King Saud University
Department of Mathematics

Midterm Exam in Math151
Semester 1, 1443 H.

Q1. (a) Without using truth tables, show that $p \wedge (q \to p) \equiv (p \to q) \to p$. (3 pts)

(b) Let n be an integer. Use a direct proof to show that if $3 \mid (n-1)$, then $3 \mid (n^2 - n + 6)$. (3pts)

(c) Let a be a real number such that a^2 is irrational. Use a proof by contradiction to show that $a+1$ is irrational. (2pts)

Q2. (a) Use induction to show that

$$4 + 8 + 12 + \ldots + 4n = 2n(n+1),$$

for all integers $n \geq 1$. (4 pts)

(b) Let $\{u_n\}$ be a sequence defined by the equations:

$$u_1 = 1,\ u_2 = 3 \text{ and } u_{n+1} = 2u_{n-1} + u_n + 2 \text{ for } n = 2, 3, 4, \ldots.$$

Show that $u_n = 2^n - 1$ for all $n \geq 1$. (4 pts)

Q3. (a) Let R be the relation from $A = \{-1, 1, 2\}$ to $B = \{2, 3, 4, 5\}$ defined by aRb iff $a + b > 4$.

(i) List the ordered pairs of R. (2pts)

(ii) Represent R by a matrix. (1pts)

(iii) Find the domain and image (range) of R. (1pts)

(b) Let $S = \{(w,y), (x,x), (x,y), (x,z), (z,w)\}$ and $T = \{(w,w), (w,z), (y,x), (z,w), (z,y)\}$ be relations on $X = \{w, x, y, z\}$.

(i) Find $S \cap T^{-1}$. (1pts)

(ii) Find $S - T^{-1}$. (1pts)

(iii) Find $\overline{S \cup T}$. (2pts)

(iv) Find $T \circ S$. (2pts)

(c) Let R be the relation defined on $\mathbb{Z}$ by mRn iff $m - n$ is odd. Determine whether R is reflexive, symmetric, antisymmetric or transitive. (4pts)

Dr Bohen

Answer Sheet Midterm Exam Math 151
First Semester 1443

Q1 (a) $p \wedge (q \to p) \overset{?}{\equiv} (p \to q) \to p$

$(p \to q) \to p \equiv (\neg p \vee q) \to p \equiv \neg(\neg p \vee q) \vee p$ 1

$\equiv (p \wedge \neg q) \vee p \equiv (p \vee p) \wedge (\neg q \vee p)$ 1

$\equiv p \wedge (q \to p)$ 1

(b) $3 \mid n-1 \overset{?}{\Longrightarrow} 3 \mid (n^2 - n + 6)$

As $3 \mid n-1$ then there exists $k \in \mathbb{Z}$ such $(n-1) = 3k$ 1

Also $n^2 - n + 6 = n(n-1) + 6$

So $n^2 - n + 6 = 3kn + 6$

$= 3(kn+2) = 3M$ with $M = (kn+2) \in \mathbb{Z}$

2

So $3 \mid n^2 - n + 6$

(c) We suppose that $(a+1)$ is a rational number.

Then a is a rational number. So a^2 is also a rational number (as product of 2 rational numbers). ↯ 2

Impossible because $a^2 \notin \mathbb{Q}$.

we deduce that $(a+1) \notin \mathbb{Q}$.

Q2 (a) Put $P(n)$: $4 + 8 + 12 + \dots + 4n = 2n(n+1)$

- Basis step: $n = 1$

$4 \overset{?}{=} 2.1(1+1) = 4$ ✓ So $P(1)$ is true

1

- Inductive step: Let $k \geq 2$. We suppose that $P(k)$ is true.

(we have: $4 + 8 + 12 + \dots + 4k = 2k(k+1)$.

Now we prove that $P(k+1)$ remains true.

$4 + 8 + \dots + 4(k+1) \overset{?}{=} 2(k+1)(k+2)$

3

$\underbrace{4 + 8 + \dots + 4k} + 4(k+1) = \underbrace{2k(k+1)} + 4(k+1)$

$= 2(k+1)(k+2)$ ✓

we deduce for $n \geq 1$; $P(n)$ is true.

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(b) We use the Second principal of mathematical Induction:

Put $P(n)$: $u_n = 2^n - 1$

- <u>Basis step</u> : $n = 1$ $\quad 1 = u_1 \overset{?}{=} 2^1 - 1 = 1$ ✓ $\;\Big|\;$ $n = 2$ $\quad 3 = u_2 \overset{?}{=} 2^2 - 1 = 3$ ✓

1

So $P(1)$ & $P(2)$ are true.

- <u>Inductive step</u>: let $k \geq 3$, we suppose that $P(3), \dots, P(k)$ are all true. Now we prove that $P(k+1)$ remains true.

As $u_{k+1} = 2u_{k-1} + u_k + 2$ $(*)$

And $P(k)$ & $P(k-1)$ are true then we have:

1

$$u_k = 2^k - 1 \quad \text{and} \quad u_{k-1} = 2^{k-1} - 1$$

By substitution in $(*)$, we get:

$$u_{k+1} = 2(2^{k-1} - 1) + (2^k - 1) + 2$$

2

$$u_{k+1} = 2^k - \cancel{2} + 2^k - 1 + \cancel{2}$$

$$u_{k+1} = 2 \cdot 2^k - 1 = 2^{k+1} - 1 \checkmark$$

We deduce that $u_n = 2^n - 1$ for $n \geq 1$.

Q3

2 (a) (i) $R = \{(1,4); (1,5); (2,3); (2,4); (2,5)\}$

1 (ii)

$$M_R = \begin{array}{c} \\ -1 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{matrix} 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

1 (iii) The domain of R is $D_R = \{1, 2\} \subset A$.
The image of R is $Im_R = \{4, 5, 3\} \subset B$

(b) (i) $S = \{(w,y); (x,x); (\underline{x,y}); (x,z); (\underline{z,w})\}$

$T^{-1} = \{(w,w); (\underline{z,w}); (\underline{x,y}); (w,z); (y,z)\}$

1 $S \cap T^{-1} = \{(x,y); (z,w)\}$

1 (ii) $S - T^{-1} = \{(w,y); (x,x); (x,z)\} \subset S$

(iii) $S \cup T = \{(w,y); (x,x); (x,y); (x,z); (z,w); (w,w); (w,z); (y,x); (z,y)\}$

$$M_{S\cup T} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{matrix} w & x & y & z \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$\overline{S \cup T} = X \times X - (S \cup T)$

2 $\overline{S \cup T} = \{(w,x); (x,w); (y,w); (y,y); (y,z); (z,x); (z,z)\}$

or Notice that $\overline{S \cup T} = \overline{S} \cap \overline{T}$

(iv) $S = \{(w,y);(x,x);(x,y);(x,z)\,(z,w)\}$

$T = \{(w,w);(w,z);(y,x);(z,w);(z,y)\}$

2 Then $T \circ S = \{(w,x);(x,x);(x,w);(x,y);(z,w);(z,z)\}$

(c) 1 • R is not reflexive because $m \not R m$ ie $m-m=0$ is even

1 • R is symmetric on $\mathbb{Z}$ because if $m R n$ then $(m-n)$ is odd

Also $(n-m)$ is odd so $n R m$.

1 • R is not antisymmetric because $5R2$ and $2R5$ but $2 \neq 5$

1 • R is not transitive because $5R2$ and $2R1$ but $5 \not R 1$.