College of Sciences
Department of
Statistics and Operations
Research

كلية العلوم
قسم الإحصاء
وبحوث العمليات

Mid 1 Exam

Monday, March 9, 2020	ACTU 466	Academic year 1441-42H
10:00 – 11:30 AM	LOSS	Second Semester

Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Classroom No.		رقم قاعة الاختبار
Teacher's Name		اسم أستاذ المقرر
Roll Number		رقم التحضير

Instructions

- Switch off your mobile and place it under your seat.
- Time allowed is 90 minutes.
- You are not asked to provide **detailed explanations** for all answers.
- For exercise 5, four responses are given, only one is correct.

Exercise 1 *Consider the random variables* $X_1 \sim$ *Pareto*($\alpha = 9, \theta = 7$) *with a density* f_{X_1} *and* $X_2 \sim$ *Exponential(mean* $= 3$) *with a density* f_{X_2} *and* $X_3 \sim$ *Exponential(mean* $= 5$) *with a density* f_{X_3}. *Construct a spliced distribution* X *from* f_{X_1} *on* $[0, 2)$, $f_{X_2}(x)$ *on* $[2, 7)$, *and* $f_{X_3}(x)$ *on* $[7, \infty)$, *such that under the new distribution the probabilities of these intervals are* $1/4$ *and* $2/3$ *and* $1/12$ *respectively.*

Hint: write $f(x) = \begin{cases} a \times f_{X_1}(x) \text{ for } x \in [0, 2) \\ b \times f_{X_2}(x) \text{for } x \in [2, 7) \\ c \times f_{X_3}(x) \text{for } x \in [7, \infty) \end{cases}$

Exercise 2 *The YANG insurance company provided earthquake insurance houses in San Francisco. For* 2015, *the claims* X *were distributed as a Pareto distribution with* $\alpha = 3$ *and* $\theta = 100,000$. *In* 2015, *the earthquake insurance did not have any deductibles, but had an upper limit* u *so that the expected cost* $E(X \wedge u)$ *for the coverage with policy limit in* 2015 *was* 80% *of the expected value* EX *of the loss. Determine the upper limit* u.

Exercise 3 *The number of losses* N *per policy for health insurance is distributed as a Negative binomial with a mean of* 6. *The amounf of each loss* Y *is follows a Weibull distribution with* $\tau = 2$ *and* $\theta = 500$. *An insurance policy covering health insurance has an ordinary deductible* d *such that the expected number of claims per policy (or the expected number of payments) is* 5. *Calculate the amount of the deductible* d.

Exercise 4 *The random variable* X *represents loss under comprehensible coverage for automobile insurance. Losses are distributed as a two point mixture distribution as follows:*

Distribution	***Weight***
Pareto distribution with $\alpha = 5$ *and* $\theta = 1,000$	0.6
Pareto distribution with $\alpha = 3$ *and* $\theta = 6,000$	0.4

Calculate the $var(X)$.

Exercise 5 *You are given that* X *is the random variable representing the amount of a claim from an automobile accident.* X *is distributed as a two parameter Pareto Distribution with mean of* 2,000 *and a variance of* 6,000,000. *The Henry Auto Insurance Company issues automobile policies with a deductible* d *of* 500. *This policy will pay for the amount of any claim in excess of* 500. *Calculate the expected value of payment per claim incurred* $E(Y^L)$. *This is the same as* $E[(X - 500)^+]$.
Hint: calculate first the values of the parameters of the Pareto distribution.
a)76,435.98 b) 5,476.76 c) 54.54 d) 1,567.05

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1}-1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

A.3.2.2 Inverse gamma (Vinci)—α, θ

$$
\begin{aligned}
f(x) &= \frac{(\theta/x)^\alpha e^{-\theta/x}}{x\Gamma(\alpha)} \qquad F(x) = 1 - \Gamma(\alpha; \theta/x) \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha \qquad \mathrm{E}[X^k] = \frac{\theta^k}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}[1 - \Gamma(\alpha - k; \theta/x)] + x^k \Gamma(\alpha; \theta/x) \\
&= \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)} G(\alpha - k; \theta/x) + x^k \Gamma(\alpha; \theta/x), \text{ all } k \\
\text{mode} &= \theta/(\alpha + 1)
\end{aligned}
$$

A.3.2.3 Weibull—θ, τ

$$
\begin{aligned}
f(x) &= \frac{\tau (x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} \qquad F(x) = 1 - e^{-(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 + k/\tau), \quad k > -\tau \\
\mathrm{VaR}_p(X) &= \theta[-\ln(1-p)]^{1/\tau} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 + k/\tau) \Gamma[1 + k/\tau; (x/\theta)^\tau] + x^k e^{-(x/\theta)^\tau}, \quad k > -\tau \\
\text{mode} &= \theta \left(\frac{\tau - 1}{\tau} \right)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3.2.4 Inverse Weibull (log Gompertz)—θ, τ

$$
\begin{aligned}
f(x) &= \frac{\tau (\theta/x)^\tau e^{-(\theta/x)^\tau}}{x} \qquad F(x) = e^{-(\theta/x)^\tau} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1 - k/\tau), \quad k < \tau \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1/\tau} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1 - k/\tau)\{1 - \Gamma[1 - k/\tau; (\theta/x)^\tau]\} + x^k \left[1 - e^{-(\theta/x)^\tau}\right], \quad \text{all } k \\
&= \theta^k \Gamma(1 - k/\tau) G[1 - k/\tau; (\theta/x)^\tau] + x^k \left[1 - e^{-(\theta/x)^\tau}\right] \\
\text{mode} &= \theta \left(\frac{\tau}{\tau + 1} \right)^{1/\tau}
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad \mathrm{E}[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
\mathrm{E}[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p) \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1) \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}
$$

A.3.3.2 Inverse exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}
$$

A.5 Other distributions

A.5.1.1 Lognormal—μ,σ (μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
\mathrm{E}[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
\mathrm{E}[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}
$$

B.2.1.2 Geometric—β

$$
\begin{aligned}
p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\
\mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}.
\end{aligned}
$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer$)$

$$
\begin{aligned}
p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\
p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\
\mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m.
\end{aligned}
$$

B.2.1.4 Negative binomial—β, r

$$
\begin{aligned}
p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\
p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\
\mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}.
\end{aligned}
$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

loss HW1 Solution

Ex1

$$f(x) = \begin{cases} \frac{1}{4} \times A f_1(x) & 0 < x < 2 \\ 2/3 \times B f_2(x) & 2 < x < 7 \\ 1/12 \times C f_3(x) & x > 7 \end{cases}$$

where $f_1(x)$ density of Pareto $(\alpha = 9, \theta = 7)$

$f_2(x)$ " " Exp (mean = 3)

$f_3(x)$ " " Exp (mean = 5)

$$\int_0^2 A f_1(x)dx = 1 \iff A\, F_1(2) = 1$$

(You should not calculate the integral. We have new did it)

$$\iff A = \frac{1}{1-(7/9)^9} = 1.11627$$

$$\int_2^7 B f_2(x) = 1 \iff B\left[S_2(2) - S(7)\right] = 1$$

$$\iff B = \frac{1}{e^{-2/3} - e^{-7/3}} = 2.401277$$

$$\int_7^\infty C f_3(x)dx = 1 \iff C \times S(7) = 1$$

$$\iff C \times e^{-7/5} = 1 \iff C = e^{7/5}$$

$$= 4.0552$$

$$f(x) = \begin{cases} 0.279 \times f_1(x) \\ 1.600 \times f_2(x) \\ 0.3379 \times f_3(x) \end{cases}$$

Ex2

$$E(X\wedge u) = 0.8 \times E(X)$$

$$E(X) = \frac{\theta}{\alpha-1} = 50{,}000$$

$$E(X\wedge u) = \frac{\theta}{\alpha-1}\left[1-\left(\frac{\theta}{u+\theta}\right)^{\alpha-1}\right] = 0.8\,\frac{\theta}{\alpha-1}$$

$$\Leftrightarrow\ 50{,}000 - 50{,}000\left(\frac{\theta}{u+\theta}\right)^2 = 0.8 \times 50{,}000$$

$$\Leftrightarrow\ 0.2 \times 50{,}000 = 50{,}000\left(\frac{\theta}{u+\theta}\right)^2$$

$$\Leftrightarrow\ \left(\frac{\theta}{u+\theta}\right) = \sqrt{0.2} \Rightarrow u = \frac{(1-\sqrt{0.2})\,100{,}000}{\sqrt{0.2}}$$

(all calculations should be done)

$$= 123{,}607$$

Ex3

Solution 1

For a given value of N, the number of losses, N^P, is the number of losses for which loss $Y > d$.

(= the number of success)

Then $N^P | N \sim \text{Binomial}(N,\ p = P(Y > d))$

$$\Rightarrow E(N^P|N) = Np \Rightarrow E(N^P) = E(E(N^P|N)) = E(Np) \underset{\text{indep}}{=} p\,E(N) \Rightarrow p = 5/6$$

---- $d = 213.4956$

Solution 2 ([redacted]

$$N^P = \sum_{i=1}^{N} \mathbb{1}_{Y_i > d}$$

$\Rightarrow N^P$ is a compound distribution

$$E(N^P) = E(N) \times \underbrace{E(\mathbb{1}_{Y>d})}_{= P(Y>d)}$$

⋮

Ex 4

$E(X) = 0.6\ E(X_1) + 0.4\ E(X_2)$

$= 0.6 \times \frac{1000}{4} + 0.4 \times 3000 = 1350$

$E(X^2) = 0.6\ E(X_1^2) + 0.4\ E(X_2^2)$

$= 0.6 \times 2 \times \frac{1000^2}{12} + 0.4 \times 2 \times \frac{6000^2}{2}$

$= 14,500,000$

$V(X) = E(X^2) - [E(X)]^2 = 12,677,500$

Ex 5

$E(X) = \frac{\theta}{\alpha - 1} = 2000 \Rightarrow \theta = (\alpha-1)2000$

$E(X^2) = [E(X)]^2 + V(X) = 10^7$

$\Rightarrow E(X^2) = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 10^7$

$\Rightarrow \frac{2 \times 4 \times 10^6 \times (\alpha-1)^2}{(\alpha-1)(\alpha-2)} = 10^7$

$\Leftrightarrow 8 \times 10^6 \times \frac{\alpha-1}{\alpha-2} = 10^7$

$\Rightarrow \frac{\alpha-1}{\alpha-2} = \frac{10}{8} = 5/4$

$\Rightarrow 4(\alpha-1) = 5(\alpha-2) \Rightarrow \alpha = 6 ; \theta = 2000(\alpha-1)$
$= 10,000$

(all calculations should be given)

~~E(X) =~~

$\underbrace{E(X)}_{2000} = E(X \wedge d) + E(Y^L)$

$E(X \wedge d) = \left(\frac{\theta}{\alpha-1}\right)\left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right] = 432.95$

$\Rightarrow E(Y^L) = 2000 - 432.95$

$= 1567.05$