King Saud University	College of Sciences	Department of Statistics

Final Examination	Math 466	Semester II	1441-1442	Time: 4H

الضوابط التنظيمية للاختبارات المنزلية Take Home Exam الخاصة بالطلاب.

1- على الطلاب ضرورة الاطلاع على كل ما يخص الاختبارات من تعليمات سواء عن طريق البريد الإلكتروني وصفحة التعليمات في الاختبار.

2- الالتزام بوقت بداية الاختبار المدرجة في البوابة الاكاديمية ونهاية الاختبار التي ستحدد من قبل عضو هيئة التدريس.

3- تسليم اجابة الاختبار دون تأخير. ومن يتأخر عن موعد التسليم لن يقبل منه.

4- تسليم الإجابة عبر البلاك بورد أو البريد الإلكتروني أو كليهما (إما مستند pdf أو صورة ممسوحة ضوئيًا).

5- يجب أن يتم إتمام الاختبار بشكل فردي. ويُحظر عرضه أو مناقشته مع أي شخص آخر، بما في ذلك (على سبيل المثال لا الحصر) الطلاب الآخرين في نفس المقرر.

6- يمكن للطالب استخدام أي مادة متاحة يريدها، بما في ذلك العروض التقديمية ومذكرة المحاضرات والكتب والإنترنت، ولا يجب نسخ المعلومة كما هي ولكن تكتب حسب فهم الطالب وإلا ستعتبر إقتباسا يؤثر على درجة الطالب.

9- سوف يتم النظر في جميع الحالات الطلابية التي لم يتمكنوا من أداء الاختبار المنزلي بعقد اختبار بديل لها في بداية الفصل الدراسي الأول من العام ١٤٤٢هـ.

Explain your reasoning for why you have answered a certain value.

Exercise 1 *You may use calculator (no need to calculate primitives). Assume the following data set:*

$$8 \quad 12 \quad 15 \quad 25 \quad 40$$

If the data is smoothed using a triangular kernel with a bandwidth of 6*, calculate the variance of the smoothed distribution.*

(A) 37.6 *(B)* 237.6 *(C)* 137.6 *(D)* 337.6 *(E)* 437.6

Exercise 2 *Losses are distributed as a Pareto distribution with parameters* $\alpha = 2$ *and* θ*. If losses are subject to an ordinary deductible of* $20,000$*, the expected cost* ***per payment*** *is* $30,000$*. If losses are subject to a franchise deductible of* $20,000$*, calculate the the expected cost* ***per-loss****.*

Exercise 3 *You are given the following ages at time of death of* 10 *individuals*

$$10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \quad 70 \quad 80 \quad 90 \quad 100.$$

Using a triangular kernel with bandwidth 20*, find the kernel estimate of* $F(51)$*.*

A) 0.40 *B)* 0.42 *C)* 0.44 *D)* 0.46 *E)* 0.48

Exercise 4 *You are given:*
(i) Losses are uniformly distributed on $(0, \theta)$ *with* $\theta > 15$
(ii) The policy limit u *is* 15*.*
(iii) A sample of payments is:

$$2, 3, 4, 5, 8, 8, 9, 10, 11, 11, 12, 12, 15, 15, 15.$$

Determine the method-of-moments estimate of θ*.*

A) 16.85 *B)* 15.85 *C)* 17.85 *D)* 19.85 *E)* 21.85

Exercise 5 (Bonus question (ii)) *There are two independent insurance policies. The claim frequency of each policy follows a Binomial*$(2, 0.1)$ *distribution. When there is a claim, the claim amount follows the zero-truncated Binomial*$(5, 0.4)$ *distribution. What is the probability that the aggregate claim* $S_1 + S_2$ *of these two policies is*
(i) What is the probability that the aggregate claim $S_1 + S_2$ *of these two policies is zero?*
(A) 0.6561 *(B)* 0.0561
(ii) What is the probability that the aggregate claim $S_1 + S_2$ *of these two policies is one?*

Loss - Final Solution

Ex 1 (5 marks)

y-b	y	y+b
2	8	14
6	12	18
9	15	21
19	25	31
34	40	46

$$f(x) = \frac{1}{5} \sum_{i=1}^{5} k_{y_i}(x)$$

where

$$k_{y_i}(x) = \begin{cases} \frac{1}{36}\left(x-(y_i-b)\right) & \text{for } x \text{ in } (y_i-b, y) \\ \frac{1}{36}\left(y_i+b-x\right) & \text{for } x \text{ in } (y, y_i+b) \end{cases}$$

$$E(X) = \frac{1}{5\times 36}\Big[\int_2^8 x(x-2)\,dx + \int_8^{14} x(14-x)\,dx$$

$$+ \int_6^{12} x(x-6)\,dx + \int_{12}^{18} x(18-x)\,dx$$

$$+ \int_9^{15} x(x-9)\,dx + \int_{15}^{21} x(21-x)\,dx$$

$$+ \int_{19}^{25} x(x-19)\,dx + \int_{25}^{31} x(31-x)\,dx$$

$$+ \int_{34}^{40} x(x-34)\,dx + \int_{40}^{46} x(46-x)\,dx$$

$$= \frac{3600}{5\times 36} = 20.$$

$$E(X^2) = \frac{1}{5\times 36}\Big[\int_2^8 x^2(x-2)\,dx + \int_8^{14} x^2(14-x)\,dx + \cdots\Big]$$

$$= \frac{96768}{5\times 36} = 537.6$$

$$V(X) = 537.6 - 400 = 137.6.$$

Or

X is a 5-point mixture distribution
let Y the discrete uniform with support $\{8, 12, 15, 25, 40\}$
$X|_{Y=y_i}$ is the r.v with density $k_{y_i}(x)$.
let $Z_i = X|_{Y=y_i}$
We have $$X = \sum_{i=1}^{5} \mathbb{1}_{Y=y_i} \times Z_i \qquad (1)$$

We show that
$$E(Z_i) = \frac{1}{b^2}\left[\int_{y_i-b}^{y_i} x(x-y_i+b)\,dx + \int_{y_i}^{y_i+b} x(y_i+b-x)\,dx\right.$$
$$= y_i$$
$$E(Z_i^2) = \frac{1}{b^2}\left[\int_{y_i-b}^{y_i} x^2(x-y_i+b)\,dx + \int_{y_i}^{y_i+b} x^2(y_i+b-x)\,dx\right]$$
$$= \frac{b^2}{6} + y_i^2 .$$

$$(1) \Rightarrow E(X) = \sum_{i=1}^{5} \underbrace{P(Y=y_i)}_{1/5} \cdot E(Z_i) = \frac{\sum y_i}{5} = 20$$

$$E(X^2) = \sum_{i=1}^{5} P(Y=y_i)\, E(Z_i^2) = \frac{b^2}{6} + \frac{\sum y_i^2}{5}$$
$$= \frac{36}{6} + 531.6 = 537.6$$

$$\Rightarrow VX = \underbrace{\left[\frac{\sum y_i^2}{5} - \bar{y}^2\right]}_{V_e(x)} + \frac{b^2}{6} = 137.6$$

Ex 2 (4 marks) (3

$X \sim \text{Pareto}(\alpha = 2, \theta)$; $d = 20{,}000$; $E(Y_1^P) = 30{,}000$

franchise $d = 20{,}000$ $E(Y_2^L)$?

$$E(Y_1^L) = E(X) - E(X \wedge d)$$

$$= \frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right]$$

$$= \frac{\theta}{\alpha-1}\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}$$

$$E(Y_1^P) = \frac{E(Y_1^L)}{P(X>d)} = \frac{\frac{\theta}{\alpha-1}\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} = \frac{d+\theta}{\alpha-1}$$

$$= \frac{20{,}000+\theta}{2-1} = 30{,}000 \Rightarrow \theta = 10{,}000$$

franchise $E(Y_2^L) = \underbrace{E(X) - E(X \wedge d)}_{E(Y_1^L)} + d\, P(X>d)$

$$= 30{,}000\, P(X>d) + d\, P(X>d)$$

$$= (30{,}000 + 20{,}000)\left(\frac{\theta}{d+\theta}\right)^{\alpha}$$

$$= 50{,}000\left(\frac{10{,}000}{30{,}000}\right)^2 = 5{,}555.556$$

Ex 3 (5 marks)

y_i-b	y_i	y_i+b	$k_{y_i}(51)$
-10	10	30	1
0	20	40	1
10	30	50	1
20	40	60	
30	50	70	
40	60	80	
50	70	90	
60	80	100	0
70	90	110	0
80	100	120	0

$$F_{y_i}(51) = \begin{cases} \frac{1}{2b^2}\left(51-(y_i-b)\right)^2 & \text{for } 51 \text{ in } (y_i-b, y_i) \\ 1 - \frac{(y_i+b-51)^2}{2b^2} & \text{for } 51 \text{ in } (y_i, y_i+b) \end{cases}$$

$$\hat{F}(51) = \frac{1}{10}\Big[1+1+1+\left(1-\frac{(60-51)^2}{2b^2}\right) + \left(1-\frac{(70-51)^2}{2b^2}\right) + \frac{(51-40)^2}{2b^2} + \frac{(51-50)^2}{2b^2}\Big] = \frac{4.6}{10}$$

$$= 0.46.$$

Ex 4 (5 marks)

$u = 15$.

$$E(X \wedge u) = \int_0^u S(x)\,dx = \int_0^u \frac{\theta - x}{\theta}\,dx$$

$$= \frac{1}{\theta}\left[\theta x - \frac{x^2}{2}\right]_0^u = \frac{1}{\theta}\left[\theta u - u^2/2\right]$$

A method of moment estimate of θ is $\hat{\theta}$ solution of

$$\frac{1}{\theta}\left(\theta u - u^2/2\right) = \bar{X} \quad (= 9.33)$$

$$\Leftrightarrow \quad \theta u - u^2/2 = \theta \bar{X}$$

$$\Rightarrow \quad \theta(u - \bar{X}) = u^2/2$$

$$\Rightarrow \quad \hat{\theta} = \frac{u^2}{2(u - \bar{X})} = \frac{15^2}{2(15 - 9.33)} = 19.85.$$

Ex 5 (1 + 2 marks)

$N \sim \text{Bin}(2, 0.1)$

$Y \sim$ Zero-truncated $\text{Bin}(5, 0.4)$

$$\begin{cases} S = \sum_1^N Y_i \\ S = 0 \text{ if } N = 0. \end{cases}$$

(a) $P(S_1 + S_2 = 0) = P(S_1 = 0) \times P(S_2 = 0)$

$= P(S_1 = 0)^2$

$P(S_1 = 0) = P(N_1 = 0) = 0.9^2 = 0.81$

$\Rightarrow P(S_1 + S_2 \neq 0) = 0.81^2 = 0.6561$

(b)

$$P(S_1+S_2=1) = P(S_1=1, S_2=0) + P(S_1=0, S_2=1)$$

$$= 2 \times P(S_1=1)\underbrace{P(S_2=0)}_{0.81}$$

$$P(S_1=1) = \underbrace{P(S_1=1 \mid N=0)}_{=0} P(N=0)$$

$$+ P(S_1=1 \mid N=1)\underbrace{P(N=1)}_{=2\times 0.1\times 0.9 = 0.18}$$

$$+ \underbrace{P(S_1=1 \mid N=2)}_{=0} P(N=2)$$

$$= P(Y=1)\, P(N=1) = P_1^T\,(0.18)$$

with $P_1^T = \dfrac{m(1-q)^{m-1}q}{1-(1-q)^m} = 0.281 \; (m=5, q=0.4)$

$$\Rightarrow P(S_1+S_2=1) = 2\times 0.281\times 0.18\times 0.81$$

$$= 0.0819.$$