College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

Final Exam
Academic Year 1442-1443 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	توزيعات الخسارة		اسم المقرر
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Exam Duration	3 hours	ثلاث ساعات	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 6 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 6 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
-

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
4				
5				
6				
7				
8				

EXAM COVER PAGE

Explain your reasoning for why you have answered a certain value.

Exercise 1 *You know that ages in a certain population follow a single-parameter Pareto distribution with*

$$f(x) = \frac{\alpha}{x^{\alpha+1}},\ x > 1$$

A random sample of size six produced four losses with values 19, 23, 75 *and* 95, *and two losses exceeding* 100. *Use the method of maximum likelihood estimation to estimate the parameter* α.

$A)$ 0.3655 $B)$ 0.2655 $C)$ 0.1655 $D)$ 0.0655

Exercise 2 *For a sample of* 100 *pieces of data, you know the following:* $\Sigma x_i = 97,000$; $\Sigma x_i^2 = 890,340,000$. *A lognormal distribution with parameters* μ *and* σ *is used to fit the data. Estimate* μ *and* σ *using the method of moments.*
The value of μ *is among the following* $A)$ 4.7588 $B)$ 5.7536 $C)$ 6.7588 $D)$ 3.7588
The value of σ *is among the following* $A)$ 2.0584 $B)$ 1.4991 $C)$ 0.4868 $D)$ 2.4974

Exercise 3 *A certain Burr distribution has parameters* $\alpha = 1$, *and* θ *and* γ *unknown (use distribution in A.2.2.2). You know the following about a random variable that follows a Burr distribution.*
60% *of values are greater than* 50.
10% *of values are greater than* 9,200.
Find the value of θ *for this Burr distribution using this data. The method you will be using is known as percentile matching.*

$A)$ 113 $B)$ 213 $C)$ 313 $D)$ 413

Exercise 4 (Bonus) *You are given:*
(i) Losses follow an exponential distribution with mean θ.
(ii) A random sample of 20 *losses is distributed as follows:*

Loss Range	*Frequency*
$[0, 1000]$	7
$[1000, 2000]$	6
$[2000, \infty)$	7

Calculate the maximum likelihood estimate of θ.

Exercise 5 *You observe the following sample of losses:*

$$600 \quad 700 \quad 900$$

No information is available about losses of 500 *or less. Losses are assumed to follow an exponential disttribution with mean* θ*.*
Determine the maximum likelihood estimate of θ*.*

$A)$ 233 $B)$ 400 $C)$ 500 $D)$ 733 $E)$ 1233

Exercise 6 *A group dental policy has a negative binomial claim count* N *distribution with mean* 300 *and variance* 800*. Ground-up severity* X *is given by the following table:*

Severity	*Probability*
40	1/4
80	1/4
120	1/4
200	1/4

You expect severity to increase 50% *with no change in frequency. You decide to impose a per claim deductible of* 100*. Calculate the expected total claim payment* ES *after these changes.*

$A)$ 7500 $B)$ 10500 $C)$ 14500 $D)$ 19500 $E)$ 22500

A.2 Transformed beta family

A.2.2 Three-parameter distributions

A.2.2.1 Generalized Pareto (beta of the second kind)—α, θ, τ

$$
\begin{aligned}
f(x) &= \frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\frac{\theta^\alpha x^{\tau-1}}{(x+\theta)^{\alpha+\tau}} \qquad F(x) = \beta(\tau,\alpha;u), \quad u = \frac{x}{x+\theta} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}, \quad -\tau < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\tau(\tau+1)\cdots(\tau+k-1)}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(\tau)}\beta(\tau+k,\alpha-k;u) + x^k[1-F(x)], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{\alpha+1}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.2.2 Burr (Burr Type XII, Singh-Maddala)—α, θ, γ

$$
\begin{aligned}
f(x) &= \frac{\alpha\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^{\alpha+1}} \qquad F(x) = 1-u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}, \quad -\gamma < k < \alpha\gamma \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha}-1]^{1/\gamma} \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(1+k/\gamma)\Gamma(\alpha-k/\gamma)}{\Gamma(\alpha)}\beta(1+k/\gamma,\alpha-k/\gamma;1-u) + x^k u^\alpha, \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\alpha\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

A.2.2.3 Inverse Burr (Dagum)—τ, θ, γ

$$
\begin{aligned}
f(x) &= \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^\gamma]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}, \quad -\tau\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau}-1)^{-1/\gamma} \\
\mathrm{E}[(X\wedge x)^k] &= \frac{\theta^k\Gamma(\tau+k/\gamma)\Gamma(1-k/\gamma)}{\Gamma(\tau)}\beta(\tau+k/\gamma,1-k/\gamma;u) + x^k[1-u^\tau], \quad k > -\tau\gamma \\
\text{mode} &= \theta\left(\frac{\tau\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \tau\gamma > 1, \text{ else } 0
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad \mathrm{E}[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
\mathrm{E}[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p) \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}$$

A.3.3.2 Inverse exponential—θ

$$\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}$$

A.5 Other distributions

A.5.1.1 Lognormal—μ,σ (μ can be negative)

$$\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
\mathrm{E}[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
\mathrm{E}[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}$$

A.5.1.2 Inverse Gaussian—μ, θ

$$\begin{aligned}
f(x) &= \left(\frac{\theta}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta z^2}{2x}\right), \quad z = \frac{x-\mu}{\mu} \\
F(x) &= \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] + \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right], \quad y = \frac{x+\mu}{\mu} \\
M(t) &= \exp\left[\frac{\theta}{\mu}\left(1-\sqrt{1-\frac{2t\mu^2}{\theta}}\right)\right], \quad t < \frac{\theta}{2\mu^2}, \qquad \mathrm{E}[X] = \mu, \quad \mathrm{Var}[X] = \mu^3/\theta \\
\mathrm{E}[X \wedge x] &= x - \mu z \Phi\left[z\left(\frac{\theta}{x}\right)^{1/2}\right] - \mu y \exp\left(\frac{2\theta}{\mu}\right)\Phi\left[-y\left(\frac{\theta}{x}\right)^{1/2}\right]
\end{aligned}$$

A.5.1.3 log-t—r, μ, σ (μ *can be negative*)

Let Y have a t distribution with r degrees of freedom. Then $X = \exp(\sigma Y + \mu)$ has the log-t distribution. Positive moments do not exist for this distribution. Just as the t distribution has a heavier tail than the normal distribution, this distribution has a heavier tail than the lognormal distribution.

$$\begin{aligned}
f(x) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{x\sigma\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\left[1+\frac{1}{r}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]^{(r+1)/2}}, \\
F(x) &= F_r\left(\frac{\ln x - \mu}{\sigma}\right) \text{ with } F_r(t) \text{ the cdf of a } t \text{ distribution with } r \text{ d.f.,}
\end{aligned}$$

$$F(x) = \begin{cases} \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r+\left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & 0 < x \le e^{\mu}, \\ 1 - \frac{1}{2}\beta\left[\frac{r}{2}, \frac{1}{2}; \frac{r}{r+\left(\frac{\ln x - \mu}{\sigma}\right)^2}\right], & x \ge e^{\mu}. \end{cases}$$

A.5.1.4 Single-parameter Pareto—α, θ

$$\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta \qquad F(x) = 1 - (\theta/x)^\alpha, \quad x > \theta \\
\mathrm{VaR}_p(X) &= \theta(1-p)^{-1/\alpha} \qquad \mathrm{TVaR}_p(X) = \frac{\alpha\theta(1-p)^{-1/\alpha}}{\alpha - 1}, \quad \alpha > 1 \\
\mathrm{E}[X^k] &= \frac{\alpha\theta^k}{\alpha - k}, \quad k < \alpha \qquad \mathrm{E}[(X \wedge x)^k] = \frac{\alpha\theta^k}{\alpha - k} - \frac{k\theta^\alpha}{(\alpha - k)x^{\alpha - k}}, \ x \ge \theta \\
\text{mode} &= \theta
\end{aligned}$$

Note: Although there appears to be two parameters, only α is a true parameter. The value of θ must be set in advance.

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}. \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer$)$

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m. \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ \mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}. \end{aligned}$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a+b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

Exponential, Lognormal, Single Pareto, Burr, Neg. binomial

Ex 1

$$L(\alpha|x_1,\dots,x_n) = f(19)\times f(23)\times f(75)\times f(95) \times \left(S(100)^2\right)$$

$$= \left(\frac{\alpha}{19^{\alpha+1}}\right)\left(\frac{\alpha}{23^{\alpha+1}}\right)\left(\frac{\alpha}{75^{\alpha+1}}\right)\left(\frac{\alpha}{95^{\alpha+1}}\right)\left(\frac{1}{100^{\alpha}}\right)^2$$

$$= \frac{\alpha^4}{(\Pi x_i)^{\alpha+1}} \frac{1}{100^{2\alpha}}.$$

$$\ell = \log L = 4\log\alpha - (\alpha+1)\log(\prod_i x_i) - \alpha\log(100^2)$$

$$\frac{d\ell}{d\alpha} = \frac{4}{\alpha} - \log(\Pi x_i) - \log(100^2) = 0$$

$$\Leftrightarrow \alpha = \frac{4}{\log(\Pi x_i)+\log(100^2)} = \frac{4}{24.16} = 0.1655. = \alpha^*$$

$\alpha < \alpha^*$ $\frac{d\ell}{d\alpha} > 0$ $\Rightarrow$ α^* is the

$\alpha > \alpha^*$ $\frac{d\ell}{d\alpha} < 0$ maximum LE.

Ex 2

$$\begin{cases} \exp\{\mu+\sigma^2/2\} = 970 \Rightarrow \exp\{2\mu+\sigma^2\} = 970^2 & (1) \\ \exp\{2\mu+2\sigma^2\} = 8903400 & (2) \end{cases}$$

$$(2)/(1) \Rightarrow \exp\{\sigma^2\} = \frac{8903400}{970^2}$$

$$\Rightarrow \sigma = \sqrt{\log(8903400/970^2)} = 1.4991$$

$$\mu = \log(970) - \sigma^2/2 = 5.7536$$

OR

$$\begin{cases} \mu + \sigma^2/2 = \log(970) & (3) \\ 2\mu + 2\sigma^2 = 8903400 & (4) \end{cases}$$

$(4) - 2\times(3) \Rightarrow$ $\sigma^2 = \log(8903400) - 2\log(970) \Rightarrow \sigma = 1.4991$

$(3) \Rightarrow$ $\mu = \log(970) - \sigma^2/2 = 5.7536$

Ex3

Burr

$$S(x) = \frac{1}{(1+(x/\theta)^{\gamma})}$$

$$\begin{cases} S(50) = 0.6 \\ S(9200) = 0.1 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{1+(50/\theta)^{\gamma}} = 0.6 \\ \frac{1}{1+(9200/\theta)^{\gamma}} = 0.1 \end{cases}$$

$$\Leftrightarrow \begin{cases} (50/\theta)^{\gamma} = 2/3 & (1) \\ (9200/\theta)^{\gamma} = 9 & (2) \end{cases}$$

$$(2)/(1) \Rightarrow \left(\frac{9200}{50}\right)^{\gamma} = \frac{9}{2/3} = \frac{27}{2} \Rightarrow \gamma = \frac{\log(27/2)}{\log(9200/5)} = 0.4991$$

$$(1) \Rightarrow \frac{50}{\theta} = \left(\frac{2}{3}\right)^{1/\gamma} \Rightarrow \theta = 50 \times (2/3)^{-1/\gamma} = 112.6646$$

Ex 4 Bonus $S(x) = e^{-x/\theta}$; $F(x) = 1 - e^{-x/\theta}$

$$L(\theta|x) = F(1000)^7 \times (S(1000) - S(2000))^6 (S(2000))^7$$

$$= \left(1 - e^{-1000/\theta}\right)^7 \left(e^{-1000/\theta} - e^{-2000/\theta}\right)^6 \left(e^{-2000/\theta}\right)^7$$

let $a = e^{-1000/\theta}$ $(a<1)$ a is increasing with θ.

$$L(\theta|x) = (1-a)^7 (a-a^2)^6 (a^2)^7$$

$$= (1-a)^7 a^6 (1-a)^6 (a^{14}) = a^{20}(1-a)^{13}$$

$$l = 20 \log a + 13 \log(1-a)$$

$$\frac{dl}{da} = \frac{20}{a} - \frac{13}{1-a} = 0 \Leftrightarrow a = \frac{20}{33}$$

$$a = e^{-1000/\theta} \Rightarrow \log a = -\frac{1000}{\theta} \Rightarrow \theta = \frac{-1000}{\log(a)} = 1996.9$$

$$\frac{dl}{da} = \frac{20-33a}{a(1-a)} \begin{cases} <0 \text{ if } a > 20/33 \\ >0 \text{ if } a < 20/33 \end{cases} \text{ (and } a<1)$$

Ex5 (similar to Pb 60.1)

$d = 500$ (No deductible)

distribution of $X|_{X>d}$ is $\frac{f(x)}{S(d)} = \frac{\frac{1}{\theta} e^{-x/\theta}}{e^{-d/\theta}}, \; x > d$

$$L(\theta|x) = \frac{\prod_{i=1}^{3} \left(\frac{1}{\theta} e^{-x_i/\theta} \right)}{\left(e^{-500/\theta}\right)^3} = \frac{1}{\theta^3} \exp\left\{ -\left(\Sigma x_i - 500\right)/\theta \right\}$$

$$= \frac{1}{\theta^3} e^{-700/\theta}$$

$$l = \log L = -3 \log \theta - \frac{700}{\theta}.$$

$$\frac{dl}{d\theta} = -\frac{3}{\theta} + \frac{700}{\theta^2} = 0 \iff \theta^* = \frac{700}{3} = 233$$

$$\frac{dl}{d\theta} < 0 \text{ if } \theta > \theta^*$$
$$> 0 \text{ if } \theta < \theta^*.$$

$$EX = \theta^* = 73.$$

Ex6

$$S = \sum_{i=1}^{N} Y^L \quad \text{where } Y^L = \begin{cases} 0 & \text{if } X_2 < d = 100 \\ X_2 - 100 & \text{if } X_2 > d = 100 \end{cases}$$

$$X_2 = (1 + 0.5) X = \begin{cases} 60 & 0.25 \\ 220 & 0.25 \\ 180 & 0.25 \\ 300 & 0.25 \end{cases}$$

$$EY^L = \sum_{X_2 > 100} (X_2 - 100)\, P(X_2) = \frac{20 + 80 + 200}{4} = 75$$

$$ES = EN \times EY^L = 300 \times 75 = 22{,}500$$