King Saud University,
College of Sciences
Mathematical Department.

Mid-Term Exam/S1/2022
Full Mark:30. Time 2H
05/10/2022

Question 1. [5,4] **a)** Determine and sketch the largest local region of the $xy-$plane for which the initial value problem

$$\begin{cases} (x^2-1)dy+\left(3+y+\sqrt{y-4x}\right)dx=0 \\ y(0)=2, \end{cases}$$

has a unique solution.

b) Find the general solution of the differential equation

$$(xy+x)dx+(x^2y^2+x^2+y^2+1)dy=0.$$

Question 2. [4, 4]. **a)** Show that $\mu(x,y)=xy$ is an integrating factor for the differential equation

$$\left(\frac{y}{x^2}+\frac{2\ln y}{y}\right)dx+\left(\frac{x}{y^2}+\frac{2\ln x}{x}\right)dy=0,\ x>0,\ y>0,$$

and hence solve the differential equation.

b) obtain the general solution of the differential equation

$$(2x+y)\frac{dy}{dx}-1-(2x+y)^2=0,\ 2x+y\neq 0.$$

Question 3. [4, 4]. **a)** Solve the initial value problem

$$\begin{cases} (6xy+2y^2-5)dx+(3x^2+4xy-6)dy=0 \\ y(1)=1 \end{cases}$$

b) Solve the differential equation

$$5xy^2y'+y^3=32(1+\ln x)y^{-2},\quad x>0,\ y\neq 0.$$

Question 4. [5] Assume that the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 290 K (here K stands for Kelvin) and the substance cools from 370 K to 330 K in 10 minutes, find when the temperature will be 295 K.

Answer Sheet Midterm Exam Math 204
Sem I – 1444H

Question 1 a) Determine and sketch the largest local region for which the following initial value problem admits unique solution:

$$\begin{cases}(x^2-1)\,dy + (3+y+\sqrt{y-4x})\,dx = 0 \\ y(0) = 2\end{cases}$$

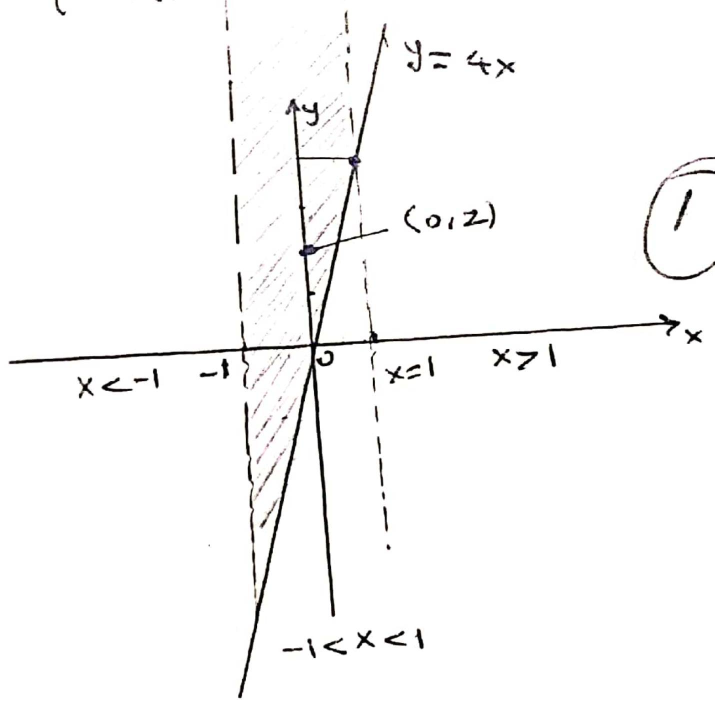
Solution: $(x^2-1)\,y' = -3xy - \sqrt{y-4x}$

$$y' = \frac{-3x}{x^2-1}\,y - \frac{\sqrt{y-4x}}{x^2-1} = f(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{-3x}{x^2-1} - \frac{1}{x^2-1}\cdot\frac{1}{2\sqrt{y-4x}}$$ 1

f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y);\ x \neq \mp 1,\ y > 4x\}$$



1

$$R = \{(x,y);\ x<-1\ \&\ y>4x\} \cup \{(x,y);\ -1<x<1,\ y>4x\} \cup \{(x,y);\ x>1,\ y>4x\}$$

But $y(0)=2 \Rightarrow (0,2) \in R_1 = \{(x,y);\ -1<x<1,\ y>4x\}$ 3

Then R_1 is the largest region for which the IVP admits unique solution.

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Midterm Exam Math 204

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Q.1 b) Solve the following differential equation:

$$(xy+x)\,dx = -(x^2y^2+x^2+y^2+1)\,dy$$

Ans

$$x(y+1)\,dx = -\left(x^2(y^2+1)+(y^2+1)\right)dy$$

$$x\,(y+1)\,dx = -(x^2+1)(y^2+1)\,dy$$

$$\boxed{\frac{x}{x^2+1}\,dx = -\frac{y^2+1}{y+1}\,dy}$$ it's a separable equation (1)

Integrating both sides,

$$\frac{1}{2}\int\frac{2x}{x^2+1}\,dx = -\int\frac{y^2+1}{y+1}\,dy = -\int\left[(y-1)+\frac{2}{y+1}\right]dy$$ (1)

$$\ln(x^2+1) + c = -y^2/2 + y - 2\ln|y+1|$$ (2)

The general solution is: $y^2/2 - y + 2\ln|y+1| + \ln(1+x^2) = c$ with $c \in \mathbb{R}$.

Question2 a)

$$\left(\frac{y}{x^2}+\frac{2}{y}\ln y\right)dx+\left(\frac{2}{x}\ln x+\frac{x}{y^2}\right)dy=0 \quad , x>0\ y>0$$

Multiply by $\mu(x,y)=xy$, we have

$$\left(\underbrace{\frac{y^2}{x}+2x\ln y}_{M}\right)dx+\left(\underbrace{2y\ln x+\frac{x^2}{y}}_{N}\right)dy=0 \quad (*)$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=\frac{2y}{x}+\frac{2x}{y}\\ \frac{\partial N}{\partial x}=\frac{2y}{x}+\frac{2x}{y}\end{array}\right\} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow Eq(*) \text{ is exact}$$

1

$$\Rightarrow \exists\ F(x,y): \begin{cases}\frac{\partial F}{\partial x}=\frac{y^2}{x}+2x\ln y \longrightarrow (1)\\ \frac{\partial F}{\partial y}=\frac{x^2}{y}+2y\ln x \longrightarrow (2)\end{cases}$$

1

From Eq(1): $F(x,y)=y^2\ln x+x^2\ln y+C(y)$

$$\frac{\partial F}{\partial y}=2y\ln x+\frac{x^2}{y}+C'(y) \longrightarrow (3)$$

2

From (2) and (3), it follows that $C'(y)=0$

$$\Rightarrow C(y)=C^*$$

Hence $y^2\ln x+x^2\ln y=C^*$

Q_2 b) $(2x+y)y' - 1 - (2x+y)^2 = 0$

Let $u = 2x+y \Rightarrow u' = 2 + y' \Rightarrow y' = u' - 2$

Hence $u(u'-2) = 1 + u^2$ 1

$$\Rightarrow \frac{du}{dx} - 2 = \frac{1+u^2}{u}$$

$$\Rightarrow \frac{du}{dx} = 2 + \frac{1+u^2}{u} = \frac{u^2+2u+1}{u}$$

$$\Rightarrow \int \frac{u}{(u+1)^2}\,du = \int dx = x + c$$ 1

Let $u+1 = z \Rightarrow du = dz$

$$\int \frac{z-1}{z^2}\,dz = x + c$$

$$\Rightarrow \ln|z| + \frac{1}{z} = x + c$$

$$\Rightarrow \ln|u+1| + \frac{1}{1+u} - x = c$$

$$\Rightarrow \ln|2x+y+1| + \frac{1}{1+2x+y} - x = c$$

Solve the following D. E

$(6xy+2y^2-5)dx+(3x^2+4xy-6)dy=0$

Solution:

$$\begin{cases} M(x,y) = 6xy+2y^2-5 \\ N(x,y) = 3x^2+4xy-6 \end{cases}$$

$M_y = 6x+4y = N_x$

Then the D.E is exact. 1

$F(x,y)$? $\begin{cases} \dfrac{\partial F}{\partial x} = 6xy+2y^2-5 \\ \dfrac{\partial F}{\partial y} = 3x^2+4xy-6 \end{cases}$

$\Rightarrow F(x,y) = \int (6xy+2y^2-5)dx + h(y)$ 1

$\Rightarrow F(x,y) = 3x^2y+2y^2x-5x+h(y)$

Then $\dfrac{\partial F}{\partial y} = 3x^2+4xy+h'(y)$

$= 3x^2+4xy-6$ 1

Thus $h'(y) = -6 \Rightarrow h(y) = -6y$

So, we conclude that the Solution is $\{(x,y) : 3x^2y+2y^2x-5x-6y = c ; c \in \mathbb{R}\}$.

$y(1)=1 \Rightarrow 3+2-5-6=c \Rightarrow c=-6$ 1 b)

$5xy^2y' + y^3 = 32(1+\ln x)y^{-2}, \quad x>0, \quad y \neq 0$

Divide by $5xy^2$: $\quad y' + \frac{1}{5x}y = \frac{32}{5x}(1+\ln x)y^{-4}$ (BE) 1

Divide again by y^{-4}: $\quad y^4y' + \frac{1}{5x}y^5 = \frac{32}{5x}(1+\ln x)$

Now let $u = y^5 \Rightarrow 5y^4y' = u'$. Hence

$$\frac{u'}{5} + \frac{1}{5x}u = \frac{32}{5x}(1+\ln x) \quad \text{(LE)}$$

1

Standard form: $\quad u' + \frac{1}{x}u = \frac{32}{x}(1+\ln x)$

$$\mu(x) = e^{\int \frac{dx}{x}} = x$$

1

Then $\quad \frac{d}{dx}(xu) = 32(1+\ln x)$

$\Rightarrow \quad xu = 32x + 32\int \ln x\,dx = 32x + 32(x\ln x - x) +$

$\Rightarrow xu = 32x\ln x + C$ 1

$\Rightarrow y^5 = 32\ln x + \frac{C}{x}$, is the solution of the D.E.

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Question 4

Assume that the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 290 K (here K stands for Kelvin) and the substance cools from 370 K to 330 K in 10 minutes, find when the temperature will be 295 K.

Solution :

Let T denote the temperature of the substance at time t.

We have $$\frac{dT}{dt} = -k(T - T_s), \quad k>0 \text{ constant of proportionality} \quad (1$$

T_s is the temperature of the air : $T_s = 290$

So $$\frac{dT}{T-290} = -k\,dt \Rightarrow \ln|T-290| = -kt + c_1$$

$$\Rightarrow T - 290 = e^{c_1 - kt} = \underbrace{e^{c_1}}_{=c} e^{-kt} = c e^{-kt}$$

$$\Rightarrow T(t) = 290 + c e^{-kt} \quad 1$$

At $t=0$, we have $T(0) = 370 \Rightarrow T(0) = 370 = 290 + c e^{0}$

$$\Rightarrow c = 80 \Rightarrow T(t) = 290 + 80 e^{-kt}$$

At $t = 10$, we have $T(10) = 330 = 290 - 80 e^{-kt}$ 2

$$\Rightarrow k = \frac{\ln 2}{10} \Rightarrow \boxed{T(t) = 290 + 80 e^{-t\frac{\ln 2}{10}}}$$

when $T = 295$, then $295 = 290 + 80 e^{-t\frac{\ln 2}{10}}$

$$\Rightarrow e^{-t\frac{\ln 2}{10}} = \frac{5}{80} \Rightarrow \ln e^{-t\frac{\ln 2}{10}} = \ln\frac{1}{16} = -\ln 2^4 = -4\ln 2 \quad 1$$

$$\to -t\ln 2 = -4\ln 2 \Rightarrow t = 40 \text{ mn}$$

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