College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

Second Midterm Exam
Academic Year 1442-1443 Hijri- SecondSemester

Exam Information معلومات الامتحان			
Course name	توزيعات الخسارة		اسم المقرر
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Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
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Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 5 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 5 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
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5				
6				
7				
8				

EXAM COVER PAGE

Exercise 1 *Claims in a portfolio of house contents policies have been modeled by a Pareto distribution with parameters* $\alpha = 6$ *and* $\theta = 1500$. *Inflation for next year is expected to be* 5%, *but a* \$100 *deductible is to be introduced for all claims as well. What will be the resulting decrease in average claim payment for next year?*

$A)$ 58.45 $B)$ 68.45 $C)$ 78.45 $D)$ 88.45 $E)$ 98.45

Exercise 2 *The claim frequency of a good driver is distributed as Poisson* ($\lambda = 1$)*, and the claim frequency of a bad driver is distributed as Poisson* ($\lambda = 4$). *A town consists of* 80% *good drivers and* 20% *bad drivers. What are the mean and variance of the claim frequency of a randomly selected driver from the town?*

Exercise 3 *If the number of claims is distributed as a zero modified Poison distribution with* $\lambda = 3$ *and* $p_0^M = 0.5$, *calculate: a.* $Pr(N = 0)$ *b.* $Pr(N = 1)$ *c.* $Pr(N = 2)$ *d.* $E(N)$ *e.* $var(N)$.

Exercise 4 *In year 2007, claim amounts have the following Pareto distribution*

$$F(x) = 1 - \left(\frac{800}{x + 800}\right)^3$$

The annual inflation rate is 8%. *A franchise deductible of 300 will be implemented in 2008. Calculate the loss elimination ratio of the franchise deductible.*

$A)$ 0.065 $B)$ 0.165 $C)$ 0.265 $D)$ 0.365 $E)$ 0.465

Exercise 5 (Bonus) *Losses follow a Pareto distribution with* $\alpha = 5$ *and* $\theta = 2000$*. An insurance policy covering these losses has a deductible of 100 and makes payments directly to the physician. Additionally, the physician is entitled to a bonus if the loss is less than 500. The bonus is 10% of the difference between 500 and the amount of the loss.*
The following table should help clarify the arrangement:

Amount of Loss	***Loss Payment***	***Bonus***
50	*0*	*45*
100	*0*	*40*
250	*150*	*25*
400	*300*	*10*
500	*400*	*0*

Calculate the expected total payment (loss payment plus bonus) from the insurance policy to the physician per loss.

$A)$ 431.83 $B)$ 451.83 $C)$ 481.83 $D)$ 511.83 $E)$ 551.83

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)} \beta[k+1, \alpha-k; x/(x+\theta)] + x^k \left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k (-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \tau \int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k} dy + x^k \left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta \frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1} - 1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} kn_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$p_0 = e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\mathrm{E}[N] = \lambda, \quad \mathrm{Var}[N] = \lambda \qquad P(z) = e^{\lambda(z-1)}$$

B.3.1.4 Zero-truncated binomial—$q, m, (0 < q < 1,\ m$ **an integer**)

$$
\begin{aligned}
p_1^T &= \frac{m(1-q)^{m-1}q}{1-(1-q)^m}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q}, \\
p_k^T &= \frac{\binom{m}{k} q^k (1-q)^{m-k}}{1-(1-q)^m}, \quad k = 1, 2, \ldots, m, \\
\mathrm{E}[N] &= \frac{mq}{1-(1-q)^m}, \\
\mathrm{Var}[N] &= \frac{mq[(1-q)-(1-q+mq)(1-q)^m]}{[1-(1-q)^m]^2}, \\
\tilde{q} &= \frac{\hat{\mu}}{m}, \\
P(z) &= \frac{[1+q(z-1)]^m - (1-q)^m}{1-(1-q)^m}.
\end{aligned}
$$

B.3.1.5 Zero-truncated negative binomial—$\beta, r, (r > -1, r \neq 0)$

$$
\begin{aligned}
p_1^T &= \frac{r\beta}{(1+\beta)^{r+1} - (1+\beta)}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta}, \\
p_k^T &= \frac{r(r+1)\cdots(r+k-1)}{k![(1+\beta)^r - 1]} \left(\frac{\beta}{1+\beta}\right)^k, \\
\mathrm{E}[N] &= \frac{r\beta}{1-(1+\beta)^{-r}}, \\
Var[N] &= \frac{r\beta[(1+\beta)-(1+\beta+r\beta)(1+\beta)^{-r}]}{[1-(1+\beta)^{-r}]^2}, \\
\tilde{\beta} &= \frac{\hat{\sigma}^2}{\hat{\mu}} - 1, \quad \tilde{r} = \frac{\hat{\mu}^2}{\hat{\sigma}^2 - \hat{\mu}}, \\
P(z) &= \frac{[1-\beta(z-1)]^{-r} - (1+\beta)^{-r}}{1-(1+\beta)^{-r}}.
\end{aligned}
$$

This distribution is sometimes called the extended truncated negative binomial distribution because the parameter r can extend below 0.

B.3.2 The zero-modified subclass

A zero-modified distribution is created by starting with a truncated distribution and then placing an arbitrary amount of probability at zero. This probability, p_0^M, is a parameter. The remaining probabilities are adjusted accordingly. Values of p_k^M can be determined from the corresponding zero-truncated distribution as $p_k^M = (1-p_0^M)p_k^T$ or from the corresponding $(a, b, 0)$ distribution as $p_k^M = (1-p_0^M)p_k/(1-p_0)$. The same recursion used for the zero-truncated subclass applies.

The mean is $1 - p_0^M$ times the mean for the corresponding zero-truncated distribution. The variance is $1-p_0^M$ times the zero-truncated variance plus $p_0^M(1-p_0^M)$ times the square of the zero-truncated mean. The probability generating function is $P^M(z) = p_0^M + (1-p_0^M)P(z)$, where $P(z)$ is the probability generating function for the corresponding zero-truncated distribution.

The maximum likelihood estimator of p_0^M is always the sample relative frequency at 0.

Ex 1

$X \sim$ Pareto $(\alpha = 6;\ \theta = 1500)$

$r = 5\%$ $\quad d = 100$

$\text{payment}_1 = E(X) = \frac{1500}{5} = 300$

$\text{payment}_2 = E(Y^L) = \ldots$

$= (1 + 5\%)\left[E(X) - E\left(X \wedge \frac{100}{1.05}\right)\right]$

$E(X) = \frac{\theta}{\alpha - 1} = 300$

$E\left(X \wedge \frac{100}{1.05}\right) = \frac{1500}{5}\left[1 - \left(\frac{1500}{1500 + \frac{100}{1.05}}\right)^5\right]$

$= 79.48$

$\Rightarrow E(Y^L) = 1.05\,[300 - 79.48] = 231.55.$

Variation in payment $= 300 - 231.55 = 68.45$

Ex2 $\quad N_B \sim \mathcal{P}(\lambda_1 = 4)$; $N_G \sim \mathcal{B}(\lambda_2 = 1)$

$P(\lambda_1 = 4) = 0.2 \quad P(\lambda_2 = 1) = 0.8$

$E(N) = E(N|\lambda_1)P(\lambda_1) + E(N|\lambda_2)P(\lambda_2)$

$= 0.2(4) + 0.8(1) = 1.6$

$E(N^2) = E(N^2|\lambda_1)P(\lambda_1) + E(N^2|\lambda_2)P(\lambda_2)$

$= (4 + 4^2)(0.2) + (1 + 1^2)(0.8) = 5.6$

$V(N) = 5.6 - 1.6^2 = 3.04.$

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Ex3

(Exo 33)

$$p_k^M = \frac{1-p_0^M}{1-p_0} p_k \quad , \quad k = 1, \ldots$$

a) ~~p_0^M~~ $P(N=0) = p_0^M = \boxed{0.5}$

$$P(N=1) = \frac{1-0.5}{1-p_0} p_1$$

~~X~~ $\sim \mathcal{P}(\lambda=3) \Rightarrow p_0 = e^{-3} \quad p_1 = e^{-3} \cdot 3 \quad p_2 = e^{-3} \frac{3^2}{2}$

$$P(N=1) = \frac{1-0.5}{1-p_0} p_1 = \boxed{0.078}$$

$$P(N=2) = \frac{1-0.5}{1-p_0} p_2 = 0\boxed{.11781}$$

$$E(N) = \sum_0^{\infty} k p_k^M = \sum_1^{\infty} k p_k^M = \frac{1-p_0^M}{1-p_0} \underbrace{\sum_1 k p_k}_{E(X)=3}$$

$$= \frac{1-0.5}{1-p_0} \times 3 = \boxed{1.578}$$

$$E(N^2) = \sum k^2 p_k^M = \frac{1-p_0^M}{1-p_0} \underbrace{E(X^2)}_{3^2+3} = 6.314$$

$$V(X) = E(N^2) - (E(N))^2 = 6.314 - 1.578^2 = \boxed{3.822}$$

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Ex 4 (5b)

$d = 300$; $r = 8\%$.

$X \sim \text{Pareto}(\alpha = 3;\ \theta = 800)$

$X' = (1+r)X$.

$$Y^L = \begin{cases} 0 & X' \le d \\ X' & X' > d \end{cases}$$

$$\because F_{X'}(y) = P\left(X \le \frac{y}{1+r}\right) = 1 - \left(\frac{800}{\frac{y}{1+r} + 800}\right)^3$$

$$= 1 - \left(\frac{(1+r)800}{y + 800(1+r)}\right)^3$$

$\Rightarrow X' \sim \text{Pareto}(\alpha = 3;\ \theta = 800(1+.08) = 864)$

$$E(Y^L) = E(X') - E(X' \wedge d) + d\, S_{X'}(d)$$

$$= \frac{864}{2} - \frac{864}{2}\left[1 - \left(\frac{864}{864+300}\right)^2\right] + 300\left(\frac{864}{864+300}\right)^3 = 360.7039$$

$\therefore E(X') = \frac{864}{2} = 432$

$$LER = \frac{432 - 360.7039}{432} = 0.165$$

Ex 5 (64)

$X \sim \text{Pareto}(\alpha = 5 ; \theta = 2000)$

$d = 500$

$$E(Y^L) = E(X) - E(X \wedge d) = \frac{\theta}{\alpha - 1}\left(\frac{\theta}{d+\theta}\right)^{\alpha-1} = 411.3512$$

$$\text{Bonus} = \begin{cases} 0 & X > 500 \\ 10\%(500 - X) & X < 500 \end{cases}$$

$$= 10\% \begin{cases} 500 - 500 & X > 500 \\ 500 - X & X < 500 \end{cases}$$

$$\text{Bonus} = 10\% \left[500 - \min(X, 500) \right]$$

$$EB = 10\% \left[500 - E(X \wedge 500) \right]$$

Table $E(X \wedge 500) = \frac{2000}{4}\left[1 - \left(\frac{2000}{2500}\right)^4\right] =$

$$\Rightarrow EB = 10\% * 500 \left(20/25\right)^4 = 20.48$$

$$\text{Total} = 411.35 + 20.48 = 431.83.$$

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