King Saud University **Academic Year (G) 2019–2020**
College of Sciences **Academic Year (H) 1441**
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Model Answer of the second midterm exam ACTU–362 (25%)

December 2, 2019 (Fall 2019)

Problem 1. (6 marks)

1. **(3 marks)** A man, age 46, purchases a 3–year endowment insurance with benefits payable at the end of the year of death. The benefit is given by the formula:

x	46	47	48	49	50
ℓ_x	60000	59000	57,500	55,000	50000

k	0	1	2
b_{k+1}	60000×1.06	$60000 \times (1.06)^2$	$60000 \times (1.06)^3$

Calculate the net single premium for this insurance for $i = 0.06$.

2. **(3 marks)** A whole life insurance on (40) pays a benefit of 50,000 at the end of the year of death. The force of mortality is given by:

$$\mu_{40+t} = \begin{cases} 0.002 & \text{if} \quad t \leq 10 \\ \dfrac{1}{50-t} & \text{if} \quad 10 < t < 50 \end{cases}$$

Calculate NSP the net single premium for this insurance when the effective interest rate is 6%.

Solution:

1. The NSP is given by

$$\begin{aligned} \text{APV(FB)}_0 &= \sum_{k=0}^{2} b_{k+1} v^{k+1} \ {}_{k|}q_{46} + b_3 v^3 \ {}_3p_{46} = 60000 \left(\sum_{k=0}^{2} (v(1+i))^{k+1} \ {}_{k|}q_{46} + (v(1+i))^3 \ {}_3p_{46} \right) \\ &= 60000 \left(\sum_{k=0}^{2} {}_{k|}q_{46} + {}_3p_{46} \right) = 60000 \left({}_3p_{46} + \sum_{k=0}^{2} ({}_{k+1}q_{46} - {}_kq_{46}) \right) \\ &= 60000 ({}_3p_{46} + {}_3q_{46} - {}_0q_{46}) = 60000 ({}_4p_{46} + {}_3q_{46}) = \mathbf{60000}. \end{aligned}$$

2. We can split the insurance into two components: a 10–year term insurance and a 10–year deferred insurance. In the initial 10–year period, $p_x = p = e^{-0.02}$. The APV of the future benefits is

$$\text{NSP} = 50000(A^1_{40:\overline{10}|} + {}_{10|}A_{40}) = 50000(A^1_{40:\overline{10}|} + {}_{10}E_{40} A_{50})$$

Moreover

$$\begin{aligned} A^1_{40:\overline{10}|} &= \frac{1-p}{1+i-p} \left(1 - (vp)^{10}\right) = \frac{q}{q+i} \left(1 - (vp)^{10}\right) \\ &= \frac{1-e^{-0.02}}{1.06 - e^{-0.02}} \left(1 - \left(\frac{e^{-0.02}}{1.06}\right)^{10}\right) = 0.13469 \end{aligned}$$

and

$$
\begin{aligned}
{}_{10}E_{40}A_{50} &= (vp)^{10}\left(\frac{a_{\overline{10}|}}{50}\frac{1-v^{10}}{i}\right) = \frac{1}{50}(vp)^{10}\left(\frac{1-v^{10}}{i}\right) \\
&= \frac{1}{50}\left(\frac{e^{-0.02}}{1.06}\right)^{10}\left(\frac{1-(1.06)^{-10}}{0.06}\right) = 06.7297
\end{aligned}
$$

Thus NSP $= 50000\,(0.13469{+}06.7297) =$ **343220**.

Problem 2. (6 marks)

1. **(3 marks)** A special insurance on (45) pays 5000 at the end of the year of death if death occurs between ages 50 and 60, and 3000 at the end of the year of death if death occurs after age 60. Calculate the net single premium for this insurance using ILT for $i = 0.06$.

2. **(3 marks)** You are given: $i = 0.05$, $q_{40} = 0.05$ and $q_{41} = 0.08$. Calculate $\bar{A}^{1}_{40:\overline{2}|}$, assuming uniform distribution of deaths for fractional ages.

Solution.

1. Denote by NSP the APV of this insurance, then NSP $= 5000\ {}_{5|}A^{1}_{45:\overline{10}|} + 3000\ {}_{15|}A_{45}$. It can be also written as

$$
\text{NSP} = 5000\ {}_{5|}A_{45} - 2000\ {}_{15|}A_{45} = 5000\ {}_{5}E_{45}A_{50} - 2000\ {}_{15}E_{45}A_{60}
$$

The net single premium for the insurance is

$$
\begin{aligned}
\text{NSP} &= 5000\frac{1}{1.06^5}\frac{\ell_{50}}{\ell_{45}}0.24905 - 2000\frac{1}{1.06^{15}}\frac{\ell_{60}}{\ell_{45}}0.36913 \\
&= 5000\frac{1}{1.06^5}\frac{8950901}{9164051}0.24905 - 2000\frac{1}{1.06^{15}}\frac{8188074}{9164051}0.36913 = \mathbf{633.64}
\end{aligned}
$$

2. Under UDD

$$
\begin{aligned}
\bar{A}^{1}_{40:\overline{2}|} &= \frac{i}{\delta}A^{1}_{40:\overline{2}|} = \frac{i}{\ln(1+i)}\left(vq_{40} + v^2 p_x q_{x+1}\right) \\
&= \frac{0.05}{\ln(1.05)}\left(\frac{0.05}{1.05} + \frac{0.95 \times 0.08}{1.05^2}\right) = \mathbf{0.119\,44}
\end{aligned}
$$

Problem 3. (6 marks)

1. **(3 marks)** For a 1–year term insurance on (66.2), you are given: (i) The insurance pays $10,000$ at the end of the half–year of death. (ii) $q_{66} = 0.06$ and $q_{67} = 0.08$. (iii) Mortality is uniformly distributed between integral ages. (iv) $i = 0.03$. Calculate the expected present value of the insurance.

2. **(3 marks)** We assume that $i = 0.06$ and $\mu_{x+t} = \ln(1.02)$ for $t > 0$, calculate $A_x^{(4)}$

Solution:

1. The APV of the insurance is

$$10^4 A^{(2)\,1}_{66.2:\overline{1|}} = 10^4 \left(v^{0.5}\ {}_{0.5}q_{66.2} + v\ {}_{0.5|0.5}q_{66.2}\right).$$

   Now, we have

$$\begin{aligned} {}_{0.5}q_{66.2} &= 1 - {}_{0.5}p_{66.2} = 1 - \frac{{}_{0.7}p_{66}}{{}_{0.2}p_{66}} = 1 - \frac{1-0.7q_{66}}{1-0.2q_{66}} \\ &= \frac{0.5q_{66}}{1-0.2q_{66}} = \frac{0.5\times 0.06}{1-0.2\times 0.06} = 0.030364 \end{aligned}$$

   and

$$\begin{aligned} {}_{0.5|0.5}q_{66.2} &= {}_{0.5}p_{66.2} - p_{66.2} = {}_{0.5}p_{66.2} - \frac{{}_{1.2}p_{66}}{{}_{0.2}p_{66}} \\ &= {}_{0.5}p_{66.2} - \frac{{}_{0.2}p_{67}}{{}_{0.2}p_{66}} p_{66} = {}_{0.5}p_{66.2} - \frac{1-0.2q_{67}}{1-0.2q_{66}} p_{66} \\ &= 1 - 0.030364 - \frac{1-0.2\times 0.08}{1-0.2\times 0.06} 0.94 = 0.033442 \end{aligned}$$

   Finally

$$10^4 A^{(2)\,1}_{66.2:\overline{1|}} = 10^4 \left(\frac{0.030364}{1.03^{0.5}} + \frac{0.033442}{1.03}\right) = \mathbf{623.86}.$$

2. By definition

$$\begin{aligned} A_x^{(4)} &= \sum_{k=0}^{\infty} v^{\frac{k+1}{4}}\ {}_{\frac{k}{4}|\frac{1}{4}}q_x = \sum_{k=0}^{\infty} v^{\frac{k+1}{4}}\ {}_{\frac{k}{4}}p_x\ {}_{\frac{1}{4}}q_{x+\frac{k}{4}} \\ &= \sum_{k=0}^{\infty} v^{\frac{k+1}{4}}\ e^{-\frac{k}{4}\ln(1.02)} \left(1 - e^{-\frac{1}{4}\ln(1.02)}\right) \quad (1.06\times 1.02)^{-0.25} \\ &= \sum_{k=0}^{\infty} v^{\frac{k+1}{4}}\ 1.02^{-\frac{k}{4}} \left(1 - 1.02^{-\frac{1}{4}}\right) = \frac{(1-1.02^{-0.25})}{(1.06)^{0.25}} \sum_{k=0}^{\infty} r^k \end{aligned}$$

   where $r = (1.06\times 1.02)^{-0.25} = 0.98067$, thus $A_x^{(4)} = \frac{(1-1.02^{-0.25})}{(1.06)^{0.25}} \frac{1}{1-0.98067} = \mathbf{0.25179}$.

Problem 4. (6 marks)

1. **(3 marks)** Mortality follows ILT for $i = 0.06$. Determine the 10–year certain and life annuity due of 1 on (65)

2. **(3 marks)** An annuity on (x) pays continuously at a rate of 1 per year until the later of the death of (x) and 10 years. Calculate the expected present value of this annuity when it is subject to the force of mortality $\mu_{x+t} = \frac{1}{30-t}$ for $t < 30$ and the force of interest of 5%.

Solution:

1. We know that
$$\ddot{a}_{\overline{65:\overline{10|}}} = \ddot{a}_{\overline{10|}} + {}_{10|}\bar{a}_{65} = \bar{a}_{\overline{10|}} + {}_{10}E_{65}\bar{a}_{75}.$$
The 10–year certain annuity has present value
$$\ddot{a}_{\overline{10|}} = \frac{1-v^{10}}{d} = \frac{1-(1.06)^{-10}}{\frac{0.06}{1.06}} = \frac{1-0.55842}{0.056604} = 7.8012.$$
Therefore
$$\bar{a}_{\overline{65:\overline{10|}}} = 7.8012 + 0.39994 \times 7.2170 = \mathbf{10.688}.$$

2. This is certain and life annuity of 1 on (x), so its APV is $\bar{a}_{\overline{x:\overline{10|}}} = \bar{a}_{\overline{10|}} + {}_{10|}\bar{a}_x$. The 10–year certain annuity has present value $\bar{a}_{\overline{10|}} = \frac{1-e^{-\delta 10}}{\delta} = \frac{1-e^{-0.5}}{0.05} = 7.8694$. The 10–year deferred annuity will be evaluated as the difference between a whole life and a 10–year temporary life annuity, with each annuity evaluated using insurance formulas
$$\begin{aligned}\bar{a}_x - \bar{a}_{x:\overline{10|}} &= \frac{\bar{A}_{x:\overline{10|}} - \bar{A}_x}{\delta} = \frac{\bar{A}^{1}_{x:\overline{10|}} + {}_{10}E_x - \bar{A}_x}{\delta} \\ &= \frac{\bar{a}_{\overline{10|}}}{30\delta} + \frac{{}_{10}E_x}{\delta} - \frac{\bar{a}_{\overline{30|}}}{30\delta} = \frac{1-e^{-0.5}}{30(0.05)^2} + \frac{2e^{-0.5}}{3\times 0.05} - \frac{1-e^{-1.5}}{30(0.05)^2} = 2.9751\end{aligned}$$
Finally $\bar{a}_{\overline{x:\overline{10|}}} = 7.8694 + 2.9751 = \mathbf{10.845}$.

Problem 5. (6 marks)

1. **(3 marks)** Use ILT for $i = 0.06$ to find $a^{(4)}_{25:\overline{20|}}$ under UDD over each year of age.

2. **(3 marks)** Use Woolhouse's formula with three terms to find $\ddot{a}^{(4)}_{25}$ given: (i) $\ddot{a}_{25} = 26$ (ii) $\delta = 0.02$ (iii) $q_{24} = q_{25} = 0.1$.

(Hint $a^{(m)}_x = \ddot{a}^{(m)}_x - \frac{1}{m}$; $a^{(m)}_{x:\overline{n|}} = \ddot{a}^{(m)}_{x:\overline{n|}} - \frac{1}{m} + \frac{nE_x}{m}$ and ${}_{n|}a^{(m)}_x = {}_{n|}\ddot{a}^{(m)}_x - \frac{nE_x}{m}$, $\mu_x \simeq -\frac{1}{2}\left(\ln\left(p_{x-1}\right) + \ln\left(p_x\right)\right)$).

Solution:

1. First observe that $a^{(4)}_{25:\overline{20|}} = \ddot{a}^{(4)}_{25:\overline{20|}} - \frac{1}{4} + \frac{{}_{20}E_{25}}{4}$. Moreover, Under UDD we have
$$\ddot{a}^{(4)}_{25:\overline{20|}} = \alpha(4)\ddot{a}_{25:\overline{20|}} - \beta(4)\left(1 - {}_{20}E_{25}\right) = \alpha(4)\left(\ddot{a}_{25} - {}_{20}E_{25}\ddot{a}_{45}\right) - \beta(4)\left(1 - {}_{20}E_{25}\right)$$
where
$$\alpha(4) = \frac{0.06\frac{0.06}{1.06}}{4\left((1.06)^{\frac{1}{4}} - 1\right) 4\left(1 - \left(1 - \frac{0.06}{1.06}\right)^{\frac{1}{4}}\right)} = 1.0003$$
and
$$\beta(4) = \frac{0.06 - 4((1.06)^{\frac{1}{4}} - 1)}{4^2((1.06)^{\frac{1}{4}} - 1)(1 - \left(1 - \frac{0.06}{1.06}\right)^{\frac{1}{4}})} = 0.38424$$
Thus
$a^{(4)}_{25:\overline{20|}} = 1.0003\left(16.2242 - 0.29873 \times 14.1121\right) - 0.38424\left(1 - 0.29873\right) - 0.25\left(1 - 0.29873\right) = \mathbf{11.567}$.

2. We know that
$$\ddot{a}^{(4)}_{25} = \ddot{a}_{25} - \frac{4-1}{2\times 4} - \frac{4^2-1}{12\times 4^2}\left(\mu_{25} + 0.06\right),$$
but $\mu_{25} \simeq -\frac{1}{2}\ln\left(p_{24}p_{25}\right) = -\ln\left(p_{24}\right)) = -\ln(0.9) = 0.10536$, hence
$$\ddot{a}^{(4)}_{25} = 26 - \frac{4-1}{2\times 4} - \frac{4^2-1}{12\times 4^2}\left(0.10536 + 0.06\right) = \mathbf{25.612}.$$