to $yy' = \dfrac{y^2}{x} + 1$. The function $z = y^2$ fulfills the linear equation: $z' = 2\dfrac{z}{x} + 2$.
$z = \lambda x^2$ is the general solution of the homogeneous equation $z' = 2\dfrac{z}{x}$ and the general solution fulfills $y^2 = \lambda x^2 - 2x$.

5) $xy' + 2y - y^2 = 0$,

6) $y' = \dfrac{y}{x} + \dfrac{1}{y}$.

6-8 1) $y = x$ is a solution of the differential equation $y^{'} = 1 - x^2 + y^2$. Let $z = y - x$, then the equation is equivalent to $z' - 2xz - z^2 = 0$. Let $u = \frac{1}{z}$, we get $u' - 2xu - 1 = 0$ Hence $u = \lambda e^{x^2} + e^{-x^2} \displaystyle\int_0^x e^{t^2} dt$, $\lambda \in \mathbb{R}$.

2) $y = x^2$ is a solution of the differential equation $xy' - 2y + y^2 = x^4$. Let $z = y - x^2$, then the equation is equivalent to $xz' + 2(x^2 - 1)z + z^2 = 0$. Let $u = \frac{1}{z}$, we get $xu' + 2(x^2 - 1)u + 1 = 0$. Hence $u = \lambda x e^{-x^2} - x e^{-x^2} \displaystyle\int \frac{1}{x^2} e^{x^2} dx$, $\lambda \in \mathbb{R}$.

Solutions of Exercises on Chapter 2

1-1 The Wronskian of this system is $W = e^{3x} \begin{vmatrix} 1 & 1 & x \\ 1 & 2 & 1 \\ 1 & 4 & 0 \end{vmatrix} = (2x - 3)e^{3x}$.

1-2 1) The Wronskian of this system is $W = e^x \begin{vmatrix} 1 & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = e^x(\frac{1}{x} - \ln x)$.

2) Let y be a solution, then the Wronskian of the system $\{y, y_1, y_2\}$ is 0. Then $e^x \begin{vmatrix} y & 1 & \ln x \\ y' & 1 & \frac{1}{x} \\ y'' & 1 & -\frac{1}{x^2} \end{vmatrix} = 0$. Hence y fulfills the following differential equation: $x(1 - x\ln x)y'' + (1 + x^2 \ln x)y' - y(1 + x) = 0$.

1-3 $y = e^x = y' = y''$, then e^x is a solution to $y'' - 2y' + y = 0$. If $y = e^x z$, $y' = e^x z + e^x z'$ and $y'' = e^x z + 2e^x z' + e^x z''$. Hence $y'' - 2y' + y = 0 = e^x z''$. Then $\{e^x, xe^x\}$ is a fundamental set of solutions.

1-4 If $y = (e^{2x} \cos x)z$, $y' = (e^{2x} \cos x)z' + e^{2x}(2\cos x - \sin x)z$ and $y'' = e^{2x} \left(z'' \cos x + 2z'(2\cos x - \sin x) + z(3\cos x - 4\sin x \right)$. Hence $y'' - 4y' + 5y = e^{2x} \left(z'' \cos x - 2z' \sin x \right))$. Then $e^{2x} \cos x$ is a solution, we take $z = 1$ and $y = (e^{2x} \cos x)z$ is solution of the differential equation if and only if, $z'' \cos x - 2z' \sin x = 0$. Then $z' = \lambda \sec^2 x$ and $z = \tan x$. We deduce that $\{e^{2x} \cos x, e^{2x} \sin x\}$ is a fundamental set of solutions.

1-5 Let $y = xz$, $y' = xz' + z$ and $y'' = xz'' + 2z'$. Then $(x-1)(x-2)y'' - xy' + y = x(x-1)(x-2)z'' + 2(2-3x)z'$. Hence $y = x$ is a solution and $y = xz$ is a solution to $(x-1)(x-2)y'' - xy' + y = 0$ if and only if $x(x-1)(x-2)z'' + (x^2 - 6x + 4)z' = 0$. Then $z' = \lambda \dfrac{(x-2)^2}{x^2(x-1)}$ and $z = -\dfrac{4}{x} + \ln|x - 1|$.
The set $\{x, -4 + x\ln|x - 1|\}$ is a fundamental set of solutions.

1-6 We know that $W(x) = W(a)e^{-\int_a^x \frac{dt}{2t}} = W(a)\dfrac{\sqrt{a}}{\sqrt{x}} = \dfrac{c}{\sqrt{x}}$ on any interval which do not contain 0. $W(2) = \frac{c}{2}$.

1-7 $Y_2 = \sqrt{x}$, $y_2' = \frac{1}{2}x^{-\frac{1}{2}}$ and $y_2'' = -\frac{1}{4}x^{-\frac{3}{2}}$. $y_2 y_2'' + (y_2')^2 = -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0$. $a + b\sqrt{x}$ is not in general a solution of this equation. This result doesn't contradict the method of linear superposition since the equation is not linear and homogeneous.

1-8 $W(y_3, y_4) = \begin{vmatrix} Ay_1 + By_2 & Ay_1' + By_2' \\ Cy_1 + Dy_2 & Cy_1' + Dy_2' \end{vmatrix} = (AC - BD)W(y_1, y_2)$. The necessary and sufficient conditions are such that the functions $y_3 = Ay_1 + By_2$ and also $y_4 = Cy_1 + Dy_2$ form a linearly independent set of solutions is $AC - BD \neq 0$.

1-9 1) $y = x, y' = 1$ and $y'' = 0$, then $y = x$ is a solution of the following differential equation $x^2 y'' - (x^2 + 2x)y' + (x + 2)y = 0$.

2) Let $y = xz$, $y' = xz' + z$ and $y^{''} = xz^{''} + 2z'$. Then $x^2y^{''} - (x^2 + 2x)y^{'} + (x+2)y = x^3z^{''} + 2x^2z' - x^2(x+2)z^{'} - (x^2+2x)z + x(x+2)z = x^3(z^{''} - z^{'})$. Then $z = \lambda e^x$ and the general solution of this differential equation is $y = ax + bxe^x$, $a, b \in \mathbb{R}$.

1-10

1) $y = e^x = y' = y^{''}$, then $y = e^x$ is a solution of the following differential equation $(x-1)y^{''} - xy^{'} + y = 0$.

2) Let $y = e^x u$, $y' = e^x u' + e^x u$ and $y^{''} = e^x u^{''} + 2e^x u' + e^x u$. Then $(x-1)y^{''} - xy^{'} + y = (x-1)e^x(u^{''} + 2u' + u) - xe^x(u' + u) + e^x u = e^x((x-1)u^{''} + (x-2)u^{'})$. Then $u = \lambda xe^{-x}$ and the general solution of this differential equation is $ax + be^x$, $a, b \in \mathbb{R}$. The Wronskian of x and e^x is $(x-1)e^x$.

2-1

$y(0) = 9$ and $y(x) + 8\sin x \int_0^x y(t)\cos t dt - 8\cos x \int_0^x y(t)\sin t dt = 9$,

$y'(x) + 8\cos x \int_0^x y(t)\cos t dt + 8\sin x \int_0^x y(t)\sin t dt = 0$, $y'(0) = 0$,

$y^{''} + 9y - 9 = 0$, $y(0) = 9$ and $y'(0) = 0$. Then $y = \frac{9}{2}e^{(3+3\sqrt{2})x} + \frac{9}{2}e^{(3-3\sqrt{2})x}$.

2-2

1) i. $h'(x) = \cos x \int_0^x g(t)\cos t dt + \sin x \int_0^x g(t)\sin t dt$ and $h^{''}(x) = -h(x) + g(x)\cos^2 x + g(x)\sin^2 x = g(x) - h(x)$. Then h is a solution of the differential equation (2.10).

ii. Since $\sin(x-t) = \sin x\cos t - \cos x \sin t$, then $h(x) = \int_0^x g(t)\sin(x-t)dt$.

iii.

$$\begin{aligned} h(x) + h(x+\pi) &= \int_0^x g(t)\sin(x-t)dt - \int_0^{x+\pi} g(t)\sin(x-t)dt \\ &= -\int_x^{x+\pi} g(t)\sin(x-t)dt \\ &\overset{x-t=u}{=} \int_0^{\pi} g(x+u)\sin u du. \end{aligned}$$

$h(x) + h(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$ because $g(x) \geq 0$, $\forall x \in \mathbb{R}$.

2) 1) Let f be a solution of the differential equation (2.10) on $\mathbb{R}$. The function $k = f - h$ fulfills the differential equation $y^{''} + y = 0$. Then there is $a, b \in \mathbb{R}$ such that $f = h + a\cos x + b\sin x$. It results that $f(x) + f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

2) The function $g = F^{''} + F$ is non negative and the function F is a solution of the differential equation (2.10). Then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

2-3 The general solution of the homogenous differential equation is $y = a\cos x + b\sin x$. Using the classical method for solving this differential equation, $y = U\cos x + V\sin x$, we find: $U'\cos x + V'\sin x = 0$ and $-U'\sin x + V'\cos x = \dfrac{1}{3+\cos(2x)}$. Then

$$U = -\frac{1}{2}\tan^{-1}(\cos x) + a, \qquad V = \frac{1}{2\sqrt{2}}\tan^{-1}(\frac{\sin x}{\sqrt{2}}) + b.$$

2-4

1) The characteristic equation is $r^2 + \lambda = 0$.
   - If $\lambda = 0$, the general solutions of this equation is $y = ax + b$, with $a, b \in \mathbb{R}$.
   - If $\lambda > 0$, the general solutions of this equation is $y = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, with $a, b \in \mathbb{R}$.
   - If $\lambda < 0$, the general solutions of this equation is $y = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$, with $a, b \in \mathbb{R}$.
2) To have a solution y of the differential equation $y'' + \lambda y = 0$ such that $y(0) = y(1) = 0$, we must have $\lambda > 0$ and $\lambda = k^2\pi^2$, with $k \in \mathbb{Z}$.

2-5

1) By integration by parts

$$\int e^{\alpha x}\sin^2 x dx = \frac{e^{\alpha x}}{2\alpha} - \frac{e^{\alpha x}}{2(\alpha^2+4)}(\alpha\cos(2x) + 2\sin(2x)) + c$$

and

$$\begin{aligned}\int e^{\alpha x}\sin x\cos x dx &= \frac{e^{\alpha x}}{2}\sin^2(x) + \frac{\alpha}{2}\int e^{\alpha x}\sin^2 x dx \\ &= \frac{e^{\alpha x}}{2}\sin^2(x) + \frac{\alpha e^{\alpha x}}{4} - \frac{\alpha e^{\alpha x}}{4(\alpha^2+4)}(\alpha\cos(2x) + 2\sin(2x)) + c\end{aligned}$$

2) The equation $r^2 - 2kr + (1+k^2)$ is the characteristic equation of the differential equation. Then $e^{(k+1)x}$ and $e^{(k-1)x}$ are solutions linearly independent of the differential equation.
3) Using the variation of the constant method, the general solution of the differential equation has the form $y = Ue^{(k+1)x} + Ve^{(k-1)x}$, with $U'e^{(k+1)x} + V'e^{(k-1)x} = 0$ and $(k+1)U'e^{(k+1)x} + (k-1)V'e^{(k-1)x} = e^x\sin x$. Then

$$U' = \frac{1}{2}e^{-kx}\sin x, \qquad U = \frac{1}{2}e^{-kx}\left(-\frac{1}{1+k^2}\cos x - \frac{k}{1+k^2}\sin x\right) + c_1,$$

$$V' = -\frac{1}{2}e^{(2-k)x}\sin x, \qquad V = \frac{1}{2}e^{(2-k)x}\left(\frac{1}{1+(k-2)^2}\cos x + \frac{2-k}{1+(k-2)^2}\sin x\right) + c$$

2-6 1) The characteristic equation of the differential equation $y''-5y'+6y=0$ is $r^2-5r+6=0=(r-2)(r-3)$. The general solution of the differential equation is $y=ae^{3x}+be^{2x}$, $a,b\in\mathbb{R}$.

2) The characteristic equation of the differential equation $4y''+4y'+y=0$ is $4r^2+4r+1=0=(2r+1)^2$. The general solution of the differential equation is $y=(ax+b)e^{-\frac{1}{2}x}$, $a,b\in\mathbb{R}$.

3) The characteristic equation of the differential equation $y''+y'+y=0$ is $r^2+r+1=0=(r-e^{-\frac{1+i\sqrt{3}}{2}x})(r-e^{\frac{-1+i\sqrt{3}}{2}x})$. The general solution of the differential equation is $y=ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x)+be^{-\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x)$, $a,b\in\mathbb{R}$.

4) The characteristic equation of the differential equation $y''+y'-2y=2x^2-3x+1$ is $r^2+r-2=0=(r-1)(r+2)$. The general solution of the homogeneous differential equation is $y=ae^{x}+be^{-2x}$, $a,b\in\mathbb{R}$. $-\dfrac{1}{2}x^2-\dfrac{3}{2}x-\dfrac{3}{4}$ is a particular solution. Then $y=ae^{x}+be^{-2x}-\frac{1}{2}x^2-\frac{3}{2}x-\frac{3}{4}$, $a,b\in\mathbb{R}$ is the general solution of the differential equation.

5) The characteristic equation of the differential equation $2y''+2y'+3y=x^2+2x-1$ is $2r^2+2r+3=0=(r-e^{-\frac{1+i\sqrt{5}}{2}x})(r-e^{\frac{-1+i\sqrt{5}}{2}x})$. The general solution of the homogeneous differential equation is $y=ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x)+be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)$, $a,b\in\mathbb{R}$.
$\dfrac{1}{3}x^2+\dfrac{2}{9}x-\dfrac{25}{27}$ is a particular solution. Then $y=ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x)+be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)+\frac{1}{3}x^2+\frac{2}{9}x-\frac{25}{27}$, $a,b\in\mathbb{R}$ is the general solution of the differential equation.

6) The characteristic equation of the differential equation $y''-2y'+y=e^{-x}$ is $r^2-2r+1=0=(r-1)^2$. The general solution of the homogeneous differential equation is $y=(ax+b)e^{x}$, $a,b\in\mathbb{R}$.
$\dfrac{1}{4}e^{-x}$ is a particular solution. Then $y=(ax+b)e^{x}+\frac{1}{4}e^{-x}$, $a,b\in\mathbb{R}$ is the general solution of the differential equation.

7) The characteristic equation of the differential equation $y''-y'-2y=x^2e^{-3x}$ is $r^2-r-2=0=(r+1)(r-2)$. The general solution of the homogeneous differential equation is $y=ae^{-x}+be^{2x}$, $a,b\in\mathbb{R}$.
$\dfrac{1}{10}x^2+\dfrac{7}{50}x+\dfrac{39}{500}$ is a particular solution. Then $y=ae^{-x}+be^{2x}+\frac{1}{10}x^2+\frac{7}{50}x+\frac{39}{500}$, $a,b\in\mathbb{R}$ is the general solution of the differential equation.

8) The characteristic equation of the differential equation $y''-2y'+2y=e^{x}+x$ is $r^2-2r+2=0=(r-e^{(1+i)})(r-e^{(1-i)})$. The general solution of the homogeneous differential equation is $y=$

$ae^x\cos(x) + be^x\sin(x)$, $a, b \in \mathbb{R}$.
$e^x + \dfrac{1}{2}(x+1)$ is a particular solution. Then $y = ae^x\cos(x) + be^x\sin(x) + e^x + \frac{1}{2}(x+1)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

9) The characteristic equation of the differential equation $y'' + 4y = \sin(3x)$ is $r^2 + 4 = 0 = (r - 2\mathrm{i})(r + 2\mathrm{i})$. The general solution of the homogeneous differential equation is $y = a\cos(2x) + b\sin(2x)$, $a, b \in \mathbb{R}$.
$\sin(3x)$ is a particular solution. Then $y = a\cos(2x) + b\sin(2x) + \sin(3x)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

10) The characteristic equation of the differential equation $y'' + 4y = \cos(2x) + \cos(4x)$ is $r^2 + 4 = 0 = (r - 2\mathrm{i})(r + 2\mathrm{i})$. The general solution of the homogeneous differential equation is $y = a\cos(2x) + b\sin(2x)$, $a, b \in \mathbb{R}$.
$\dfrac{x}{4}\sin(2x) - \dfrac{1}{12}\cos(4x)$ is a particular solution. Then $y = a\cos(2x) + b\sin(2x) + \frac{x}{4}\sin(2x) - \frac{1}{12}\cos(4x)$, $a, b \in \mathbb{R}$ is the general solution of the differential equation.

11) The characteristic equation of the differential equation $y'' + y = \dfrac{1}{1 + \sin^2 x}$ is $r^2 + 1 = (r + \mathrm{i})(r - \mathrm{i})$.The general solution of the homogeneous differential equation is $y = a\cos(x) + b\sin(x)$, $a, b \in \mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y = U\cos(x) + V\sin(x)$, with $U'\cos(x) + V'\sin(x) = 0$ and $-U'\sin(x) + V'\cos(x) = \dfrac{1}{1 + \sin^2 x}$. Then $U = \dfrac{1}{2\sqrt{2}}\ln\left(\dfrac{\sqrt{2} + \cos(x)}{\sqrt{2} - \cos(x)}\right) + a$ and $V = \tan^{-1}(\sin(x)) + b$

12) The characteristic equation of the differential equation $y'' + 4y' + 5y = \cosh(2x)\cos x$ is $r^2 + 4r + 5 = (r + 2 + \mathrm{i})(r + 2 - \mathrm{i})$. The general solution of the homogeneous differential equation is $y = e^{-2x}\left(a\cos(2x) + b\sin(2x)\right)$, $a, b \in \mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y = Ue^{-2x}\cos(x) + Ve^{-2x}\sin(x)$, with $U'e^{-2x}\cos(x) + V'e^{-2x}\sin(x) = 0$ and $U'e^{-2x}\left(-\sin(x) - 2\cos(x)\right) + V'e^{-2x}\left(\cos(x) - 2\sin(x)\right) = \cosh(2x)\cos(x)$. Then $U = -\dfrac{1}{8}\cos(2x) + \dfrac{1}{20}e^{4x}\sin(2x) - \dfrac{1}{40}e^{4x}\cos(2x) + a$ and $V = \frac{1}{40}e^{4x}\sin(2x) + \frac{1}{20}e^{4x}\cos(2x) + \frac{x}{4} + \frac{1}{8}\sin(2x) + \frac{1}{16}e^{4x} + b$.

13) The characteristic equation of the differential equation $y''-6y'+9y = \sinh^3 x$ is $r^2-6r+9=(r-3)^3$. The general solution of the homogeneous differential equation is $y=(ax+b)e^{3x}$, $a,b\in\mathbb{R}$.
Using the change of parameter method, the general solution of the equation takes the form: $y=(U+xV)e^{3x}$, with $U'+xV'=0$ and $3U'+(1+3x)V'=e^{-3x}\sinh^3 x$. Then $V=\frac{1}{8}\left(x+3e^{-x}-\frac{3}{4}e^{-4x}+\frac{1}{6}e^{-6x}\right)$ and
$U=\frac{1}{8}\left(-\frac{1}{2}x^2-3(1+x)e^{-x}+\frac{3}{16}(1+4x)e^{-4x}-\frac{1}{36}(1+6x)e^{-6x}\right)$.

2-7

1) The general solution of the differential equation : $y''-y=1$ is $y=-1+ae^x+be^{-x}$, $a,b\in\mathbb{R}$.
The general solution of the differential equation : $y''+y=1$ is $y=1+a\cos x+b\sin x$, $a,b\in\mathbb{R}$.

2) The bounded solutions on $\mathbb{R}^+$ of the differential equation : $y''-y=1$ are $y=-1+be^{-x}$, $b\in\mathbb{R}$.
All solutions of the differential equation : $y''+y=1$ are bounded on $\mathbb{R}^+$.

3) The even solutions of the differential equation : $y''-y=1$ are $y=-1+a\cosh(x)$, $a\in\mathbb{R}$.
The even solutions of the differential equation : $y''+y=1$ are $y=1+a\cos x$.

4) Let $y=-1+be^x+ce^{-x}$. $y(0)=0=y(a)$ yields that $b+c=1$ and $be^a+ce^{-a}-1=0$. It results that $c=\frac{1-e^a}{2\sinh(a)}$.
Let $y=1+b\cos x+c\sin x$. $y(0)=0=y(a)$ yields that $b=-1$ and $c\sin a=1-\cos a$.
If $a\in 2\pi\mathbb{Z}$, $b=1$ and $c\in\mathbb{R}$.
If $a\notin 2\pi\mathbb{Z}$, $b=1$ and $c=\tan(\frac{a}{2})$.

2-8

1) The general solution of the differential equation $y''-(\alpha+\beta)y'+\alpha\beta y=0$ is $y=ae^{\alpha x}+be^{\beta x}$, with $a,b\in\mathbb{R}$.

2) The solutions of the differential equation $y''+y=\cos x$ is $y=a\cos x+(b+\frac{x}{2})\sin x$, with $a,b\in\mathbb{R}$.

3) In use of the variation of parameter method, the general solution of the differential equation $y''+y=\frac{1}{3+\cos(2x)}$ is

$$y=a\cos+b\sin x-\frac{1}{2}\tan^{-1}(\cos x)\cos x+\frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+\sin x}{\sqrt{2}-\sin x}\right)\sin x,\qquad a,b\in\mathbb{R}.$$

2-9 1) The general solutions of (5.11) are $y = -1+ae^x+be^{-x}$, with $a, b \in \mathbb{R}$. The general solutions of (5.12) are $y = 1 + a\cos x + b\sin x$, with $a, b \in \mathbb{R}$.
If $y(0) = \alpha, y'(0) = \beta$, the solution of (5.11) is $y = -1 + \frac{\alpha+\beta+2}{2}e^x + \frac{\alpha-\beta}{2}e^{-x}$ and the solution of (5.12) is $y = 1 + (\alpha - 1)\cos x + (\beta - 1)\sin x$.

2) 1) The bounded solutions on $\mathbb{R}^+$ for the differential equation (5.11) are $y = -1 + be^{-x}$, with $b \in \mathbb{R}$ and the solutions of the differential equation (5.12) are all bounded.
   2) The even solutions on $\mathbb{R}$ for the differential equation (5.11) are $y = -1 + a\cosh x$, with $a \in \mathbb{R}$ and the even solutions on $\mathbb{R}$ for the differential equation (5.12) are $y = 1 + a\cos x$, with $a \in \mathbb{R}$.

3) Let $y = -1 + ue^x + ve^{-x}$ be a solution of the differential equation (5.11). $y(0) = y(a) = 0 \iff u+v-1 = 0$ and $-1+ue^a+ve^{-a} = 0$, for $a \neq 0$. This system is Crammer and has a unique solution.
Let $y = 1+u\cos x+v\sin x$ be a solution of the differential equation (5.12). $y(0) = y(a) = 0 \iff u = 1$ and $v\sin a = 1 - \cos a$. This equation has solutions if and only if $a \neq (2n + 1)\pi$, $n \in \mathbb{Z}$.

4) 1) $h'(x) = e^{\lambda x}g(x)$. Since $f' + \lambda f \leq g$, then $h'(x) \geq (f'(x) + \lambda f(x))e^{\lambda x} = (f(x)e^{\lambda x})'$. After integration, we get $f(x) \leq e^{-\lambda x}f(0)+ e^{-\lambda x}h(x)$, $\forall\, x \in \mathbb{R}^+$.
   2)
$\psi(x) = \frac{1}{2}(1 + \varphi(0) + \varphi'(0))e^x + \frac{1}{2}(1 + \varphi(0) - \varphi'(0))e^{-x} - 1.$
Let $f = \varphi' - \varphi$ and $\lambda = 1$, we have $f' + f = \varphi'' - \varphi \leq 1$, then using question a), we get

$$\varphi'(x) - \varphi(x) \leq -e^{-x}(1 + \varphi(0) - \varphi'(0)) + 1.$$

Also using the same question with $\lambda = -1$ and $g = -e^{-x}(1+\varphi(0)-\varphi'(0)) + 1$, we find

$$\varphi(x) \leq \frac{1}{2}(1+\varphi(0)+\varphi'(0))e^x+\frac{1}{2}(1+\varphi(0)-\varphi'(0))e^{-x}-1 = \psi(x), \quad \forall\, x \in \mathbb{R}^+.$$

5) $\varphi'(x) = e^{-x}$, $\varphi''(x) = -e^{-x}$. Then $\varphi'' + \varphi = 1 - 2e^{-x} \leq 1$.
$\psi(x) = -\cos x + \sin x + 1$.
$\varphi(x) - \psi(x) = -e^{-x} + \cos x - \sin x$.
$\varphi(2n\pi) - \psi(2n\pi) = 1 - e^{-2n\pi} \geq 0$ and $\varphi(\frac{\pi}{2} + 2n\pi) - \psi(\frac{\pi}{2} + 2n\pi) = -1 - e^{-2n\pi} \leq 0$.

2-10

1) If $\lambda > 0$, the real general solution of the equation is $y = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, with $a, b \in \mathbb{R}$.
If $\lambda < 0$, the real general solution of the equation is $y = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$, with $a, b \in \mathbb{R}$.
If $\lambda = 0$, the real general solution of the equation is $y = ax + b$, with $a, b \in \mathbb{R}$.

2) There exists a non zero real solution of the equation such that $y(0) = y(1) = 0$ on ly for $\lambda = (k\pi)^2$, with $k \in \mathbb{N}$.

2-11

1)

$$\begin{aligned}\int e^{\alpha x}\sin^2 x dx &= \frac{1}{2}\int e^{\alpha x}(1-\cos(2x))dx \\ &= \frac{1}{2\alpha}e^{\alpha x} + \frac{e^{\alpha x}}{\alpha^2+4}(\alpha\cos(2x) + 2\sin(2x)) + c,\end{aligned}$$

with $c \in \mathbb{R}$.
By integration by parts, $\int e^{\alpha x}\sin^2 x dx = \frac{e^{\alpha x}}{\alpha}\sin^2 x - \frac{2}{\alpha}\int e^{\alpha x}\sin x\cos x dx$.
Then

$$\int e^{\alpha x}\sin x\cos x dx = \frac{e^{\alpha x}}{2}\sin^2 x - \frac{1}{4}e^{\alpha x} - \frac{\alpha e^{\alpha x}}{2(\alpha^2+4)}(\alpha\cos(2x) + 2\sin(2x)) + c$$

2) $y_1 = e^{kx}\cos x$ and $y_2 = e^{kx}\sin x$ are linearly independent solutions of the differential equation: $y'' - 2ky' + (k^2+1)y = 0$.
In use of the variation of constants method, the general solution of the equation is $y = Ue^{kx}\cos x + Ve^{kx}\sin x$, with
$U'e^{kx}\cos x + V'e^{kx}\sin x = 0$ and $U'(e^{kx}(k\cos x - \sin x) + V'e^{kx}(k\sin x + \cos x) = e^x\sin x$. Then $U' = -e^{(1-k)x}\sin^2 x$ and $V' = e^{(1-k)x}\sin x\cos x$.
If $k = 1$, $e^x\sin x$, $U = -\frac{x}{2} + \frac{1}{4}\sin(2x) + c_1$ and $V = -\frac{1}{4}\cos(2x) + c_2$.
If $k \neq 1$, $U = -\frac{1}{2(1-k)}e^{(1-k)x} - \frac{e^{(1-k)x}}{(1-k)^2+4}((1-k)\cos(2x) + 2\sin(2x)) + c_1$ and $V = \frac{e^{(1-k)x}}{2}\sin^2 x - \frac{1}{4}e^{(1-k)x} - \frac{(1-k)e^{(1-k)x}}{2((1-k)^2+4)}((1-k)\cos(2x) + 2\sin(2x)) + c_2$

2-12

1) The general solution of this homogeneous equation is $y = Ae^{3x} + Be^{2x}$, with $A, B \in \mathbb{R}$.

2) The general solution of this homogeneous equation is $y = (Ax + B)e^{-\frac{1}{2}x}$, with $A, B \in \mathbb{R}$.

3) The general solution of this homogeneous equation is
$y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x)$, with $A, B \in \mathbb{R}$.

4) The general solution of the homogeneous equation is $y = Ae^{x} + Be^{-2x}$, with $A, B \in \mathbb{R}$.
The polynomial $P = -x^2 + \frac{1}{2}x - \frac{5}{4}$ is a particular solution of the equation $y'' + y' - 2y = 2x^2 - 3x + 1$.
Then the general solution of this equation is: $y = Ae^{x} + Be^{-2x} - x^2 + \frac{1}{2}x - \frac{5}{4}$, with $A, B \in \mathbb{R}$.

5) The general solution of the homogeneous equation is $y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x)$, with $A, B \in \mathbb{R}$.
The polynomial $P = -\frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{9}$ is a particular solution of the equation $2y'' + 2y' + 3y = x^2 + 2x - 1$. Then the general solution of this equation is: $y = Ae^{-\frac{1}{2}x}\cos(\frac{\sqrt{5}}{2}x) + Be^{-\frac{1}{2}x}\sin(\frac{\sqrt{5}}{2}x) - \frac{1}{3}x^2 + \frac{2}{9}x - \frac{25}{9}$, with $A, B \in \mathbb{R}$.

6) The general solution of the homogeneous equation is $y = (Ax+B)e^{x}$, with $A, B \in \mathbb{R}$.
$\frac{1}{4}e^{-x}$ is a particular solution of the equation $y'' - 2y' + y = e^{-x}$.
Then the general solution of this equation is: $y = (Ax+B)e^{x} + \frac{1}{4}e^{-x}$, with $A, B \in \mathbb{R}$.

7) The general solution of the homogeneous equation is $y = Ae^{-x} + Be^{2x}$, with $A, B \in \mathbb{R}$.
$e^{-3x}(\frac{1}{10}x^2 + \frac{7}{50}x + \frac{39}{500})$ is a particular solution of the equation $y'' - y' - 2y = x^2e^{-3x}$. Then the general solution of this equation is: $y = Ae^{-x} + Be^{2x} + e^{-3x}(\frac{1}{10}x^2 + \frac{7}{50}x + \frac{39}{500})$, with $A, B \in \mathbb{R}$.

8) The general solution of the homogeneous equation is $y = Ae^{x}\cos x + Be^{x}\sin x$, with $A, B \in \mathbb{R}$.
$e^{x} + \frac{1}{2}x + \frac{1}{2}$ is a particular solution of the equation $y'' - 2y' + 2y = e^{x} + x$. Then the general solution of this equation is: $y = Ae^{x}\cos x + Be^{x}\sin x + e^{x} + \frac{1}{2}x + \frac{1}{2}$, with $A, B \in \mathbb{R}$.

9) The general solution of the homogeneous equation is $y = A\cos(2x) + B\sin(2x)$, with $A, B \in \mathbb{R}$.
$-\frac{1}{5}\sin(3x)$ is a particular solution of the equation $y'' + 4y = \sin(3x)$.

Then the general solution of this equation is: $y = A\cos(2x) + B\sin(2x) - \dfrac{1}{5}\sin(3x)$, with $A, B \in \mathbb{R}$.

10) The general solution of the homogeneous equation is $y = A\cos(2x) + B\sin(2x)$, with $A, B \in \mathbb{R}$.
$-\dfrac{1}{12}\cos(4x) + \dfrac{1}{4}\sin(2x)$ is a particular solution of the equation $y'' + 4y = \cos(2x) + \cos(4x)$. Then the general solution of this equation is: $y = A\cos(2x) + B\sin(2x) - \dfrac{1}{12}\cos(4x) + \dfrac{1}{4}\sin(2x)$, with $A, B \in \mathbb{R}$.

11) The general solution of the homogeneous equation is $y = A\cos x + B\sin x$, with $A, B \in \mathbb{R}$.
Using the variation of constants method, the general solution of the equation is written as $y = U\cos x + V\sin x$, with

$$U' = \frac{-\sin x}{1+\sin^2 x}, \quad V' = \frac{\cos x}{1+\sin^2 x}.$$

Then $U = \dfrac{1}{2\sqrt{2}}\ln\left|\dfrac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}\right| + A$ and $V = \tan^{-1}(\sin x) + B$
Then the general solution of the equation is

$$y = A\cos x + B\sin x + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}\right|\cos x + (\sin x)\tan^{-1}(\sin x),$$

with $A, B \in \mathbb{R}$.

12) The general solution of the homogeneous equation is $y = Ae^{-2x}\cos x + Be^{-2x}\sin x$, with $A, B \in \mathbb{R}$.
Using the variation of constants method, the general solution of the equation is written as $y = Ue^{-2x}\cos x + Ve^{-2x}\sin x$, with

$$U' = -\frac{1}{2}e^{4x}\cosh(2x)\sin(2x), \quad V' = e^{4x}\cosh(2x)\cos^2 x = \frac{1}{4}(e^{6x}+e^{2x})(1+\cos(2x)).$$

Then

$$U = -\frac{1}{80}e^{6x}\left(-\cos(2x) + 3\sin(2x)\right) - \frac{1}{32}e^{2x}\left(-2\cos(2x) + \sin(2x)\right) + A$$

and

$$V = -\frac{1}{24}\left(e^{6x} + 3e^{2x}\right) - \frac{1}{80}e^{6x}\left(3\cos(2x) + \sin(2x)\right) - \frac{1}{16}e^{2x}\left(\cos(2x) + \sin(2x)\right) + B$$

13) The general solution of the homogeneous equation is $y = (Ax + B)e^{3x}$, with $A, B \in \mathbb{R}$.
The differential equation is $y''-6y'+9y = \sinh^3 x = \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)$.
$-\dfrac{3}{32}e^x - \dfrac{3}{128}e^{-x} - \dfrac{1}{288}e^{-3x} + \dfrac{1}{48}xe^{3x}$ is a particular solution of the equation. Then the general solution of the equation is

$$y = (Ax + B)e^{3x} - \frac{3}{32}e^x - \frac{3}{128}e^{-x} - \frac{1}{288}e^{-3x} + \frac{1}{48}xe^{3x},$$

with $A, B \in \mathbb{R}$.

2-13 The general solution of the homogenous equation is

$$y_c = (ax + b)e^x, \quad a, b \in \mathbb{R}$$

Let $z = e^{-x}y$, $y^{''} - 2y' + y = e^x z^{''} = e^x(x + \cos x) \iff z^{''} = x + \cos x$.
Then $y_p = (\dfrac{x^3}{6} - \cos x)e^x$ and $y = e^x(ax + b + \dfrac{x^3}{6} - \cos x)$.

2-14 The general solution of the homogenous equation is

$$y_c = e^{-\frac{1}{2}x}(a\cos(\frac{\sqrt{3}}{2}x) + b\sin(\frac{\sqrt{3}}{2}x)), \quad a, b \in \mathbb{R}$$

$y_p = \sin x$ and

$$y = e^{-\frac{1}{2}x}(a\cos(\frac{\sqrt{3}}{2}x) + b\sin(\frac{\sqrt{3}}{2}x)) + \sin x, \quad a, b \in \mathbb{R}.$$

2-15 The general solution of the homogenous equation is

$$y_c = ae^x + be^{2x}, \quad a, b \in \mathbb{R}$$

$y_p = -xe^x + (\frac{1}{2}x^2 - x)e^{2x}$ and

$$y = (a - x)e^x + (\frac{1}{2}x^2 - x + b)e^{2x}, \quad a, b \in \mathbb{R}.$$

2-16 1) Let $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:

$$h(x) = \sin x \int_0^x g(t)\cos t dt - \cos x \int_0^x g(t)\sin t dt.$$

i. $h'(x) = \sin x \int_0^x g(t)\sin t dt + \cos x \int_0^x g(t)\cos t dt = \int_0^x g(t)\cos(x-t)dt$. $h''(x) = g(x) - h(x)$, which proves that h fulfills the equation (2.13).

ii. $h(x) = \int_0^x g(t)(\sin x \cos t - \cos x \sin t)dt = \int_0^x g(t)\sin(x-t)dt$.

iii.

$$\begin{aligned} h(x) + h(x+\pi) &= \int_0^x g(t)\sin(x-t)dt - \int_0^{x+\pi} g(t)\sin(x-t)dt \\ &= -\int_x^{x+\pi} g(t)\sin(x-t)dt \\ &\overset{t=s+x}{=} \int_0^{\pi} g(s+x)\sin(s)ds. \end{aligned}$$

Since $g \geq 0$, $h(x) + h(x+\pi) \geq 0$.

2) i. The general solution of the homogenous equation is $y = a\cos x + b\sin x$. By the variation of constant method any solution f of (2.13) on $\mathbb{R}$ has the form

$$f(x) = a\cos x + b\sin x + \sin x \int_0^x g(t)\cos t dt - \cos x \int_0^x g(t)\sin t dt = a\cos x + b\sin$$

which proves that $f(x) + f(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

ii. Let $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^2$ such that $F''(x) + F(x) \geq 0$. We denote $g = F'' + F$. Since $g \geq 0$ and F is a solution of the differential equation $y'' + y = g$ with $g \geq 0$, then $F(x) + F(x+\pi) \geq 0$, $\forall x \in \mathbb{R}$.

2-17 Let E_k be the vector space of complex functions of class $\mathcal{C}^k$ defined on $\mathbb{R}$ and let $\alpha \neq \beta \in \mathbb{C}$. We consider the linear map $D\colon E_2 \longrightarrow E_0$ defined by

$$D(y) = y" - (\alpha+\beta)y' + \alpha\beta y.$$

1) α and β are solutions of the characteristic equation. Then the kernel of D is vector space generated by $e^{\alpha x}$ and $e^{\beta x}$.

2) If $\alpha = \mathrm{i} = -\beta$, $D(y) = y" + y$. The solutions of $y" + y = e^{\mathrm{i}x}$ are $\{ae^{\mathrm{i}x} + be^{-\mathrm{i}x} - \frac{\mathrm{i}x}{2}e^{\mathrm{i}x},\quad a, b \in \mathbb{C}\}$.

3) The general solutions of the homogenous equations are $ae^{\mathrm{i}x} + be^{-\mathrm{i}x}$, with $a, b \in \mathbb{C}$. By the variation of the constant, $\frac{1}{2}\tan^{-1}(\cos x)e^{\mathrm{i}x} + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+\sin x}{\sqrt{2}-\sin x}\right)e^{-\mathrm{i}x}$ is a particular solution.

2-18 We consider the following differential equations

$$y" - y = 1 \quad (5.11)$$

and

$$y" + y = 1 \quad (5.12)$$

1) The general solutions of (5.11) are $y = -1 + ae^x + be^{-x}$, with $a, b \in \mathbb{R}$. The general solutions of (5.12) are $y = 1 + a\cos x + b\sin x$, with $a, b \in \mathbb{R}$.
   If $y(0) = \alpha, y'(0) = \beta$, the solution of (5.11) is $y = -1 + \frac{\alpha+\beta+2}{2}e^x + \frac{\alpha-\beta}{2}e^{-x}$ and the solution of (5.12) is $y = 1 + (\alpha - 1)\cos x + (\beta - 1)\sin x$.
2) a) the bounded solutions on $\mathbb{R}^+$ for the differential equation (5.11) are $y = -1 + be^{-x}$, with $b \in \mathbb{R}$ and the solutions of the differential equation (5.12) are all bounded.
   b) the even solutions on $\mathbb{R}$ for the differential equation (5.11) are $y = -1 + a\cosh x$, with $a \in \mathbb{R}$ and the even solutions on $\mathbb{R}$ for the differential equation (5.12) are $y = 1 + a\cos x$, with $a \in \mathbb{R}$.
3) Let $y = -1 + ue^x + ve^{-x}$ be a solution of the differential equation (5.11). $y(0) = y(a) = 0 \iff u + v - 1 = 0$ and $-1 + ue^a + ve^{-a} = 0$, for $a \neq 0$. This system is Crammer and has a unique solution.
   Let $y = 1 + u\cos x + v\sin x$ be a solution of the differential equation (5.12). $y(0) = y(a) = 0 \iff u = 1$ and $v\sin a = 1 - \cos a$. This equation has solutions if and only if $a \neq (2n+1)\pi$, $n \in \mathbb{Z}$.
4) a) Let $\lambda \in \mathbb{R}$, f and g two differentiable functions on $\mathbb{R}^+$ such that $f' + \lambda f \leq g$. We set $h(x) = \displaystyle\int_0^x e^{\lambda t} g(t)dt$.
   Compute the differential of h in term of the function $x \longmapsto e^{\lambda x} f(x)$.
   Deduce that $\forall\ x \in \mathbb{R}^+$, $f(x) \leq e^{-\lambda x} f(0) + e^{-\lambda x} h(x)$.
   b) Let φ be function twice differentiable on $\mathbb{R}^+$ such that

$$\forall t \in \mathbb{R}^+, \quad \varphi"(t) - \varphi(t) \leq 1.$$

   Let ψ be the solution of (5.11) such that $\psi(0) = \varphi(0)$, $\psi'(0) = \varphi'(0)$. Prove that $\forall\ t \in \mathbb{R}^+$, $\varphi(t) \leq \psi(t)$. (Hint: we can use the question a) in the case where $f = \varphi' - \varphi$ and $\lambda = 1$).
5) Let $\varphi(t) = 1 - e^{-t}$. verify that $\varphi" + \varphi \leq 1$.
   Let ψ be the solution of (5.12) such that $\psi(0) = \varphi(0) = 0$ and $\psi'(0) = \varphi'(0) = 1$. Do we have $\varphi(t) \leq \psi(t)$, $\forall\ t \in \mathbb{R}^+$?

3-1 1) Consider the differential equation: $x^2y'' - 2xy' + 2y = 0$.
If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $z'' - 3z' + 2z = 0$. Then $z = ae^t + be^{2t}$ and $y = ax + bx^2$.

2) Consider the differential equation: $x^2y'' - xy' + y = 0$.
If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $z'' - 2z' + z = 0$. Then $z = (at + b)e^t$ and $y = (a \ln x + b)x$.

3) Consider the differential equation: $x^2y'' - xy' + 10y = 0$. If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $z'' - 2z' + 10z = 0$. Then $z = x(a\cos(3\ln x) + b\sin(3\ln x))$.

4) Consider the differential equation: $x^2y'' + xy' + y = 0, \qquad x > 0$.
If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $z'' + z = 0$. Then $z = x(a\cos(\ln x) + b\sin(\ln x))$.

5) Consider the differential equation: $2x^2y'' + 5xy' + y = 0, \qquad x > 0$.
If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $2z'' + 3z' + z = 0$. Then $z = \frac{a}{x} + b\sqrt{x}$.

6) If $x = e^t$ and $z(t) = y(e^t)$. The differential equation is equivalent to $z'' + z = 0$. Then $z = a\cos(t) + b\sin(t)$ and

$$y = a\cos(\ln(x)) + b\sin(\ln(x)).$$

7) If $x = e^t$ and $z(t) = y(e^t)$. The differential equation is equivalent to $2z'' + 3z' + z = 0$. Then $z = ae^{-t} + be^{-\frac{1}{2}t}$ and

$$y = \frac{a}{x} + \frac{b}{\sqrt{x}}.$$

Consider the differential equation: $9x^2y'' + 15xy' + y = 0$.
If $x = e^t$ and $z(t) = y(e^t)$, the differential equation is equivalent to $9z'' + 6z' + z = 0$. Then $z = (at + b)e^{-\frac{1}{3}t}$ and $y = (a\ln x + b)x^{-\frac{1}{3}}$.

4-1 (a) $y^{(6)} - 5y^{(4)} - 36y'' = (D^6 - 5D^4 - 36D^2)(y) = D^2(D^2 + 4I)(D^2 - 9I)y$.
Then $y = ax + b + ce^{3x} + de^{-3x} + h\sin(2x) + k\cos(2x)$.

(b) $y^{(6)} - 2y^{(4)} + y'' = (D^6 - 2D^4 + D^2)(y) = D^2(D - I)^2(D + I)^2y$.
Then $y = ax + b + (cx + d)e^x(hx + k)e^{-x}$.

(c) $y_p = \frac{1}{40}e^x + \frac{1}{30}\sin x$

(d) $y_p = \frac{1}{20}e^{2x} - \frac{1}{6}\cos x$.

Solutions of Exercises on Chapter 3

1-1 1)

$$\begin{aligned}\mathcal{L}(f)(s) &= -\frac{d}{ds}(\mathcal{L}(e^x \sin x)(s)) = -\frac{d}{ds}(\mathcal{L}(\sin x)(s-1)) \\ &= -\frac{d}{ds}\left(\frac{1}{(s-1)^2+1}\right) = \frac{2(s-1)}{((s-1)^2+1)^2}.\end{aligned}$$

This Laplace transform is valid for $s > 1$.

2) We have:

$$\begin{aligned}\mathcal{L}(f)(s) &= -\frac{d}{ds}(\mathcal{L}(2\cos^2 x)(s)) = -\frac{d}{ds}(\mathcal{L}(1+\cos(2x))(s)) \\ &= -\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right) = \frac{1}{s^2}+\frac{s^2-4}{(s^2+4)^2} \\ &= \frac{2(s^4+2s^2+8)}{s^2(s^2+4)^2}.\end{aligned}$$

This Laplace transform is valid for $s > 0$.

3) We have:

$$\begin{aligned}\mathcal{L}(f)(s) &= (-\frac{d}{ds})^3(\mathcal{L}(e^{-3x})(s)) = (-\frac{d}{ds})^3\left(\frac{1}{s+3}\right) \\ &= \frac{6}{(s+3)^4}.\end{aligned}$$

This Laplace transform is valid for $s > -3$.

4) $\mathcal{L}\left(6\sin(2x) - 5\cos(2x)\right) = \dfrac{12}{s^2+4} - \dfrac{5s}{s^2+4} = \dfrac{12-5s}{s^2+4}$, for $s > 0$.

5)

$$\begin{aligned}\mathcal{L}\{(\sin x - \cos x)^2\} &= \mathcal{L}\{\sin^2 x - 2\sin x\cos x + \cos^2 x\} \\ &= \mathcal{L}\{1-\sin(2x)\} \\ &= \frac{1}{s} - \frac{2}{s^2+4}, \quad s > 0\end{aligned}$$

6) $\mathcal{L}\{(x^2+1)^2\} = \mathcal{L}\{x^4+2x^2+1\} = \dfrac{24}{s^5}+\dfrac{4}{s^3}+\dfrac{1}{s} = \dfrac{24+4s^2+s^4}{s^5}$, for $s > 0$.

7) $\mathcal{L}\{e^{-4x}\cosh(2x)\} = F(s+4)$ where $F(s) = \mathcal{L}\{\cosh(2x)\} = \dfrac{s}{s^2-4}$.
Hence $\mathcal{L}\{e^{-4x}\cosh(2x)\} = \dfrac{s+4}{(s-4)^2-4}$, for $s > -4$.

8) $e^{-5x}\sin(4x)\cos(4x) = \dfrac{1}{2}e^{-5x}\sin(8x)$. Then
$\mathcal{L}\left(e^{-5x}\sin(4x)\cos(4x)\right) = \dfrac{1}{2}\dfrac{8}{(s+5)^2+8^2}$;

9) $6\sin(8x)\sin(2x) = 3\cos(6x) - 3\cos(10x)$. Then
$\mathcal{L}\left(6\sin(8x)\sin(2x)\right) = 3\dfrac{s}{s^2+36} - 3\dfrac{s}{s^2+100}$;

2-1

1) $\mathcal{L}^{-1}(\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)})$.
$\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\dfrac{16}{5}\dfrac{1}{s-1} + \dfrac{25}{6}\dfrac{1}{s-2} + \dfrac{1}{30}\dfrac{1}{s+4}$. Then
$\mathcal{L}^{-1}(\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}) = -\dfrac{16}{5}e^{t} + \dfrac{25}{6}e^{2t} + \dfrac{1}{30}e^{-4t}$.

2) $\dfrac{s}{(s-3)^3} = \dfrac{1}{(s-3)^2} + \dfrac{3}{(s-3)^3} = \mathcal{L}\left(xe^{3x} + \dfrac{3}{2}x^2e^{3x}\right)$.

3) $\dfrac{s+1}{s^2+2s+10} = \dfrac{s+1}{(s+1)^2+9} = \mathcal{L}(e^{-x}\cos(3x))$.

4)

$$\begin{aligned}\frac{s^2+4s-15}{(s-1)(s^2+9)} &= -\frac{1}{s-1} + \frac{2s}{s^2+9} + \frac{6}{s^2+9}\\ &= -\frac{1}{s-1} + 2\left(\frac{s}{s^2+9}\right) + 2\left(\frac{3}{s^2+9}\right)\\ &= -\mathcal{L}(e^x) + 2\mathcal{L}(\cos(3x)) + 2\mathcal{L}(\sin(3x)).\end{aligned}$$

5) $\mathcal{L}^{-1}\left(\dfrac{4}{(s-3)^3}\right) = 2x^2e^{3x}$

6)

$$\begin{aligned}\frac{2s-3}{s^2+2s+10} &= \frac{2(s+1)}{(s+1)^2+3^2} - 5\frac{1}{(s+1)^2+3^2}\\ &= \mathcal{L}\left(2e^{-x}\cos(3x) - \frac{5}{3}e^{-x}\sin(3x)\right)\end{aligned}$$

7)

$$\begin{aligned}\frac{9s^2-12s+28}{s(s^2+4)} &= \frac{7}{s} + \frac{2s-12}{s^2+4}\\ &= \mathcal{L}\left(7 + 2\cos(2x) - 6\sin(2x)\right).\end{aligned}$$

8)

$$\begin{aligned}\frac{(s-2)e^{-s}}{s^2-4s+3} &= \frac{(s-2)e^{-s}}{(s-2)^2-1}\\ &= e^{-s}\mathcal{L}\left(e^{2x}\cosh(2x)\right)\\ &= \mathcal{L}\left(e^{2(x-1)}\cosh(2(x-1))H(x-1)\right).\end{aligned}$$

Then $\mathcal{L}^{-1}\left(\dfrac{(s-2)e^{-s}}{s^2-4s+3}\right) = \begin{cases}\frac{1}{2}(e^{4(x-1)}+1) & \text{if } x\geq 1\\ 0 & \text{if } 0\leq x<1\end{cases}.$

9)

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+4)}\right) &= \frac{1}{5}\mathcal{L}^{-1}\left(-\frac{1}{s+1}+\frac{s}{s^2+4}-4\frac{1}{s^2+4}\right)\\ &= \frac{1}{5}\left(e^{-x}+\cos(2x)-2\sin(2x)\right).\end{aligned}$$

10)

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{3s}{(s^2+9)^2}\right) &= -\frac{1}{2}\mathcal{L}^{-1}\left(\frac{d}{ds}\frac{3}{s^2+3^2}\right)\\ &= -\frac{1}{2}\mathcal{L}^{-1}\left(\frac{d}{ds}\mathcal{L}(\sin(3x))\right)\\ &= \frac{1}{2}\mathcal{L}^{-1}\left(\mathcal{L}(x\sin(3x))\right) = \frac{x}{2}\sin(3x).\end{aligned}$$

11)

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+12s+32}\right) &= \mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s+6)^2-4}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(e^{-2s}\mathcal{L}(e^{-6x}\sinh(2x)\right)\\ &= \frac{1}{2}e^{-6(x-2)}\sinh(2(x-2))H(x-2)\\ &= \begin{cases}\frac{1}{2}e^{-6(x-2)}\sinh(2(x-2)) & \text{if } x\geq 2\\ 0 & \text{if } 0\leq x<2\end{cases}\end{aligned}$$

12) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+6s+13}\right) = \mathcal{L}^{-1}\left(\dfrac{s}{(s+3)^2+4}\right) = e^{-3x}\cos(2x)$

13)

$$\begin{aligned}\mathcal{L}^{-1}\left(e^{-s}\frac{7-3s}{s^2-8s+20}\right) &= -\mathcal{L}^{-1}\left(e^{-s}\frac{3(s-4)+5}{(s-4)^2+4}\right)\\ &= -3\mathcal{L}^{-1}\left(e^{-s}\frac{(s-4)}{(s-4)^2+4}\right)-\frac{5}{2}\left(e^{-s}\frac{2}{(s-4)^2+4}\right)\\ &= -3\mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(e^{4x}\cos(2x))\right)-\frac{5}{2}\left(e^{-s}\mathcal{L}(e^{4x}\sin(2x))\right)\\ &= -H(x-1)e^{4(x-1)}\left(3\cos(2x-2))+\frac{5}{2}\sin(2x-2))\right)\end{aligned}$$

14)

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s^2+5}{(s-2)^3}\right) &= \mathcal{L}^{-1}\left(\frac{(s-2)^2+2(s-2)+5}{(s-2)^3}\right)\\ &= \mathcal{L}^{-1}\left(\frac{1}{s-2}+2\frac{1}{(s-2)^2}\frac{5}{(s-2)^3}\right)\\ &= e^{2x}+2xe^{2x}+\frac{5}{2}x^2e^{2x}.\end{aligned}$$

15)

$$\begin{aligned}\mathcal{L}^{-1}\left(e^{-3s}\frac{s}{s^4-16}\right) &= \frac{1}{8}\mathcal{L}^{-1}\left(\frac{1}{2}e^{-3s}\frac{1}{s-2}+\frac{1}{2}e^{-3s}\frac{1}{s+2}-e^{-3s}\frac{s}{s^2+4}\right)\\ &= \frac{1}{16}H(x-3)\left(e^{2(x-3)}+e^{-2(x-3)}-2\cos(2(x-3))\right)\end{aligned}$$

16)

$$\begin{aligned}\mathcal{L}^{-1}\left(e^{-\pi s}\frac{1}{s^4+16s^2}\right) &= \frac{1}{16}\mathcal{L}^{-1}\left(e^{-\pi s}(\frac{1}{s^2}-\frac{1}{s^2+16})\right)\\ &= \frac{1}{16}H(x-\pi)\left((x-\pi)-\frac{1}{16}\sin(4x)\right).\end{aligned}$$

3-1

1) $f(x)=e^{-2x}H(x-1)=e^{-2}e^{-2(x-1)}H(x-1)$. Then $\mathcal{L}(f)(s)=\dfrac{e^{-3}}{s+2}$.

2) $f(x)=\begin{cases}x^2, & 0\le x\le 1\\ 5-x, & 1<x\le 2\\ 6, & 2<x\end{cases}$

$$\begin{aligned}f(x) &= x^2(H(x-1)-H(x))+(5-x)(H(x-2)-H(x-1))+6H(x-2)\\ &= (x^2+x-5)H(x-1)-x^2H(x)+(11-x)H(x-2)\\ &= ((x-1)^2-(x-1)-7)H(x-1)-x^2H(x)+(9-(x-2))H(x-2).\end{aligned}$$

Then $\mathcal{L}(f)(s)=e^{-s}(\frac{2}{s^3}-\frac{1}{s^2}-\frac{7}{s})-\frac{2}{s^3}-e^{-2s}(\frac{9}{s}-\frac{1}{s^2})$.

3) $f(x)=(x^2-6x+18)H(x-3)=((x-3)^2+9)H(x-3)$.
Then $\mathcal{L}(f)(s)=e^{-3s}(\frac{2}{s^3}+\frac{9}{s})$.

4) $f(x)=\left((x-3)^2+6(x-3)+9\right)H(x-3)$.
$\mathcal{L}(f)(s)=e^{-3s}(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s})$.

5)

$$\begin{aligned}\mathcal{L}\left(H(x-5)xe^{-6x}\right) &= e^{-30}\mathcal{L}\left(H(x-5)(x-5)e^{-6(x-5)}\right)\\ &\quad +5e^{-30}\mathcal{L}\left(H(x-5)e^{-6(x-5)}\right)\\ &= e^{-30}e^{-5s}\frac{1}{(s+6)^2}+5e^{-30}e^{-5s}\frac{1}{s+6};\end{aligned}$$

6) $\mathcal{L}\left(H(x-\frac{\pi}{4})\cos(2x)\right) = -\mathcal{L}\left(H(x-\frac{\pi}{4})\sin(2(x-\frac{\pi}{4}))\right) = -e^{-\frac{\pi}{4}s}\frac{\frac{\pi}{4}}{s^2+\frac{\pi^2}{16}};$

7)

$$\begin{aligned}\mathcal{L}\left(H(x-3)(x^2-x+4)\right) &= \mathcal{L}\left(H(x-3)((x-3)^2+5(x-3)+10)\right)\\ &= e^{-3s}\left(\frac{10}{s}+\frac{5}{s^2}+\frac{2}{s^3}\right).\end{aligned}$$

8)

$$\begin{aligned}f(x) &= (2x-2)(H(x)-H(x-3))+(10-2x)(H(x-3)-H(x-6))\\ &= (2x-2)H(x)-4(x-3)H(x-3)+2(x-6+1)H(x-6).\end{aligned}$$

Then, using the Laplace transform formula: we have:

$$\begin{aligned}\mathcal{L}(f)(s) &= \mathcal{L}((2x-2)H(x)-4(x-3)H(x-3)+2(x-6+1)H(x-6))(s)\\ &= \frac{2}{s^2}-\frac{2}{s}-4\frac{e^{-3s}}{s^2}+2\frac{e^{-6s}}{s^2}+\frac{2}{s}e^{-6s}\\ &= \frac{2}{s^2}(1-e^{-3s})(1-s-e^{-3s}(1+s)).\end{aligned}$$

4-1

1) The general solution of the homogeneous equation $y^{'}+3y=0$ is $y=\lambda e^{-3x}$, with $\lambda\in\mathbb{R}$. Using variation of parameters method, $y=Ue^{-3x}$, the general solution of the initial value problem $y^{'}+3y=e^{2x}$, $y(0)=-2$ is $y=-\frac{11}{5}e^{-3x}+\frac{1}{5}e^{2x}$.
Using Laplace transforms, we get:
$sY+2+3Y=\dfrac{1}{s-2} \iff Y=\dfrac{1}{5}\dfrac{1}{s-2}-\dfrac{1}{5}\dfrac{1}{s+3}-\dfrac{2}{s+3}$. Then $y=-\dfrac{11}{5}e^{-3x}+\dfrac{1}{5}e^{2x}$.

2) The general solution of the homogeneous equation $y^{'}+4y=0$ is $y=\lambda e^{-4x}$, with $\lambda\in\mathbb{R}$. Using variation of parameters method,

$y = Ue^{-4x}$, the general solution of the initial value problem $y^{'} + 4y = \sin(3x)$, $y(0) = 5$ is $y = \frac{78}{25}e^{-4x} + \frac{1}{25}(-3\cos(3x) + 4\sin(3x))$.
Using Laplace transforms, we get:
$sY - 5 + 4Y = \frac{3}{s^2+9} \iff Y = \frac{5}{s+4} + \frac{3}{25}\frac{1}{s+4} - \frac{3}{25}\frac{s-4}{s^2+9}$. Then
$y = \frac{78}{25}e^{-4x} - \frac{3}{25}\cos(3x) + -\frac{4}{25}\sin(3x)$.

3) The general solution of the homogeneous equation $y^{'} + y = 0$ is $y = \lambda e^{-x}$, with $\lambda \in \mathbb{R}$. Using variation of parameters method, $y = Ue^{-x}$, the general solution of the initial value problem $y^{'} + y = xe^{x}$, $y(0) = -1$ is $y = -\frac{1}{4}e^{-x} + \frac{1}{4}(2x-1)e^{x}$.
Using Laplace transforms, we get:
$sY + 1 + Y = \frac{1}{(s-1)^2} \iff Y = -\frac{3}{4(s+1)} - \frac{1}{4(s-1)} + \frac{1}{2(s-1)^2}$.
Then $y = -\frac{3}{4}e^{-x} - \frac{1}{4}e^{x} + \frac{1}{2}xe^{x}$.

4) The general solution of the homogeneous equation $y^{''} + 9y = 0$ is $y = a\cos(3x) + b\sin(3x)$, with $a.b \in \mathbb{R}$. Using variation of parameters method, $y = U\cos(3x) + V\sin(3x)$, the general solution of the initial value problem $y^{''} + 9y = 2$, $y(0) = 0$, $y^{'}(0) = 1$ is $y = -\frac{2}{9}\cos(3x) + \frac{1}{3}\sin(3x) + \frac{2}{9}$.
Using Laplace transforms, we get:
$s^2Y - 1 + 9Y = \frac{2}{s} \iff Y = \frac{2}{9s} - \frac{2}{9}\frac{s}{s^2+9} + \frac{1}{s^2+9}$. Then
$y = \frac{2}{9} - \frac{2}{9}\cos(3x) + \frac{1}{3}\sin(3x)$.

5) The general solution of the homogeneous equation $y^{''} + 9y = 0$ is $y = a\cos(3x) + b\sin(3x)$, with $a.b \in \mathbb{R}$. Using variation of parameters method, $y = U\cos(3x) + V\sin(3x)$, the general solution of the initial value problem $y^{''} + 9y = 5\cos(3x)$, $y(0) = 0$, $y^{'}(0) = 0$ is $y = \frac{5}{6}x\sin(3x)$.
Using Laplace transforms, we get:
$s^2Y + 9Y = 5\frac{s}{s^2+9} \iff Y = 5\frac{s}{(s^2+9)^2}$.
Since $\frac{d}{ds}\frac{1}{s^2+9} = -2\frac{s}{(s^2+9)^2}$, then $y = \frac{5}{6}x\sin(3x)$.

6) The general solution of the homogeneous equation $y^{''} + 2y^{'} + y = 0$ is $y = (ax+b)e^{-x}$, with $a.b \in \mathbb{R}$. $y = 3x - 6$ is a particular solution, then the general solution of the initial value problem $y^{''} + 2y^{'} + y = 3x$, $y(0) = 0$, $y^{'}(0) = 0$ is $y = 3(x+2)e^{-x} + 3(x-2)$.
Using Laplace transforms, we get:

$$(s+1)^2 Y = \frac{3}{s^2} \iff Y = \frac{3}{s^2(s+1)^2} = -\frac{6}{s} + \frac{3}{s^2} + \frac{6}{s+1} + \frac{3}{(s+1)^2}.$$

Then $y = -6 + 3x + 6e^{-x} + 3xe^{-x}$.

4-2 1) Using Laplace transform of both sides, we get:

$$sY - 1 + Y = \frac{1}{s+1} + \frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{s}{(s+1)(s^2+1)} \\ &\quad + \frac{1}{(s+1)(s^2+1)} \\ &= \frac{1}{2(s+1)} + \frac{1}{(s+1)^2} + \frac{1}{2(s-1)} + \frac{1}{s^2+1}. \end{aligned}$$

The solution of the differential equation is:

$$y = \frac{1}{2}e^{-x} + xe^{-x} + \frac{1}{2}e^{x} + \sin x.$$

2) By taking the Laplace transform of both sides of the differential equation, we get: $sY - 4 - 2Y = \frac{5}{s} + \frac{s}{s^2+1} + \frac{1}{s-2} + \frac{1}{s+1}$. Then

$$\begin{aligned} Y(s) &= \frac{4}{s-2} + \frac{5}{s(s-2)} + \frac{s}{(s-2)(s^2+1)} + \frac{1}{(s-2)^2} + \frac{1}{(s+1)(s-2)} \\ &= \frac{217}{30(s-2)} - \frac{5}{2s} - \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} + \frac{1}{(s-2)^2} - \frac{1}{3}\frac{1}{s+1} \end{aligned}$$

and

$$y(x) = \frac{217}{30}e^{2x} - \frac{5}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x + xe^{2x} - \frac{1}{3}e^{-x}.$$

3) Using the Laplace transform of both sides of the differential equation, we get:

$$\begin{aligned} (s+1)Y - 2 &= \mathcal{L}\left[5H(x-1) + e^{x}H(x-1) + H(x-1)\cos x\right] \\ &= \frac{5}{s}e^{-s} + \frac{e^{-(s-1)}}{s-1} + e^{-s}\frac{\sin 1 + \cos 1}{s^2+1} \end{aligned}$$

Then

$$\begin{aligned} Y(s) &= \frac{2}{s+1} + \frac{5e^{-s}}{s(s+1)} + \frac{e^{-(s-1)}}{(s-1)(s+1)} + \frac{(\sin 1 + \cos 1)e^{-s}}{(s+1)(s^2+1)} \\ &= \frac{2}{s+1} + \frac{5e^{-s}}{s} - \frac{e^{-s}}{s+1} + \frac{1}{2}\frac{e^{-(s-1)}}{s-1} - \frac{1}{2}\frac{e^{-(s-1)}}{s+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s+1} \\ &\quad - \frac{(\sin 1 + \cos 1)}{2}\frac{se^{-s}}{s^2+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s^2+1} \end{aligned}$$

and

$$y(x) = 4e^{-x} + 5H(x-1) - 5H(x-1)e^{-(x-1)}.$$

4) Using Laplace transform of both sides of the differential equation, we get: $sY - 3 + 5Y = \frac{20}{s}$, then $Y = \dfrac{20}{s(s+5)} + \dfrac{3}{s+5} = \dfrac{4}{s} - \dfrac{1}{s+5}$ and $y = -e^{-5x} + 4$.

5) Using Laplace transform of both sides of the differential equation, we get: $sY + 2 + 3Y = \frac{1}{s-2}$, then $Y = \dfrac{1}{5}\dfrac{1}{s-2} - \dfrac{11}{5}\dfrac{1}{s+3}$ and $y = \dfrac{1}{5}e^{2x} - \dfrac{11}{5}e^{-3x}$.

6) Using Laplace transform of both sides of the differential equation, we get: $sY + 4 - Y = \frac{1}{s^2}e^{3s}$. Then $Y = -\dfrac{4}{s-1} - e^{3s}\left(\dfrac{1}{s} + \dfrac{1}{s^2} - \dfrac{1}{s-1}\right)$ and

$$y = -4e^{x} + (e^{x} - x - x)H(x-3).$$

7) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 5 + 9Y = 0$, then $Y = \dfrac{5}{s^2+9}$ and $y = \frac{5}{3}\sin(3x)$.

8) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 2s - 9Y = 0$, the $Y = \dfrac{2s}{s^2-9} = \dfrac{1}{s-3} + \dfrac{1}{s+3}$ and $y = 2\cosh(3x)$.

9) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 1 + 9Y = \dfrac{2}{s}$, then $Y = \dfrac{2s}{s^2-9} = \dfrac{2}{9s} - \dfrac{2}{9}\dfrac{s}{s^2+9} + \dfrac{1}{s^2+9}$ and $y = \dfrac{2}{9} - \dfrac{2}{9}\cos(3x) + \dfrac{1}{3}\sin(3x)$.

10) Using Laplace transform of both sides of the differential equation, we get: $s^2Y + 9Y = \dfrac{5s}{s^2+1}$, then $Y = \dfrac{5s}{(s^2+1)(s^2+9)} = \dfrac{5}{4}\left(\dfrac{s}{s^2+1} - \dfrac{s}{s^2+9}\right)$ and $y = \frac{5}{4}\left(\cos(x) - \cos(3x)\right)$.

11) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 1 + 4Y = \dfrac{2}{s^2+4}$, then $Y = \dfrac{1}{s^2+4} + \dfrac{2}{(s^2+4)^2}$ and $y = \dfrac{5}{8}\sin(2x) - \dfrac{1}{4}x\cos(2x)$.

12) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 1 + 9Y = \frac{s}{s^2+9}$, then $Y = \dfrac{1}{s^2+9} + \dfrac{s}{(s^2+9)^2}$ and $y = \dfrac{1}{3}\sin(3x) + \dfrac{1}{6}x\sin(3x)$

13) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 1 + 2sY + 5Y = \frac{2}{(s+1)^2+4}$, then $Y = \dfrac{1}{(s+1)^2+4} + \dfrac{2}{((s+1)^2+4)^2}$ and

$$y = \frac{5}{8}e^{-x}\sin(2x) - \frac{1}{16}xe^{-x}\sin(2x) - \frac{1}{8}xe^{-x}\cos(2x).$$

This is because

$$\begin{aligned}\mathcal{L}(xe^{ax}\sin(bx)) &= -\frac{d}{dx}\frac{b}{(s-a)^2+b^2} = \frac{2b(s-a)}{((s-a)^2+b^2)^2}\\ &= \frac{2bs}{((s-a)^2+b^2)^2} - \frac{2ab}{((s-a)^2+b^2)^2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(xe^{ax}\cos(bx)) &= -\frac{d}{dx}\frac{s-a}{(s-a)^2+b^2} = \frac{(s-a)^2-2b^2}{((s-a)^2+b^2)^2}\\ &= \frac{1}{(s-a)^2+b^2} - \frac{2b^2}{((s-a)^2+b^2)^2}.\end{aligned}$$

We deduce that:

$$2b^3\mathcal{L}^{-1}\left[\frac{1}{((s-a)^2+b^2)^2}\right] = e^{ax}\sin(bx) - bxe^{ax}\cos(bx)$$

and

$$2b^3\mathcal{L}^{-1}\left[\frac{s}{((s-a)^2+b^2)^2}\right] = ae^{ax}\sin(bx) - abxe^{ax}\cos(bx) + b^2xe^{ax}\sin(bx).$$

14) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - s + 2Y - 1 + 5Y = \frac{e^{-4s}}{s}$. Then $Y = \frac{s+1}{(s+1)^2+4} + \frac{e^{-4s}}{s((s+1)^2+4)} = \frac{s+1}{(s+1)^2+4} + \frac{e^{-4s}}{5s} - \frac{e^{-4s}}{2}\frac{(s+1)+1}{(s+1)^2+4}$ and

$$y = e^{-x}\cos(2x) + \frac{1}{5}H(x-4) - \frac{1}{2}e^{-x}\cos(2x)H(x-4) - \frac{1}{4}e^{-x}\sin(2x)H(x-4).$$

15) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 2s - 2Y + 2 - 3Y = \frac{e^{-3s}}{s}$. Then

$$Y = \frac{2(s-1)}{(s-3)(s+1)} + \frac{e^{-3s}}{s(s-3)(s+1)} = \frac{1}{s-3} + \frac{1}{s+1} - \frac{e^{-3s}}{3s} + \frac{e^{-3s}}{12(s-3)} + \frac{e^{-3s}}{4(s+1)}$$

and

$$y = e^{3x} + e^{-x} + \left(-\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{3}e^{-x}\right)H(x-3)$$

16) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - 1 + 2Y + 5Y = \frac{2}{(s+1)^2+4} + \frac{s+1}{(s+1)^2+4)}e^{\pi s}$. Then

$$Y = \frac{1}{(s+1)^2+4} + \frac{2}{((s+1)^2+4)^2} + \frac{s+1}{((s+1)^2+4)^2}e^{\pi s}$$

and

$$y = \frac{5}{8}e^{-x}\sin(2x) - \frac{1}{4}xe^{-x}\cos(2x) + \frac{1}{4}xe^{-x}\sin(2x)$$

17) Using Laplace transform of both sides of the differential equation, we get: $s^2Y - s + Y - 1 - 2Y = \frac{4}{s-1} + \frac{1}{s}e^{3s}$. Then

$$\begin{aligned} Y &= \frac{1}{s-1} + \frac{4}{(s+1)(s-1)^2} + \frac{1}{s(s+1)(s-1)}e^{3s} \\ &= \frac{1}{s+1} + \frac{2}{(s+1)^2} + e^{3s}\left(-\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right) \end{aligned}$$

and

$$y = e^{-x} + 2xe^{-x} + (\cosh x - 1)\,H(x-3)$$

18) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - sy(0) - y^{'}(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2+2s+5\right) = s+2+\dfrac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s) = \frac{s+2}{s^2+2s+5} + e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\dfrac{s+2}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2+4}$ and
$\mathcal{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+4}\right] = e^{-x}\cos(2x)$ and $\mathcal{L}^{-1}\left[\dfrac{2}{(s+1)^2+4}\right] = e^{-x}\sin(2x)$.

$$\begin{aligned}\frac{e^{-2s}}{s\left(s^2+2s+5\right)} &= \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right)\\ &= \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}\right).\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right)\right] &= H(x-2)\left[1 - e^{-(x-2)}(\cos 2(x-2)\right.\\ &\quad \left. + \frac{1}{2}\sin 2(x-2))\right].\end{aligned}$$

The solution $y(x)$ to the initial-value problem is

$$\begin{aligned}y(x) &= e^{-x}\cos(2x) + \frac{e^{-x}}{2}\sin(2x) + \frac{1}{5}H(x-2)\left[1 - e^{-(x-2)}\cos 2(x-2)\right.\\ &\quad \left. + \frac{e^{-(x-2)}}{2}\sin 2(x-2)\right]\end{aligned}$$

19) Taking Laplace transforms,

$$\begin{aligned}s^2Y - 4s - 2sY + 8 - 3Y = 0 &\iff (s^2-2s-3)Y = 4s-8\\ &\iff Y = 4\frac{s-2}{(s+1)(s-3)}\\ &\iff Y = \frac{3}{s+1} + \frac{1}{s-3}.\end{aligned}$$

Then $y = e^{3x} + 3e^{-x}$.

20) Taking Laplace transforms, it follows that

$$(s^2+4s+13)Y - s - 4 = -2\mathcal{L}[H(x-\pi)\sin(3(x-\pi))] = -\frac{6e^{-\pi s}}{s^2+9}$$

Let $Y_1 = \dfrac{s+4}{s^2+4s+13} = \dfrac{s+2}{(s+2)^2+9} + \dfrac{2}{(s+2)^2+9}$, then $\mathcal{L}^{-1}(Y_1) = e^{-2x}(\cos(3x) + \frac{2}{3}\sin(3x))$. Let also

$$\begin{aligned} Y_2 &= \frac{6e^{-\pi s}}{(s^2+9)(s^2+4s+13)} \\ &= \frac{3}{20}e^{-\pi s}\left(\frac{s}{s^2+9} - \frac{1}{s^2+9} - \frac{s+2}{(s+2)^2+9} - \frac{1}{(s+2)^2+9}\right) \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{L}^{-1}(Y_2) &= \frac{3}{20}H(t-\pi)\Big[\cos 3(x-\pi) - \frac{1}{3}\sin 3(x-\pi) - \\ &\quad e^{-2(x-\pi)}\cos 3(x-\pi) - \frac{1}{3}e^{-2(x-\pi)}\sin 3(x-\pi)\Big] \\ &= \frac{3}{20}H(t-\pi)\Big[-\cos(3x) + \frac{1}{3}\sin(3x) + \\ &\quad e^{-2(x-\pi)}\cos(3x) + \frac{1}{3}e^{-2(x-\pi)}\sin(3x)\Big] \end{aligned}$$

and

$$\begin{aligned} y &= e^{-2x}\left(\cos(3x) + \frac{2}{3}\sin(3x)\right) + \frac{3}{20}H(x-\pi)[-\cos(3x) \\ &\quad + \frac{1}{3}\sin(3x) + e^{-2(x-\pi)}\left(\cos(3x) + \frac{1}{3}\sin(3x)\right)\Big] \end{aligned}$$

4-3

1) We begin by taking the Laplace transform of both sides to achieve

$$\mathcal{L}\left[y'\right] + \mathcal{L}[y] = \mathcal{L}(e^{-x} + e^{x} + \cos x + \sin x).$$

We know that $\mathcal{L}(e^{-x}+e^{x}+\cos x+\sin x) = \dfrac{1}{s+1} + \dfrac{1}{s-1} + \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$. Denote $Y(s) = \mathcal{L}[y]$, then

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{s}{(s+1)(s^2+1)} \\ &\quad + \frac{1}{(s+1)(s^2+1)} \\ &= \frac{1}{2(s+1)} + \frac{1}{(s+1)^2} + \frac{1}{2(s-1)} + \frac{1}{s^2+1}. \end{aligned}$$

The solution of the differential equation:

$$y = \frac{1}{2}e^{-x} + xe^{-x} + \frac{1}{2}e^{x} + \sin x.$$

2) By taking the Laplace transform of both sides of the differential equation, we get: $sY - 4 - 2Y = \frac{5}{s} + \frac{s}{s^2+1} + \frac{1}{s-2} + \frac{1}{s+1}$. Then

$$\begin{aligned} Y(s) &= \frac{4}{s-2} + \frac{5}{s(s-2)} + \frac{s}{(s-2)(s^2+1)} + \frac{1}{(s-2)^2} \\ &\quad + \frac{1}{(s+1)(s-2)} \\ &= \frac{4}{s-2} + \frac{5}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right) + \frac{2}{5}\frac{1}{s-2} + \frac{1}{5}\frac{-2s+1}{s^2+1} \\ &\quad + \frac{1}{(s-2)^2} + \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1} \\ &= \frac{217}{30(s-2)} - \frac{5}{2s} - \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} \\ &\quad + \frac{1}{(s-2)^2} - \frac{1}{3}\frac{1}{s+1} \end{aligned}$$

and

$$y(x) = \frac{217}{30}e^{2x} - \frac{5}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x + xe^{2x} - \frac{1}{3}e^{-x}.$$

3) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - s + 1 - 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+1}.$$

Then $Y(s) = \frac{s-3}{(s-1)^2+1} + \frac{s}{(s^2+1)((s-1)^2+1)}$. Solving for $Y(s)$, we find:

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} \\ &\quad - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Then

$$\begin{aligned} y &= e^x\cos x - 2e^x\sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x - \frac{1}{5}e^x\cos x + \frac{3}{5}e^x\sin x \\ &= \frac{4}{5}e^x\cos x - \frac{7}{5}e^x\sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x \end{aligned}$$

4) Using the Laplace transform of both sides of the differential equation, we get:
$(s+1)Y-2=\mathcal{L}\left[5H(x-1)+e^{x}H(x-1)+H(x-1)\cos x\right].$
Since $\mathcal{L}[H(x-1)]=\frac{1}{s}e^{-s}$, $\mathcal{L}[e^{x}H(x-1)]=\frac{e^{-(s-1)}}{s-1}$ and
$\mathcal{L}[H(x-1)\cos x]=e^{-s}\frac{\sin 1+\cos 1}{s^2+1}$. We have:

$$\begin{aligned} Y(s) &= \frac{2}{s+1}+\frac{5e^{-s}}{s(s+1)}+\frac{e^{-(s-1)}}{(s-1)(s+1)}+\frac{(\sin 1+\cos 1)e^{-s}}{(s+1)(s^2+1)} \\ &= \frac{2}{s+1}+\frac{5e^{-s}}{s}-\frac{e^{-s}}{s+1}+\frac{1}{2}\frac{e^{-(s-1)}}{s-1} \\ &\quad -\frac{1}{2}\frac{e^{-(s-1)}}{s+1}+\frac{(\sin 1+\cos 1)}{2}\frac{e^{-s}}{s+1} \\ &\quad -\frac{(\sin 1+\cos 1)}{2}\frac{se^{-s}}{s^2+1}+\frac{(\sin 1+\cos 1)}{2}\frac{e^{-s}}{s^2+1} \end{aligned}$$

We have $\mathcal{L}^{-1}\left[\frac{4}{s+1}\right]=4e^{-x}$,
$\mathcal{L}^{-1}\left[\frac{5}{s}e^{-s}\right]=5H(x-1)$ and $\mathcal{L}^{-1}\left[\frac{5}{s+1}e^{-s}\right]=5H(x-1)e^{-(x-1)}$.
Then

$$y(x)=4e^{-x}+5H(x-1)-5H(x-1)e^{-(x-1)}.$$

5) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s)-sy(0)-y^{'}(0)+2(sY(s)-y(0))+5Y(s)=\frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2+2s+5\right)=s+2+\frac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s)=\frac{s+2}{s^2+2s+5}+e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\frac{s+2}{s^2+2s+5}=\frac{s+1}{(s+1)^2+4}+\frac{1}{(s+1)^2+4}$ and
$\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right]=e^{-x}\cos(2x)$ and $\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right]=e^{-x}\sin(2x)$.
$\frac{e^{-2s}}{s\left(s^2+2s+5\right)}=\frac{1}{5}e^{-2s}\left(\frac{1}{s}-\frac{(s+1)+1}{(s+1)^2+4}\right)=\frac{1}{5}e^{-2s}\left(\frac{1}{s}-\frac{s+1}{(s+1)^2+4}+\frac{1}{(s+1)^2\right.$

$$\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s}-\frac{s+2}{s^2+2s+5}\right)\right] = H(x-2)[1 \\ -e^{-(x-2)}\left(\cos 2(x-2)+\frac{1}{2}\sin 2(x-2)\right)].$$

The solution $y(x)$ to the initial-value problem is

$$y(x) = e^{-x}\cos(2x)+\frac{e^{-x}}{2}\sin(2x)+\frac{1}{5}H(x-2)[1 \\ -e^{-(x-2)}\cos 2(x-2)+\frac{e^{-(x-2)}}{2}\sin 2(x-2)]$$

6) We take the Laplace transform of each member of the differential equation:
$\mathcal{L}(y')+3\mathcal{L}(y)=13\mathcal{L}(\sin(2t))$. Then $(s+3)Y(s)-6=6+\dfrac{26}{s^2+4}$.
$Y(s)=\dfrac{6}{s+3}+\dfrac{26}{(s+3)(s^2+4)}=\dfrac{6}{s+3}+\dfrac{-2s+6}{s^2+4}$ and
$y=6\mathcal{L}^{-1}(\dfrac{1}{s+3})-2\mathcal{L}^{-1}(\dfrac{s}{s^2+4})+6\mathcal{L}^{-1}(\dfrac{1}{s^2+4})=6e^{-3t}-2\cos(2t)+3\sin(2t)$.

4-4 1) We begin by taking the Laplace transform of both sides to achieve

$$\mathcal{L}\left[y'\right]+\mathcal{L}[y]=\mathcal{L}(e^{-x}+e^{x}+\cos x+\sin x).$$

We know that $\mathcal{L}(e^{-x}+e^{x}+\cos x+\sin x)=\dfrac{1}{s+1}+\dfrac{1}{s-1}+\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1}$. Denote $Y(s)=\mathcal{L}[y]$, then

$$Y(s) = \frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{2}(\frac{1}{s-1}-\frac{1}{s+1})+\frac{s}{(s+1)(s^2+1)} \\ +\frac{1}{(s+1)(s^2+1)} \\ = \frac{1}{2(s+1)}+\frac{1}{(s+1)^2}+\frac{1}{2(s-1)}+\frac{1}{s^2+1}.$$

The solution of the differential equation:

$$y=\frac{1}{2}e^{-x}+xe^{-x}+\frac{1}{2}e^{x}+\sin x.$$

2) By taking the Laplace transform of both sides of the differential equation, we get: $sY - 4 - 2Y = \frac{5}{s} + \frac{s}{s^2+1} + \frac{1}{s-2} + \frac{1}{s+1}$. Then

$$\begin{aligned} Y(s) &= \frac{4}{s-2} + \frac{5}{s(s-2)} + \frac{s}{(s-2)(s^2+1)} + \frac{1}{(s-2)^2} \\ &\quad + \frac{1}{(s+1)(s-2)} \\ &= \frac{4}{s-2} + \frac{5}{2}\left(\frac{1}{s-2} - \frac{1}{s}\right) + \frac{2}{5}\frac{1}{s-2} + \frac{1}{5}\frac{-2s+1}{s^2+1} \\ &\quad + \frac{1}{(s-2)^2} + \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1} \\ &= \frac{217}{30(s-2)} - \frac{5}{2s} - \frac{2}{5}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1} \\ &\quad + \frac{1}{(s-2)^2} - \frac{1}{3}\frac{1}{s+1} \end{aligned}$$

and

$$y(x) = \frac{217}{30}e^{2x} - \frac{5}{2} - \frac{2}{5}\cos x + \frac{1}{5}\sin x + xe^{2x} - \frac{1}{3}e^{-x}.$$

3) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - s + 1 - 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+1}.$$

Then $Y(s) = \frac{s-3}{(s-1)^2+1} + \frac{s}{(s^2+1)((s-1)^2+1)}$. Solving for $Y(s)$, we find:

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-1)^2+1} - \frac{2}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} \\ &\quad - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Then

$$\begin{aligned} y &= e^x\cos x - 2e^x\sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x - \frac{1}{5}e^x\cos x + \frac{3}{5}e^x\sin x \\ &= \frac{4}{5}e^x\cos x - \frac{7}{5}e^x\sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x \end{aligned}$$

4) Using the Laplace transform of both sides of the differential equation, we get:
$(s+1)Y - 2 = \mathcal{L}\left[5H(x-1) + e^x H(x-1) + H(x-1)\cos x\right].$
Since $\mathcal{L}[H(x-1)] = \frac{1}{s}e^{-s}$, $\mathcal{L}[e^x H(x-1)] = \frac{e^{-(s-1)}}{s-1}$ and
$\mathcal{L}[H(x-1)\cos x] = e^{-s}\frac{\sin 1 + \cos 1}{s^2+1}$. We get:

$$\begin{aligned}
Y(s) &= \frac{2}{s+1} + \frac{5e^{-s}}{s(s+1)} + \frac{e^{-(s-1)}}{(s-1)(s+1)} + \frac{(\sin 1 + \cos 1)e^{-s}}{(s+1)(s^2+1)} \\
&= \frac{2}{s+1} + \frac{5e^{-s}}{s} - \frac{5e^{-s}}{s+1} + \frac{1}{2}\frac{e^{-(s-1)}}{s-1} \\
&\quad -\frac{1}{2}\frac{e^{-(s-1)}}{s+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s+1} \\
&\quad -\frac{(\sin 1 + \cos 1)}{2}\frac{se^{-s}}{s^2+1} + \frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s^2+1}
\end{aligned}$$

We have $\mathcal{L}^{-1}\left[\frac{2}{s+1}\right] = 2e^{-x}$, $\mathcal{L}^{-1}\left[\frac{5e^{-s}}{s}\right] = 5H(x-1)$,
$-\mathcal{L}^{-1}\left[\frac{5e^{-s}}{s+1}\right] = -5H(x-1)e^{-(x-1)}$.
$\mathcal{L}^{-1}\left[\frac{1}{2}\frac{e^{-(s-1)}}{s-1}\right] = \frac{1}{2}e^x H(x-1)$
$-\mathcal{L}^{-1}\left[\frac{1}{2}\frac{e^{-(s-1)}}{s+1}\right] = -\frac{1}{2}e^{-x+2}H(x-1)$
$\mathcal{L}^{-1}\left[\frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s+1}\right] = \frac{(\sin 1 + \cos 1)}{2}e^{-x+1}H(x-1)$
$-\mathcal{L}^{-1}\left[\frac{(\sin 1 + \cos 1)}{2}\frac{se^{-s}}{s^2+1}\right] = -\frac{(\sin 1 + \cos 1)}{2}\cos(x-1)H(x-1)$
$\mathcal{L}^{-1}\left[\frac{(\sin 1 + \cos 1)}{2}\frac{e^{-s}}{s^2+1}\right] = \frac{(\sin 1 + \cos 1)}{2}\sin(x-1)H(x-1)$
Then

$$\begin{aligned}
y(x) &= 2e^{-x} + 5H(x-1) - 5H(x-1)e^{-(x-1)} + \frac{1}{2}e^x H(x-1) \\
&\quad -\frac{1}{2}e^{-x+2}H(x-1) + \frac{(\sin 1 + \cos 1)}{2}e^{-x+1}H(x-1) \\
&\quad -\frac{(\sin 1 + \cos 1)}{2}\cos(x-1)H(x-1) \\
&\quad +\frac{(\sin 1 + \cos 1)}{2}\sin(x-1)H(x-1).
\end{aligned}$$

5) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - s + 2(sY(s) - 1) + 5Y(s) = \frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2 + 2s + 5\right) = s + 2 + \dfrac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s) = \frac{s+2}{s^2+2s+5} + e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\dfrac{s+2}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2+4}$ and
$\mathcal{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+4}\right] = e^{-x}\cos(2x)$ and $\mathcal{L}^{-1}\left[\dfrac{2}{(s+1)^2+4}\right] = e^{-x}\sin(2x)$.

$$\begin{aligned}\frac{e^{-2s}}{s\left(s^2+2s+5\right)} &= \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right)\\ &= \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}\right).\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right)\right] &= H(x-2)[1\\ &\quad -e^{-(x-2)}\left(\cos 2(x-2) + \frac{1}{2}\sin 2(x-2)\right)].\end{aligned}$$

The solution $y(x)$ to the initial-value problem is

$$\begin{aligned}y(x) &= e^{-x}\cos(2x) + \frac{e^{-x}}{2}\sin(2x) + \frac{1}{5}H(x-2)[1\\ &\quad -e^{-(x-2)}\cos 2(x-2) + \frac{e^{-(x-2)}}{2}\sin 2(x-2)]\end{aligned}$$

6) We take the Laplace transform of each member of the differential equation:
$(s+3)Y(s) - 6 = 6 + \dfrac{26}{s^2+4}$, hence
$Y(s) = \dfrac{6}{s+3} + \dfrac{26}{(s+3)(s^2+4)} = \dfrac{6}{s+3} + \dfrac{-2s+6}{s^2+4}$ and

$$\begin{aligned}y &= 6\mathcal{L}^{-1}(\frac{1}{s+3}) - 2\mathcal{L}^{-1}(\frac{s}{s^2+4}) + 6\mathcal{L}^{-1}(\frac{1}{s^2+4})\\ &= 6e^{-3x} - 2\cos(2x) + 3\sin(2x).\end{aligned}$$

7) We take the Laplace transform of each member of the differential equation:
$\mathcal{L}(y^{''})-3\mathcal{L}(y')+2\mathcal{L}(y)=\mathcal{L}(e^{-4x})$. Then $Y(s)=\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4}$
and $y=-\dfrac{16}{5}e^x+\dfrac{25}{5}e^{2x}+\dfrac{1}{30}e^{-4x}$.

8) We take the Laplace transform of each member of the differential equation:
$Y(s)=\dfrac{2s+5}{(s-3)^2}+\dfrac{2}{(s-3)^5}$ and $y=2e^{3x}+11xe^{3x}+\frac{1}{12}x^4e^{3x}$.

9) $\mathcal{L}\left(f(x)\right)=-\frac{3s}{s^2+1}e^{-s}$. Then $sY(s)-3-2Y(s)=-\frac{3s}{s^2+1}e^{-s}$ and
$Y(s)=\dfrac{1}{s-2}\left(3-\dfrac{3s}{s^2+1}e^{-s}\right)=\dfrac{3}{s-2}+\dfrac{6}{5}\dfrac{s}{s^2+1}e^{-s}-\dfrac{3}{5}\dfrac{1}{s^2+1}e^{-s}$.
$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)(x)=3e^{2x}$, $\mathcal{L}^{-1}\left(\frac{6}{5}\frac{s}{s^2+1}e^{-s}\right)(x)=\frac{6}{5}\cos(x-1)H(x-1)$,
$\mathcal{L}^{-1}\left(\frac{3}{5}\frac{1}{s^2+1}e^{-s}\right)(x)=\frac{3}{5}\sin(x-1)H(x-1)$.

$$y(x)=3e^{2x}+\frac{6}{5}\cos(x-1)H(x-1)-\frac{3}{5}\sin(x-1)H(x-1).$$

10) Taking Laplace transforms of the differential equation, we get
$(s^2+s+1)Y(s)-s=\dfrac{1}{s^2+1}$. Then $Y(s)=\dfrac{s}{s^2+s+1}+\dfrac{1}{(s^2+s+1)(s^2+1)}$.

$$\frac{s}{s^2+s+1}=\frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}}=\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}-\frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}.$$

Finding the inverse Laplace transform. Since

$$\frac{s}{s^2+s+1}=\frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}}=\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}-\frac{1}{\sqrt{3}}\frac{\frac{1}{2}\sqrt{3}}{(s+\frac{1}{2})^2+(\frac{1}{2}\sqrt{3})^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\}=e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}x)-\frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}x).$$

Also, we have

$$\frac{1}{(s^2+s+1)(s^2+1)}=\frac{s+1}{s^2+s+1}-\frac{s}{s^2+1},$$

$$\frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}, \text{ and } \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}x}\sin(\frac{1}{2}\sqrt{3}x).$$

Then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right.$$

We obtain

$$y(x) = 2e^{-\frac{1}{2}x}\cos(\frac{1}{2}\sqrt{3}x) - \cos(x).$$

4-5 (a) Using the Laplace transform, $\mathcal{L}(y') = sY - a$, $\mathcal{L}(y'') = s^2Y - as - b$.Then $(s^2+3s+2)Y = as + (b+3a) \iff Y = \dfrac{b+2a}{s+1} - \dfrac{b+a}{s+2}$. Then $y(x) = (b+2a)e^{-x} - (b+a)e^{-2x}$.

(b) $(s^2+s+1)Y = as + (b+a) \iff Y = \dfrac{as+(b+a)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$. Then

$$y(x) = ae^{-\frac{x}{2}}\cos(\frac{\sqrt{3}}{2}x) + \frac{2}{\sqrt{3}}(b+\frac{a}{2})e^{-\frac{x}{2}}\sin(\frac{\sqrt{3}}{2}x).$$

(c)

$$y'' + 2y' + y = 0, \quad y(0) = a, \quad y'(0) = b,$$

$(s^2+2s+1)Y = as + (b+a) \iff Y = \dfrac{as+(b+2a)}{(s+1)^2} = \dfrac{a}{s+1} + \dfrac{b+a}{(s+1)^2}$. Then

$$y(x) = ae^{-x} + (a+b)xe^{-x}.$$

(d) $(s^2+1)Y = \frac{1}{s}e^{-s}$, then $Y = \dfrac{1}{s}\dfrac{e^{-s}}{s^2+1}$. Then

$$y(x) = \int_0^x \sin t H(t-1)dt = (1-\cos x)H(x-1).$$

(e) $s^3Y - s^2 + Y = e^{-s}\left(\dfrac{1}{s}+\dfrac{3}{s^2}+\dfrac{6}{s^3}+\dfrac{6}{s^4}\right)$. Then

$$Y = e^{-s}\left(\frac{1}{s^3(s-1)}+\frac{3}{s^4(s-1)}+\frac{6}{s^5(s-1)}+\frac{6}{s^6(s-1)}\right).$$

(f) $y(x) = (2e^{x-1} - x^2 - 1)H(x-1) - \frac{e^{-x}}{2} + \frac{3e^x}{2}$.

(g) Using Laplace transform, we get

$$s^2\mathcal{L}(y)(s)-sy(0)-y'(0)+3(s\mathcal{L}(y)(s)-y(0))+2\mathcal{L}(y)(s) = -\frac{5}{s^2+1}+\frac{5s}{s^2+1},$$

$$(s^2+3s+2)\mathcal{L}(y)(s) = (s+3)y(0)+y'(0)+\frac{5s-5}{s^2+1} = 5(s+3)-3+\frac{5s-5}{s^2+1},$$

$$(s+1)(s+2)\mathcal{L}(y)(s) = 5s+12+\frac{5s-5}{s^2+1},$$

$$\begin{aligned}
\mathcal{L}(y)(s) &= \frac{5s+12}{(s+1)(s+2)} + \frac{5s-5}{(s+1)(s+2)(s^2+1)} \\
&= \frac{7}{s+1} - \frac{2}{s+2} - \frac{5}{s+1} + \frac{3}{s+2} + \frac{2s+1}{s^2+1} \\
&= \frac{2}{s+1} + \frac{1}{s+2} + 2\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1} \\
&= 2\mathcal{L}(e^{-x})(s) + \mathcal{L}(e^{-2x})(s) + 2\mathcal{L}(\cos(x))(s) + \mathcal{L}(\sin(x))(s) \\
&= \mathcal{L}(2e^{-x} + e^{-2x} + 2\cos(x) + \sin(x))(s).
\end{aligned}$$

So the solution:

$$y = 2e^{-x} + e^{-2x} + 2\cos(x) + \sin(x).$$

(h) Taking Laplace transforms, we get

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{4e^{-\pi s}}{s},$$

$$(s^2+1)\mathcal{L}(y) - 2s - 4 = \frac{4e^{-\pi s}}{s},$$

$$\mathcal{L}(y) = \frac{2s+4}{s^2+1} + \frac{4e^{-\pi s}}{s(s^2+1)}.$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \text{ and } \mathcal{L}(2\cos(x)+4\sin(x))(s) = \frac{2s+4}{s^2+1}.$$

$$\mathcal{L}(H(x-\pi)) = e^{-\pi s}\mathcal{L}(1)(s) = \frac{1}{s}e^{-\pi s},$$

$$\mathcal{L}(H(x-\pi)\cos(x-\pi)) = e^{-\pi s}\mathcal{L}(\cos(x))(s) = \frac{e^{-\pi s}s}{s^2+1}.$$

So

$$\begin{aligned}
y &= 2\cos(x) + 4\sin(x) + 4H(x-\pi) - 4H(x-\pi)\cos(x-\pi) \\
&= 2\cos(x) + 4\sin(x) + 4H(x-\pi))(1+\cos(x)).
\end{aligned}$$

Then $y = 2\cos(x)+4\sin(x)$, for $x < \pi$ and $y = 4+6\cos(x)+4\sin(x)$ for $x \geq \pi$.

(i) $s^2Y(s)-sy(0)-y'(0)-5(sY(s)-y(0))-6Y(s) = (s^2-5s-6)Y(s)-s+5 = \frac{2}{s^3}+\frac{7}{s}$. Then

$$Y(s) = \frac{s^4-5s^3+7s^2+2}{s^3(s+1)(s-6)} = -\frac{1}{3s^3}+\frac{5}{18s^2}-\frac{151}{108s}+\frac{15}{7(s+1)}+\frac{235}{756(s-6)}$$

$$y = -\frac{x^2}{6}+\frac{5x}{18}-\frac{151}{108}+\frac{15}{7}e^{-x}+\frac{235}{756}e^{6x}$$

(j) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s)-s+1-2(sY(s)-1)+2Y(s) = \frac{s}{s^2+1}.$$

Then $Y(s) = \frac{s-3}{(s-1)^2+1}+\frac{s}{(s^2+1)((s-1)^2+1)}$. Solving for $Y(s)$, we find:

$$\begin{aligned} Y(s) &= \frac{s-1}{(s-1)^2+1}-\frac{2}{(s-1)^2+1}+\frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1} \\ &\quad -\frac{1}{5}\frac{s-1}{(s-1)^2+1}+\frac{3}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Then

$$\begin{aligned} y &= e^x\cos x-2e^x\sin x+\frac{1}{5}\cos x-\frac{2}{5}\sin x-\frac{1}{5}e^x\cos x+\frac{3}{5}e^x\sin x \\ &= \frac{4}{5}e^x\cos x-\frac{7}{5}e^x\sin x+\frac{1}{5}\cos x-\frac{2}{5}\sin x \end{aligned}$$

(k) Using the Laplace transform of both sides of the differential equation, we get:
$(s+1)Y-2 = \mathcal{L}\left[5H(x-1)+e^xH(x-1)+H(x-1)\cos x\right]$.
Since $\mathcal{L}[H(x-1)] = \frac{1}{s}e^{-s}$, $\mathcal{L}[e^xH(x-1)] = \frac{e^{-(s-1)}}{s-1}$ and $\mathcal{L}[H(x-1)\cos x] = e^{-s}\frac{\sin 1+\cos 1}{s^2+1}$. We have:

$$\begin{aligned} Y(s) &= \frac{2}{s+1}+\frac{5e^{-s}}{s(s+1)}+\frac{e^{-(s-1)}}{(s-1)(s+1)}+\frac{(\sin 1+\cos 1)e^{-s}}{(s+1)(s^2+1)} \\ &= \frac{2}{s+1}+\frac{5e^{-s}}{s}-\frac{e^{-s}}{s+1}+\frac{1}{2}\frac{e^{-(s-1)}}{s-1} \\ &\quad -\frac{1}{2}\frac{e^{-(s-1)}}{s+1}+\frac{(\sin 1+\cos 1)}{2}\frac{e^{-s}}{s+1} \\ &\quad -\frac{(\sin 1+\cos 1)}{2}\frac{se^{-s}}{s^2+1}+\frac{(\sin 1+\cos 1)}{2}\frac{e^{-s}}{s^2+1} \end{aligned}$$

We have $\mathcal{L}^{-1}\left[\frac{4}{s+1}\right] = 4e^{-x}$,
$\mathcal{L}^{-1}\left[\frac{5}{s}e^{-s}\right] = 5H(x-1)$ and $\mathcal{L}^{-1}\left[\frac{5}{s+1}e^{-s}\right] = 5H(x-1)e^{-(x-1)}$.
Then

$$y(x) = 4e^{-x} + 5H(x-1) - 5H(x-1)e^{-(x-1)}.$$

(l) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - sy(0) - y^{'}(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2+2s+5\right) = s+2+\frac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s) = \frac{s+2}{s^2+2s+5} + e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\dfrac{s+2}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2+4}$ and
$\mathcal{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+4}\right] = e^{-x}\cos(2x)$ and $\mathcal{L}^{-1}\left[\dfrac{2}{(s+1)^2+4}\right] = e^{-x}\sin(2x)$.
$\dfrac{e^{-2s}}{s\left(s^2+2s+5\right)} = \dfrac{1}{5}e^{-2s}\left(\dfrac{1}{s} - \dfrac{(s+1)+1}{(s+1)^2+4}\right) = \dfrac{1}{5}e^{-2s}\left(\dfrac{1}{s} - \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{(s+1)^2}\right.$

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right)\right] &= H(x-2)[1\\ &-e^{-(x-2)}\left(\cos 2(x-2) + \frac{1}{2}\sin 2(x-2)\right)].\end{aligned}$$

The solution $y(x)$ to the initial-value problem is

$$\begin{aligned}y(x) &= e^{-x}\cos(2x) + \frac{e^{-x}}{2}\sin(2x) + \frac{1}{5}H(x-2)[1\\ &-e^{-(x-2)}\cos 2(x-2) + \frac{e^{-(x-2)}}{2}\sin 2(x-2)]\end{aligned}$$

(m) $\mathcal{L}(f(x)) = -\frac{3s}{s^2+1}e^{-s}$. Then $sF(s) - 3 - 2F(s) = -\frac{3s}{s^2+1}e^{-s}$ and
$F(s) = \dfrac{1}{s-2}\left(3 - \dfrac{3s}{s^2+1}e^{-s}\right) = \dfrac{3}{s-2} + \dfrac{6}{5}\dfrac{s}{s^2+1}e^{-s} - \dfrac{3}{5}\dfrac{1}{s^2+1}e^{-s}$.
$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3e^{2t}$,
$\mathcal{L}^{-1}\left(\frac{6}{5}\frac{s}{s^2+1}e^{-s}\right) = \frac{6}{5}\cos(t-1)H(t-1)$, $\mathcal{L}^{-1}\left(\frac{3}{5}\frac{1}{s^2+1}e^{-s}\right) = \frac{3}{5}\sin(t-1)H(t-1)$.

$$y(t) = 3e^{2t} + \frac{6}{5}\cos(t-1)H(t-1) - \frac{3}{5}\sin(t-1)H(t-1).$$

(n) Using the Laplace transform of both sides of the differential equation, we get:

$$s^2Y(s) - sy(0) - y^{'}(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{e^{-2s}}{s}.$$

Then $Y(s)\left(s^2+2s+5\right) = s + 2 + \frac{e^{-2s}}{s}$. Solving for $Y(s)$, we find:

$$Y(s) = \frac{s+2}{s^2+2s+5} + e^{-2s}\frac{1}{s\left(s^2+2s+5\right)}.$$

We have $\frac{s+2}{s^2+2s+5} = \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$ and $\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] = e^{-x}\cos(2x)$ and $\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] = e^{-x}\sin(2x)$.

$$\frac{e^{-2s}}{s\left(s^2+2s+5\right)} = \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right) = \frac{1}{5}e^{-2s}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2}\right.$$

$$\mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right)\right] = H(x-2)\left[1 - e^{-(x-2)}\left(\cos 2(x-2) + \frac{1}{2}\sin 2(x-\right.\right.$$

The solution $y(x)$ to the initial-value problem is

$$y(x) = e^{-x}\cos(2x) + \frac{e^{-x}}{2}\sin(2x) + \frac{1}{5}H(x-2)\left[1 - e^{-(x-2)}\cos 2(x-2) + \frac{e^{-(x-2)}}{2}\sin 2\right.$$

Solutions of Exercises on Chapter 4

1-1 $X'(t) = \begin{pmatrix} t^2 & 3t \\ \sin t & t^2 \end{pmatrix} X(t) + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$

1-2 (a) Let $y_1 = y$, $y_2 = y'$, $y_3 = y^{''}$. Then by letting
$X(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sin(2t) & t & 0 \end{pmatrix}$ and
$F(t) = \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}$, the differential equation can be presented by the system of differential equations of first order
$X'(t) = A(t)X(t) + F(t)$.

(b) Let $y_1 = y$, $y_2 = y'$. Then by letting $X(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} 0 & 1 \\ -t & 2 \end{pmatrix}$ and $F(t) = \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$, the differential equation can be presented by the system of differential equations of first order $X'(t) = A(t)X(t) + F(t)$.

(c) $A(t) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$ and $F(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$.

(d) $y'' - 3y' + ty = 3t^2$. $A(t) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ and $F(t) = \begin{pmatrix} 0 \\ 3t^2 \end{pmatrix}$, $X_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1-3 We have $\begin{cases} x_1' &= 2x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{cases}$, then $x_1' + 3x_2' = 5x_1$ or $x_1 = \frac{1}{5}(x_1' + 3x_2')$. Let $y = \frac{1}{5}(x_1 + 3x_2)$, $y' = x_1$ and $y'' = 2y' + 3(5y - y')$.

1-4

$$
\begin{aligned}
W &= \begin{vmatrix} e^t & e^t & e^t \\ \sin t & \cos t & -\sin t \\ \sin(2t) & 2\cos(2t) & -4\sin(2t) \end{vmatrix} \\
&= e^t \begin{vmatrix} 1 & 0 & 0 \\ \sin t & \cos t - \sin t & -2\sin t \\ \sin(2t) & 2\cos(2t) - \sin(2t) & -5\sin(2t) \end{vmatrix} \\
&= e^t\left(-5\cos(2t)\cos t - 2\sin(2t)\sin t - \cos(2t)\sin t\right).
\end{aligned}
$$

$W(0) \neq 0$.

2-1 (a) Let $A = \begin{pmatrix} 2 & 7 \\ -5 & -10 \end{pmatrix}$. $q_A(\lambda) = (\lambda + 3)(\lambda + 5)$. Then the matrix is diagonalizable. The vector $X_1 = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$ is an eigenvector of the matrix A for the eigenvalue $\lambda = -3$. The vector $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix A for the eigenvalue $\lambda = -5$. Then the set of solutions of the linear system is

$$\{X = ae^{-3t}X_1 + be^{-5t}X_2, \quad a, b \in \mathbb{R}\} = \begin{pmatrix} -7ae^{-3t} + be^{-5t} \\ 5ae^{-3t} - be^{-5t} \end{pmatrix}.$$

(b) $A = \begin{pmatrix} -3 & 6 \\ -3 & 3 \end{pmatrix}$. $q_A(\lambda) = \lambda^2 + 9$. The vector $X = \begin{pmatrix} 1 - \mathrm{i} \\ 1 \end{pmatrix}$ is an eigenvector of A with respect to the eigenvalue $3\mathrm{i}$. Then the set of solutions of the linear system is

$$\{X = \begin{pmatrix} (a-b)\cos(3t) + (a+b)\sin(3t) \\ a\cos(3t) + b\sin(3t) \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

(c) $A = \begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix}$. $q_A(\lambda) = (\lambda - 6)^2$. Let $X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X_1 = (A - 6I)X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The set of solutions of the linear system is

$$\{X = e^{6t}\left((at+b)X_1 + aX_2\right) = e^{6t}\begin{pmatrix} 2at + a + 2b \\ at + b \end{pmatrix}, \ a, b \in \mathbb{R}\}$$

2-2 The characteristic polynomial of the matrix A is $q_A(\lambda) = |A - \lambda I| = (\lambda - 1)(\lambda + 2) - 4 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$.
The vectors $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ are eigenvectors of the matrix A with respect to the eigenvalues 2 and -3 respectively. The general solution of the homogenous system is

$$X = ae^{2t}X_1 + be^{-3t}X_2.$$

If $X_p = Ue^{2t}X_1 + Ve^{-3t}X_2$ be a particular solution, then $U'e^{2t}X_1 + V'e^{-3t}X_2 = \begin{pmatrix} e^{-3t} \\ e^{2t} \end{pmatrix} = \frac{1}{5}(4e^{-3t} + e^{2t})X_1 + \frac{1}{5}(e^{-3t} - e^{2t})X_2$. Then

$$\begin{cases} U' &= \frac{1}{5}(4e^{-5t} + 1) \\ V' &= \frac{1}{5}(e^{-5t} - 1) \end{cases} \quad \text{and} \quad \begin{cases} U &= \frac{t}{5} - \frac{4}{25}e^{-5t} \\ V &= -\frac{t}{5} - \frac{1}{25}e^{-5t} \end{cases}$$

Then $X_p = (\frac{t}{5}e^{2t} - \frac{4}{25}e^{-3t})X_1 - (\frac{t}{5}e^{2t} + \frac{1}{25}e^{-3t})X_2$. The general solution of the system is

$$X = (a + \frac{t}{5} - \frac{4}{25}e^{-5t})e^{2t}X_1 + (b - \frac{t}{5} + \frac{1}{25}e^{-5t})e^{-3t}X_2.$$

2-3 This system is equivalent to the following:

$$\begin{cases} x' &= x - 4y \\ y' &= 4x - 7y \end{cases}, \qquad x(0) = -2, y(0) = 1.$$

We take the Laplace transform, we get:

$$\begin{cases} s\mathcal{L}(x) + 2 &= \mathcal{L}(x) - 4\mathcal{L}(y) \\ s\mathcal{L}(y) - 1 &= 4\mathcal{L}(x) - 7\mathcal{L}(y) \end{cases} \iff \begin{cases} (s-1)\mathcal{L}(x) + 4\mathcal{L}(y) &= -2 \\ -4\mathcal{L}(x) + (s+7)\mathcal{L}(y) &= 1 \end{cases}.$$

Solving this linear system, we get

$$\mathcal{L}(y) = \frac{s-9}{(s+3)^2} = \frac{1}{s+3} - \frac{12}{(s+3)^2}, \qquad y = e^{-3t}(1 - 12t)$$

$$\mathcal{L}(x) = \frac{-2s - 18}{(s+3)^2} = \frac{-2}{s+3} - \frac{12}{(s+3)^2}, \qquad x = e^{-3t}(-2 - 12t)$$

2-4 This system is equivalent to the following:

$$\begin{cases} x' &= -4x-2y-t \\ y' &= 3x+y+2t-1 \end{cases}, \qquad x(0)=3, y(0)=-5.$$

We take the Laplace transform, we get:

$$\begin{cases} s\mathcal{L}(x)-3 &= -4\mathcal{L}(x)-2\mathcal{L}(y)-\frac{1}{s^2} \\ s\mathcal{L}(y)+5 &= 3\mathcal{L}(x)+\mathcal{L}(y)+\frac{2}{s^2}-\frac{1}{s} \end{cases} \iff \begin{cases} (s+4)\mathcal{L}(x)+2\mathcal{L}(y) &= -\frac{1}{s^2}+3 \\ -3\mathcal{L}(x)+(s-1)\mathcal{L}(y) &= \frac{2}{s^2}-\frac{1}{s}-5 \end{cases}.$$

Solving this linear system, we get

$$\mathcal{L}(x) = \frac{3s^3+7s^2+s-3}{s^2(s+2)(s+1)} = \frac{1}{4(s+2)} - \frac{3}{s^2} + \frac{11}{s}, \qquad x = \frac{1}{4}e^{-2t} - \frac{3}{2}t + \frac{11}{4}$$

$$\mathcal{L}(y) = \frac{-5s^3-12s^2-2s+5}{s^2(s+2)(s+1)} = -\frac{1}{4(s+2)} + \frac{5}{2s^2} - \frac{19}{4s}, \qquad y = -\frac{1}{4}e^{-2t} + \frac{5}{2}t - \frac{19}{4}.$$

2-5 Taking the Laplace transform of the equations, we get $\begin{cases} sX(s)-1 &= -2X(s)+Y(s), \\ sY(s)-2 &= X(s)-2Y(s), \end{cases}$ where $X(s)=\mathcal{L}\{x(x)\}, Y(s)=\mathcal{L}\{y(x)\}$. then $\begin{cases} (s+2)X(s)-Y(s) &= 1, \\ -X(s)+(s+2)Y(s) &= 2 \end{cases}$
The solutions of the linear system of equations on X and Y are $X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}$.
Using the inverse Laplace transform, we have

$$\frac{s+4}{(s+1)(s+3)} = \frac{\frac{3}{2}}{s+1} - \frac{\frac{1}{2}}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{\frac{3}{2}}{s+1} + \frac{\frac{1}{2}}{s+3},$$

we obtain

$$x(x) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(x) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

3-1 (a) This system is rewritten into the explicit form as follows:
$\begin{cases} x'(t) &= -4x(t)-2y(t)+e^{-t} \\ y'(t) &= 3x(t)+y(t)+3e^{-2t} \end{cases}$
Using Laplace transform, we get
$\begin{cases} sX(s)-2 &= -4X(s)-2Y(s)+\frac{1}{s+1} \\ sY(s)-3 &= 3X(s)+Y(s)+\frac{3}{s+2} \end{cases}$ This is equivalent to the

following system: $\begin{cases} (s+4)X(s)+2Y(s) &= 2+\frac{1}{s+1} \\ -3X(s)+(s-1)Y(s) &= 3+\frac{3}{s+2} \end{cases}$ Then $X(s) = -\dfrac{13}{s+1}+\dfrac{15}{s+2}-\dfrac{2}{(s+1)^2}+\dfrac{6}{(s+2)^2}$ and
$Y(s) = \dfrac{21}{s+1}-\dfrac{18}{s+2}+\dfrac{3}{(s+1)^2}-\dfrac{6}{(s+2)^2}$.
Taking the Laplace inverse operator, we get
$x(t) = -13e^{-t}+15e^{-2t}-2te^{-t}+6te^{-2t}$,
$y(t) = 21e^{-t}-18e^{-2t}+3te^{-t}-6te^{-2t}$.

(b) Let $X = \mathcal{L}(x)$ and $Y = \mathcal{L}(y)$. We get:

$$\begin{cases} sX-1 &= 2X-Y+\frac{1}{s-1} \\ sY+1 &= X+2Y+\frac{s}{s^2+1} \end{cases} \iff \begin{cases} (s-2)X+Y &= 1+\frac{1}{s-1} \\ -X+(s-2)Y &= -1+\frac{s}{s^2+1} \end{cases}$$

Then

$$\begin{aligned} X &= \frac{1}{(s-2)^2+1}\begin{vmatrix} \frac{s}{s-1} & 1 \\ \frac{s-s^2-1}{s^2+1} & s-2 \end{vmatrix} \\ &= \frac{s(s-2)}{(s-1)((s-2)^2+1)}+\frac{s^2-s+1}{(s^2+1)((s-2)^2+1)} \\ &= -\frac{1}{2}\frac{1}{s-1}+\frac{1}{8}\frac{13(s-2)+9}{(s-2)^2+1}+\frac{1}{8}\frac{-s+1}{s^2+1} \end{aligned}$$

$$x = -\frac{1}{2}e^t+\frac{13}{8}e^{2t}\cos t+\frac{9}{8}e^{2t}\sin t-\frac{1}{8}\cos t+\frac{1}{8}\sin t$$

$$\begin{aligned} Y &= \frac{1}{(s-2)^2+1}\begin{vmatrix} s-2 & \frac{s}{s-1} \\ -1 & \frac{s-s^2-1}{s^2+1} \end{vmatrix} \\ &= \frac{(s-2)(s-s^2-1)}{(s^2+1)((s-2)^2+1)}+\frac{s}{(s-1)((s-2)^2+1)} \\ &= \frac{1}{2}\frac{1}{s-1}-\frac{1}{8}\frac{9(s-2)-13}{(s-2)^2+1}-\frac{1}{8}\frac{3s-1}{s^2+1} \end{aligned}$$

$$y = \frac{1}{2}e^t-\frac{9}{8}e^{2t}\cos t+\frac{13}{8}e^{2t}\sin t-\frac{3}{8}\cos t+\frac{1}{8}\sin t$$

(c) Let $X = \mathcal{L}(x)$ and $Y = \mathcal{L}(y)$. We get:

$$\begin{cases} sX-1 &= 2X-Y+\frac{1}{s-1} \\ sY+1 &= -X+2Y+\frac{s}{s^2+1} \end{cases} \iff \begin{cases} (s-2)X+Y &= 1+\frac{1}{s-1} \\ X+(s-2)Y &= -1+\frac{s}{s^2+1} \end{cases}$$

Then

$$\begin{aligned}
X &= \frac{1}{(s-1)(s-3)}\begin{vmatrix} \frac{s}{s-1} & 1 \\ \frac{s-s^2-1}{s^2+1} & s-2 \end{vmatrix} \\
&= \frac{s(s-2)}{(s-1)^2(s-3)} + \frac{1}{(s-1)(s-3)} - \frac{s}{(s^2+1)((s-1)(s-3))} \\
&= \frac{1}{2}\frac{1}{(s-1)^2} + \frac{11}{10}\frac{1}{s-3} - \frac{1}{10}\frac{s}{s^2+1} + \frac{1}{5}\frac{1}{s^2+1}
\end{aligned}$$

$$x = \frac{1}{2}te^t + \frac{11}{10}e^{3t} - \frac{1}{10}\cos t + \frac{1}{5}\sin t$$

$$\begin{aligned}
Y &= \frac{1}{(s-1)(s-3)}\begin{vmatrix} s-2 & \frac{s}{s-1} \\ 1 & \frac{s-s^2-1}{s^2+1} \end{vmatrix} \\
&= \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{11}{10}\frac{1}{x-3} - \frac{2}{5}\frac{s}{s^2+1} + \frac{3}{10}\frac{1}{s^2+1}
\end{aligned}$$

$$y = \frac{1}{2}e^t + \frac{1}{2}te^t - \frac{11}{10}e^{3t} - \frac{2}{5}\cos t + \frac{3}{10}\sin t$$

(d) $\begin{cases} (s-2)X - Y = 1 + \frac{1}{s-1} \\ X + (s-2)Y = -1 + \frac{s}{s^2+1} \end{cases}$

Then

$$\begin{aligned}
X &= \frac{s(s-2)}{(s-1)((s-2)^2+1))} - \frac{1}{(s-2)^2+1} \\
&\quad + \frac{s}{(s^2+1)((s-2)^2+1))} \\
&= -\frac{1}{2}\frac{1}{s-1} + \frac{5}{4}\frac{(s-2)}{(s-2)^2+1} - \frac{1}{2}\frac{1}{(s-2)^2+1} \\
&\quad + \frac{1}{4}\frac{s}{s^2+1} - \frac{1}{4}\frac{1}{s^2+1}
\end{aligned}$$

$$x = -\frac{1}{2}e^t + \frac{5}{4}e^{2t}\cos t - \frac{1}{2}e^{2t}\sin t + \frac{1}{4}\cos t - \frac{1}{4}\sin t$$

$$\begin{aligned} Y &= \frac{-(s-2)}{(s-2)^2+1)} - \frac{s(s-2)}{(s^2+1)((s-2)^2+1)} \\ &\quad - \frac{s}{(s-1)((s-2)^2+1))} \\ &= \frac{-(s-2)}{(s-2)^2+1)} - \frac{3}{8}\frac{s}{s^2+1} + \frac{1}{8}\frac{1}{s^2+1} \\ &\quad + \frac{3}{8}\frac{(s-2)}{(s-2)^2+1} + \frac{1}{8}\frac{1}{(s-2)^2+1} \\ &\quad - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{(s-2)}{(s-2)^2+1} - \frac{3}{2}\frac{1}{(s-2)^2+1} \end{aligned}$$

$$y = -\frac{1}{8}e^{2t}\cos t - \frac{3}{8}\cos t + \frac{1}{8}\sin t - \frac{1}{2}e^t - \frac{13}{8}e^{2t}\sin t$$

(e) $X' = AX + \begin{pmatrix} e^t \\ \cos t \end{pmatrix} \iff \begin{cases} (s-3)\mathcal{X} + 2\mathcal{Y} &= \frac{s}{s-1} \\ -2\mathcal{X} + (s+1)\mathcal{Y} &= -1 + \frac{s}{s^2+1} \end{cases}.$

$$\mathcal{X} = \frac{1}{s-1} + \frac{4}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{s^2+1}$$

$$x = (t^2+4t+1)e^t + \sin t.$$

$2y = 3x - x' = 2\left((t^2+3t-\frac{1}{2})e^t + \frac{3}{2}\sin t - \frac{1}{2}\cos t\right)$ Also we have

$$\mathcal{Y} = -\frac{1}{2}\frac{1}{s-1} + \frac{3}{(s-1)^2} + \frac{2}{(s-1)^3} - \frac{1}{2}\frac{s}{s^2+1} + \frac{3}{2}\frac{1}{s^2+1}.$$

Then

$$y = (t^2+3t-\frac{1}{2})e^t - \frac{1}{2}\cos t + \frac{3}{2}\sin t$$

(f) $\begin{cases} \frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= x - 2y \end{cases}$, with the initial conditions $x(0) = 1$, $y(0) = 2$.

Taking Laplace transforms the system becomes

$$\begin{cases} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{cases}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$. We get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

We have

$$\frac{s+4}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3}.$$

Using the inverse Laplace transform, we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(g)

$$\begin{cases} \dfrac{dx}{dt} &= -2x + y, \\ \dfrac{dy}{dt} &= x - 2y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$.
Taking Laplace transform, the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{aligned}$$

where $X(s) = \mathcal{L}(x(t))$, $Y(s) = \mathcal{L}(y(t))$.
Solving for $X(s)$, $Y(s)$. The above linear system of equations, we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3},$$

Taking the inverse Laplace transform, we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(h)

$$\begin{cases} \dfrac{dx}{dt} &= -x + 2y, \\ \dfrac{dy}{dt} &= 2x + y \end{cases}$$

with the initial conditions $x(0) = 1$, $y(0) = -1$.

Taking Laplace transform, the system becomes

$$\begin{aligned} sX(s) - 1 &= -X(s) + 2Y(s), \\ sY(s) + 1 &= 2X(s) + Y(s), \end{aligned}$$

where $X(s) = \mathcal{L}(x(t))$, $Y(s) = \mathcal{L}(y(t))$.
Solving for $X(s)$, $Y(s)$. The above linear system of equations, we get

$$X(s) = \frac{s-3}{s^2-5} = \frac{a}{s-\sqrt{5}} + \frac{b}{s+\sqrt{5}}, \quad Y(s) = \frac{1-s}{s^2-5} == \frac{c}{s-\sqrt{5}} + \frac{d}{s+\sqrt{5}},$$

with $a = \dfrac{\sqrt{5}-3}{2\sqrt{5}}$, $b = \dfrac{\sqrt{5}+3}{2\sqrt{5}}$, $c = \dfrac{1-\sqrt{5}}{2\sqrt{5}}$, $d = -\dfrac{1+\sqrt{5}}{2\sqrt{5}}$.
Taking the inverse Laplace transform, we obtain

$$x(t) = ae^{\sqrt{5}t} + be^{-\sqrt{5}t}, \quad y(t) = ce^{\sqrt{5}t} + de^{-\sqrt{5}t}.$$

4-1 (a) The system has the operator form

$$\begin{cases} (D+1)x(t) + D^2y(t) = 0 \\ x(t) + (D-1)y(t) = \sin t. \end{cases}$$

Then $y(t) = -\cos t - \sin t$, $x = y - y' + \sin t = -\sin t$.

(b) The system has the operator form

$$\begin{cases} (D-1)(D-2)x(t) + (D-1)y(t) = t \\ (D-2)x(t) + (D+1)y(t) = \sin t. \end{cases}$$

Then $D(D-1)y(t) = -t + \cos t + \sin t$ and $y = (\frac{1}{2}t^2 + t + a) - \sin t + be^t$. Hence $x'(t) - 2x(t) = -y' - y + \sin t = (-\frac{1}{2}t^2 - 2t - 1 - a) - \cos t + 2\sin t$ and

$$x(t) = Ae^{2t} + (\frac{1}{4}t^2 + \frac{5}{4}t + \frac{9}{8} + \frac{1}{2}a) - \sin t.$$

Since $x(0) = 1, y(0) = 0, x'(0) = 1, y'(0) = 1$, then

$$x(t) = \frac{3}{8}e^{2t} + (\frac{1}{4}t^2 + \frac{5}{4}t + \frac{5}{8}) - \sin t, \quad y = (\frac{1}{2}t^2 + t - 1) - \sin t + e^t.$$

Solutions of Exercises on Chapter 5

1-1 (a) $\sum_{n=0}^{+\infty} \sin(\frac{n\pi}{3})\frac{x^n}{n} = \sum_{n=0}^{+\infty} a_n x^n$, with $a_{3n} = 0$ et $a_{3n+1} = \frac{(-1)^n}{3n+1}\frac{\sqrt{3}}{2}$ et $a_{3n+2} = \frac{(-1)^n}{3n+2}\frac{\sqrt{3}}{2}$. Then $R = 1$.

(b)

$$\begin{aligned}
\sum_{n=1}^{+\infty} \sin(\frac{n\pi}{3})\frac{x^n}{n} &= \frac{\sqrt{3}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{3n+1}x^{3n+1} + \frac{\sqrt{3}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{3n+2}x^{3n+2} \\
&= \frac{\sqrt{3}}{2}\int_0^x \frac{1+t}{1+t^3}dt = \frac{\sqrt{3}}{2}\int_0^x \frac{dt}{1-t+t^2}dt \\
&= \frac{1}{\sqrt{3}}\int_0^x \frac{dt}{(\frac{2t-1}{\sqrt{3}})^2+1}dt \\
&= \frac{1}{2}\tan^{-1}(\frac{2x-1}{\sqrt{3}}) + \frac{1}{2}\tan^{-1}(\frac{1}{\sqrt{3}}) \\
&= \frac{1}{2}\tan^{-1}(\frac{2x-1}{\sqrt{3}}) + \frac{\pi}{12}.
\end{aligned}$$

1-2 (a) $R = \frac{1}{2}$.

(b) $f'(x) = \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{+\infty}(2n-\lambda)a_n x^n = -\lambda f(x) + 2xf'(x)$.
The function f verifies the differential equation $(2x-1)f'(x) = \lambda f(x)$. Then $f(x) = 2\lambda\ln(1-2x)$ for $|x| < \frac{1}{2}$.

1-3 (a) $\frac{1}{1+x} = \sum_{n=0}^{+\infty}(-1)^n x^n$, $\ln(1+x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{n+1}$, then using the series product we get

$\frac{\ln(1+x)}{1+x} = x\sum_{n=0}^{+\infty}c_n x^n$, avec $c_n = (-1)^n\sum_{k=0}^{n}\frac{1}{k+1}$.

(b) The function f verifies the differential equation

$$(1-x^2)y'' - xy' = 2.$$

If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then $a_0 = a_1 = 0$ et $a_2 = 2$. Moreover $(n+1)(n+2)a_{n+2} = n^2 a_n$ for $n \geq 1$. Then $a_{2n} = 0$, $a_{2n+1} = \dfrac{(2n-1)^2 a_{2n-1}}{2n(2n+1)}$, et $a_{2n+1} = 2\dfrac{(2^n n!)^2}{(2n+1)!}$, and

$$(\sin^{-1} x)^2 = \sum_{n=0}^{+\infty} 2\frac{(2^n n!)^2}{(2n+1)!} x^{2n+2}.$$

(c) $\dfrac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}} = -\dfrac{d}{dx}(\sin^{-1}\sqrt{x})^2 = 2\sum_{n=0}^{+\infty} \dfrac{(2^n(n+1)!)^2}{(2n+1)!} x^{n+1}$

(d) If $g(x) = \ln(1 - 2x\cos\alpha + x^2)$, then $g'(x) = \dfrac{1}{x - e^{\mathrm{i}\alpha}} + \dfrac{1}{x - e^{-\mathrm{i}\alpha}}$. Since

$$\frac{1}{x - e^{\mathrm{i}\alpha}} = -e^{-\mathrm{i}\alpha} \sum_{n=0}^{+\infty} x^n e^{-\mathrm{i}n\alpha}.$$

Then

$$g(x) = \ln(1 - 2x\cos\alpha + x^2) = \sum_{n=0}^{+\infty} -2\frac{x^{n+1}}{n+1}\cos(n+1)\alpha.$$

(e) $e^{2x}\cos x = \operatorname{Re} e^{x(2+\mathrm{i})} = \operatorname{Re}\sum_{n=0}^{+\infty} \dfrac{(2+\mathrm{i})^n}{n!} x^n$. If $(2+\mathrm{i}) = \sqrt{5}e^{\mathrm{i}\theta}$, then

$$e^{2x}\cos x = \sum_{n=0}^{+\infty} \frac{(\sqrt{5})^n}{n!} x^n \cos n\theta.$$

$$f(x) = \frac{x}{1 - x - x^2} = \frac{-x}{(x-\alpha)(x-\beta)} = \frac{a}{x-\alpha} + \frac{b}{x-\beta},$$

with $a = -\dfrac{1+\sqrt{5}}{2\sqrt{5}}$, $b = -\dfrac{1-\sqrt{5}}{2\sqrt{5}}$, $\alpha = -\dfrac{1+\sqrt{5}}{2}$, and $\beta = -\dfrac{1-\sqrt{5}}{2}$.

$$\begin{aligned} f(x) = \frac{x}{1-x-x^2} &= -\frac{a}{\alpha}\sum_{n=0}^{+\infty}\frac{x^n}{\alpha^n} - \frac{a}{\beta}\sum_{n=0}^{+\infty}\frac{x^n}{\beta^n} \\ &= -\frac{1}{2\sqrt{5}}\sum_{n=0}^{+\infty}\frac{(-2)^n x^n}{(1+\sqrt{5})^n} - \frac{1}{2\sqrt{5}}\sum_{n=0}^{+\infty}\frac{(-2)^n x^n}{(1-\sqrt{5})^n}. \end{aligned}$$

2-1 Let $y = \sum_{n=0}^{+\infty} a_n x^n$.

$xy' = \sum_{n=0}^{+\infty} n a_n x^n$, $y = \sum_{n=0}^{+\infty} (n+1)(n+2)a_{n+2}x^n$. If y is a solution, then

$$(n+1)(n+2)a_{n+2} + (n+3)a_n = 0, \qquad \forall n \geq 0.$$

Let $u_n = a_{2n}$ and $v_n = a_{2n+1}$.

$$u_n = -\frac{2n+1}{2n(2n-1)} u_{n-1} = (-1)^n \frac{2n+1}{2^n n!} a_0$$

and

$$v_n = -\frac{n+1}{n(2n+1)} v_{n-1} = (-1)^n \frac{2^n (n+1)!}{(2n+1)!} a_1$$

Then

$$y = a \sum_{n=0}^{+\infty} (-1)^n \frac{2n+1}{2^n n!} x^{2n} + b \sum_{n=0}^{+\infty} (-1)^n \frac{2^n (n+1)!}{(2n+1)!} x^{2n+1}.$$

2-2 Consider the differential equation

$$x(x-4)y' + (x+2)y = 2.$$

1) For $0 < x < 4$, if $y = \sum_{n=0}^{+\infty} a_n x^n$ is a solution of the differential equation,then $x^2 y' = \sum_{n=1}^{+\infty} (n-1)a_{n-1}x^n$, $-4xy' = -4\sum_{n=0}^{+\infty} n a_n x^n$, $xy = \sum_{n=1}^{+\infty} a_{n-1}x^n$ and $2y = \sum_{n=0}^{+\infty} 2a_n x^n$.
   If y is a solution of the differential equation, then $a_0 = 1$ and
   $$a_n = \frac{n}{2(2n-1)} a_{n-1} = \frac{(n!)^2}{2n!}.$$
   The radius of convergence of this series is 4.
2) The homogeneous differential equation is equivalent to:
   $\frac{y'}{y} = -\frac{x+2}{x(x-4)} = \frac{1}{2x} - \frac{3}{2(x-4)}$. Then
   $$y = \lambda \frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}.$$

Using the change of parameter method, $y = u\frac{\sqrt{x}}{(4-x)^{\frac{3}{4}}}$, the function u satisfies: $u' = -2\frac{(4-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}}$.

$u = -2\int \frac{(4-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}}dx = -2\int \frac{1}{x}\sqrt{\frac{4-x}{x}}dx \overset{t^2=\frac{4-x}{x}}{=} 4\int \frac{t^2}{1+t^2}dt =$ $4t - 4\tan^{-1} t$.

$$\begin{aligned} y &= \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\left(\sqrt{\frac{4-x}{x}} - \tan^{-1}(\sqrt{\frac{4-x}{x}})\right) + \frac{\lambda\sqrt{x}}{(4-x)^{\frac{3}{2}}} \\ &= \frac{4}{4-x} - \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\tan^{-1}(\sqrt{\frac{4-x}{x}}) + \frac{\lambda\sqrt{x}}{(4-x)^{\frac{3}{2}}} \\ &\underset{x\to 0}{\approx} 1 + \frac{\lambda\sqrt{x}}{8} - \frac{\sqrt{x}}{2}(\frac{\pi}{2} - \frac{\sqrt{x}}{2}) \end{aligned}$$

Since $y(0) = 1$, then $\lambda = 2\pi$.

$$\begin{aligned} y &= \frac{4}{4-x} + \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\left(\frac{\pi}{2} - \tan^{-1}(\sqrt{\frac{4-x}{x}})\right) \\ &= \frac{4}{4-x} + \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\tan^{-1}(\sqrt{\frac{x}{4-x}}) \\ &= \frac{4}{4-x}\left(1 + \sqrt{\frac{x}{4-x}}\tan^{-1}(\sqrt{\frac{x}{4-x}})\right) \end{aligned}$$

$$\sum_{n=0}^{+\infty}\frac{1}{\binom{2n}{n}} = \sum_{n=0}^{+\infty}\frac{(n!)^2}{2n!} = y(1) = \frac{4}{3}\left(1 + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}})\right) = \frac{4}{3}\left(1 + \frac{1}{\sqrt{3}}\frac{\pi}{6}\right)$$

2-3 Consider the differential equation

$$(1+x^2)y'' + 2xy' - 2y = 0.$$

(a) Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $2xy' = \sum_{n=0}^{+\infty} 2na_n x^n$, $x^2y'' = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$, $y'' = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$.

Then $a_{n+2} = -\frac{n-1}{n+1}a_n = \frac{(n!)^2}{2n!}$, $a_3 = 0$, $a_2 = a_0$.

We deduce that $a_{2n+1} = 0$ for $n \geq 1$ and $a_{2n} = \dfrac{(-1)^{n-1}}{2n-1}$ for $n \geq 1$.
$y = x$ and $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^{n-1}x^{2n}}{2n-1}$ are solutions of the given differential equation.
If $z = \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1}$, $z' = \displaystyle\sum_{n=1}^{+\infty}(-1)^{n-1}x^{2n-2} = \frac{1}{1+x^2}$. Then $z = \tan^{-1}(x)$ and the general solution of the differential equation is

$$y = ax + b(1 + x\tan^{-1}(x)).$$

(b)

2-4 Let $y = \displaystyle\sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $2y' = \displaystyle\sum_{n=0}^{+\infty} 2(n+1)a_{n+1}x^n$, $4xy'' = \displaystyle\sum_{n=0}^{+\infty} 4n(n+1)a_{n+1}x^n$. Then $a_n = \dfrac{1}{2(n-1)(2n-1)}a_{n-1} = \dfrac{a_0}{2n!}$.
Then for $y = \cosh(\sqrt{|x|})$ is a solution of the differential equation. The function $\sinh(\sqrt{x})$ is also a solution for $x > 0$.

2-5 Let $y = \displaystyle\sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. The radius of convergence is 1.
$y'' = \displaystyle\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$, and $xy = \displaystyle\sum_{n=1}^{+\infty} a_{n-1}x^n$ and $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{+\infty} x^n$. We have: $y''(0) = 1$, then $a_2 = \frac{1}{2}$ and $a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)} + \dfrac{1}{(n+1)(n+2)}$.

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)}.$$

Let $u_n = a_{3n}$, $v_n = a_{3n+1}$, $w_n = a_{3n+2}$.

$$u_{n+1} = \frac{u_n}{3(3n+2)(n+1)} + \frac{1}{3(3n+2)(n+1)},$$
$$v_{n+1} = \frac{v_n}{3(n+1)(3n+4)} + \frac{1}{3(n+1)(3n+4)},$$

$$w_{n+1} = \frac{w_n}{(3n+4)(3n+5)} + \frac{1}{(3n+4)(3n+5)}.$$

2-6 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $y'' = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$, $xy' = \sum_{n=0}^{+\infty} n a_n x^n$ and $\cos x = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n}}{2n!}$. Then

$$a_{2n+1} = -\frac{a_{2n-1}}{2n+1} = (-1)^n \frac{2^n n!}{(2n+1)!} a_1$$

and

$$a_{2n} = -\frac{a_{2(n-1)}}{2n} + \frac{(-1)^n}{2n!}.$$

2-7 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $4xy'' = \sum_{n=0}^{+\infty} 4n(n+1)a_{n+1}x^n$, $-2y' = \sum_{n=0}^{+\infty} -(n+1)a_{n+1}x^n$ and $9x^2 y = \sum_{n=0}^{+\infty} 9a_{n+2}x^n$. Then $a_1 = a_2 = 0$ and

$$a_{n+1} = -\frac{9a_{n-2}}{9(n+1)(2n-1)}.$$

$$a_{3n} = \frac{(-1)^n}{2n!}, \quad a_{3n+1} = a_{3n+2} = 0.$$

For $x > 0$, $y = \cos(x^{\frac{3}{2}})$ is a solution. Also $y = \sin(x^{\frac{3}{2}})$ is a solution.

2-8 Let $y = \sum_{n=0}^{+\infty} a_n x^n$, $xy' = \sum_{n=0}^{+\infty} n a_n x^n$, $y = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$. If y is a solution, then

$$(n+1)(n+2)a_{n+2} + (n+3)a_n = 0, \qquad \forall n \geq 0.$$

Let $u_n = a_{2n}$ and $v_n = a_{2n+1}$.

$$u_n = -\frac{2n+1}{2n(2n-1)} u_{n-1} = (-1)^n \frac{2n+1}{2^n n!} a_0$$

and

$$v_n = -\frac{n+1}{n(2n+1)} v_{n-1} = (-1)^n \frac{2^n (n+1)!}{(2n+1)!} a_1$$

Then

$$y = a\sum_{n=0}^{+\infty}(-1)^n\frac{2n+1}{2^n n!}x^{2n} + b\sum_{n=0}^{+\infty}(-1)^n\frac{2^n(n+1)!}{(2n+1)!}x^{2n+1}.$$

2-9 Consider the differential equation

$$x(x-4)y' + (x+2)y = 2.$$

1) For $0 < x < 4$, if $y = \sum_{n=0}^{+\infty} a_n x^n$ is a solution of the differential equation,then $x^2y' = \sum_{n=1}^{+\infty}(n-1)a_{n-1}x^n$, $-4xy' = -4\sum_{n=0}^{+\infty} na_nx^n$, $xy = \sum_{n=1}^{+\infty} a_{n-1}x^n$ and $2y = \sum_{n=0}^{+\infty} 2a_nx^n$. If y is a solution of the differential equation, then $a_0 = 1$ and

$$a_n = \frac{n}{2(2n-1)}a_{n-1} = \frac{(n!)^2}{2n!}x^n.$$

The radius of convergence of this series is 4.

2) The homogeneous differential equation is equivalent to: $\frac{y'}{y} = -\frac{x+2}{x(x-4)} = \frac{1}{2x} - \frac{3}{2(x-4)}$. Then

$$y = \lambda\frac{\sqrt{x}}{(4-x)^{\frac{3}{2}}}.$$

Using the change of parameter method, $y = u\frac{\sqrt{x}}{(4-x)^{\frac{3}{4}}}$, the function u satisfies: $u' = -2\frac{(4-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}}$.

$u = -2\int \frac{(4-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}}dx = -2\int \frac{1}{x}\sqrt{\frac{4-x}{x}}dx \overset{t^2=\frac{4-x}{x}}{=} 4\int \frac{t^2}{1+t^2}dt =$ $4t - 4\tan^{-1}t$.

$$\begin{aligned} y &= \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\left(\sqrt{\frac{4-x}{x}} - \tan^{-1}(\sqrt{\frac{4-x}{x}})\right) + \frac{\lambda\sqrt{x}}{(4-x)^{\frac{3}{2}}} \\ &= \frac{4}{4-x} - \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\tan^{-1}(\sqrt{\frac{4-x}{x}}) + \frac{\lambda\sqrt{x}}{(4-x)^{\frac{3}{2}}} \end{aligned}$$

Since $y(0)=1$, then $\lambda = 2\pi$.

$$\begin{aligned} y &= \frac{4}{4-x} + \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\left(\frac{\pi}{2} - \tan^{-1}(\sqrt{\frac{4-x}{x}})\right) \\ &= \frac{4}{4-x} + \frac{4\sqrt{x}}{(4-x)^{\frac{3}{2}}}\tan^{-1}(\sqrt{\frac{x}{4-x}}) \\ &= \frac{4}{4-x}\left(1+\sqrt{\frac{x}{4-x}}\tan^{-1}(\sqrt{\frac{x}{4-x}})\right) \end{aligned}$$

$$\sum_{n=0}^{+\infty}\frac{1}{\binom{2n}{n}} = \sum_{n=0}^{+\infty}\frac{(n!)^2}{2n!} = y(1) = \frac{4}{3}\left(1+\frac{1}{\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}})\right) = \frac{4}{3}\left(1+\frac{1}{\sqrt{3}}\frac{\pi}{6}\right)$$

2-10 Consider the differential equation

$$(1+x^2)y^{''} + 2xy' - 2y = 0.$$

Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $2xy' = \sum_{n=0}^{+\infty} 2na_n x^n$, $x^2y^{''} = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$, $y^{''} = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$.

Then $a_{n+2} = -\dfrac{n-1}{n+1}a_n = \dfrac{(n!)^2}{2n!}$, $a_3 = 0$, $a_2 = a_0$.

We deduce that $a_{2n+1} = 0$ for $n \geq 1$ and $a_{2n} = \dfrac{(-1)^{n-1}}{2n-1}$ for $n \geq 1$.

$y = x$ and $\sum_{n=0}^{+\infty}\dfrac{(-1)^{n-1}x^{2n}}{2n-1}$ are solutions of the given differential equation.

If $z = \sum_{n=1}^{+\infty}\dfrac{(-1)^{n-1}x^{2n-1}}{2n-1}$, $z' = \sum_{n=1}^{+\infty}(-1)^{n-1}x^{2n-2} = \dfrac{1}{1+x^2}$. Then $z = \tan^{-1}(x)$ and the general solution of the differential equation is

$$y = ax + b(1 + x\tan^{-1}(x)).$$

2-11 Consider the differential equation

$$4xy^{''} + 2y' - y = 0.$$

Find a power series $\sum_{n=0}^{+\infty} a_n x^n$ solution of the differential equation.

2-12 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $2y' = \sum_{n=0}^{+\infty} 2(n+1)a_{n+1}x^n$, $4xy'' = \sum_{n=0}^{+\infty} 4n(n+1)a_{n+1}x^n$. Then $a_n = \frac{1}{2(n-1)(2n-1)}a_{n-1} = \frac{a_0}{2n!}$.
Then for $y = \cosh(\sqrt{|x|})$ is a solution of the differential equation. The function $\sinh(\sqrt{x})$ is also a solution for $x > 0$.

2-13 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. The radius of convergence is 1.
$y'' = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$, and $xy = \sum_{n=1}^{+\infty} a_{n-1}x^n$ and $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$. We have: $y''(0) = 1$, then $a_2 = \frac{1}{2}$ and $a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$.

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)}.$$

Let $u_n = a_{3n}$, $v_n = a_{3n+1}$, $w_n = a_{3n+2}$.

$$u_{n+1} = \frac{u_n}{3(3n+2)(n+1)} + \frac{1}{3(3n+2)(n+1)},$$
$$v_{n+1} = \frac{v_n}{3(n+1)(3n+4)} + \frac{1}{3(n+1)(3n+4)},$$
$$w_{n+1} = \frac{w_n}{(3n+4)(3n+5)} + \frac{1}{(3n+4)(3n+5)}.$$

2-14 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $y'' = \sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}x^n$, $xy' = \sum_{n=0}^{+\infty} n a_n x^n$ and $\cos x = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n}}{2n!}$. Then

$$a_{2n+1} = -\frac{a_{2n-1}}{2n+1} = (-1)^n \frac{2^n n!}{(2n+1)!}a_1$$

and

$$a_{2n} = -\frac{a_{2(n-1)}}{2n} + \frac{(-1)^n}{2n!}.$$

2-15 Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation. $4xy'' = \sum_{n=0}^{+\infty} 4n(n+1)a_{n+1}x^n$, $-2y' = \sum_{n=0}^{+\infty} -(n+1)a_{n+1}x^n$ and $9x^2y = \sum_{n=0}^{+\infty} 9a_{n+2}x^n$.
Then $a_1 = a_2 = 0$ and

$$a_{n+1} = -\frac{9a_{n-2}}{9(n+1)(2n-1)}.$$

$$a_{3n} = \frac{(-1)^n}{2n!}, \qquad a_{3n+1} = a_{3n+2} = 0.$$

For $x > 0$, $y = \cos(x^{\frac{3}{2}})$ is a solution. Also $y = \sin(x^{\frac{3}{2}})$ is a solution.

2-16 Consider the following differential equation

$$y'' + xy' - xy = 0. \tag{5.13}$$

1) Let $y = \sum_{n=0}^{+\infty} a_n x^n$ be a solution of the differential equation (5.13).
$xy' = \sum_{n=0}^{+\infty} na_n x^n$, $y'' = \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n$ and $xy = \sum_{n=1}^{+\infty} a_{n-1}x^n$.
Then $a_2 = 0$ and

$$(n+3)(n+2)a_{n+3} + (n+1)a_{n+1} - a_n = 0, \qquad \forall n \in \mathbb{N}.$$

2) $(n+1)b_{n+1} = (n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n = b_n$.

3) We deduce that $b_n = \dfrac{a_0 + a_1}{n!}$ for all $n \in \mathbb{N}$.

4) $R' = +\infty$ and $\sum_{n=0}^{+\infty} b_n x^n = (a_0 + a_1)e^x$.
$\sum_{n=0}^{+\infty} b_n x^n = \sum_{n=0}^{+\infty} \left((n+1)a_{n+1} + a_n + a_{n-1}\right)x^n = S'(x) + (x+1)S(x)$, $\forall x \in]-R, R[$.

5) Then the function S is solution of the differential equation

$$y'(x) + (x+1)y(x) = (a_0 + a_1)e^x, \qquad \forall x \in]-R, R[.$$

The solutions of this linear differential equation is $y = (a+b\int_0^x e^{\frac{t^2}{2}+2t}dt)e^{\frac{x^2}{2}+2x}$.

Let y, be a function of class C^2 on $\mathbb{R}$.

(a) Prove that the function y is solution of the equation (5.13) on $\mathbb{R}$ if and only if it is solution of the differential equation

$$y'(x) + (x+1)y(x) = (y'(0) + y(0))e^x, \qquad \forall x \in]-R, R[. \quad (5.14)$$

(b) Prove that the function $f = (a + b\int_0^x e^{\frac{t^2}{2}+2t}dt)e^{\frac{x^2}{2}+2x}$ is a solution of the equation (5.14).

(c) Deduce the set of solutions of the equation (5.13) on $\mathbb{R}$.

2-17 Let $y = \sum_{n=0}^{+\infty} a_n x^n$, $y^{'} = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $-2xy^{'} = -\sum_{n=1}^{\infty} 2n a_n x^n$, $y^{''} = \sum_{n=0}^{\infty}(n+1)(n+1)a_{n+2}x^n$. Then

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

and

$$a_{n+2} = \frac{2n-1}{(n+2)(n+1)} a_n, \quad \forall n \geq 0.$$

We deduce that

$$a_{2n} = \frac{-1 \cdot 3 \cdot 7 \cdots (4n-5)}{(2n)!} a_0$$

and

$$a_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdots (4n-3)}{(2n+1)!} a_1$$

The independent solution of the differential equation are:

$$y_1 = \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

and

$$y_2 = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

3-1

Bibliography

[1] D. Zill and W. Wright,*Differential equations and boundary value problems*, Brooks/Cole, Boston, (2013), 8^{th} edition.