King Saud University — **Academic Year (G) 2019–2020**
College of Sciences — **Academic Year (H) 1441**
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Model Answer of the first midterm exam ACTU–362 (25%)

October 7, 2019 (Fall 2019)

<u>**Problem 1.**</u> **(6 marks)** The survival function $S_0(t)$ of the age–at–death random variable T_0 is given by

$$S_0(t) = \left(\frac{100}{100+t}\right)^2 \quad \text{for} \quad t \geq 0,$$

1. **(2 mark)** Calculate ${}_{5|}q_{40}$.
2. **(2 mark)** You are given $S_{10}(25) = 0.9$ and $F_{20}(15) = 0.05$, calculate $S_{10}(10)$.
3. **(2 mark)** Given $S_0(t) = \sqrt{1 - \frac{t}{100}}$, for $0 \leq t \leq 100$, calculate the probability that a life age 36 will die between ages 51 and 64.

Solution:

1. We know ${}_{5|}q_{40} = {}_5p_{40} - {}_6p_{40} = S_{40}(5) - S_{40}(6) = \frac{S_0(45)}{S_0(40)} - \frac{S_0(46)}{S_0(40)} = 0.932224 - 0.919497 = \mathbf{0.012727}$.
2. We have $S_{10}(10) = {}_{10}p_{10}$. Moreover we have ${}_{25}p_{10} = {}_{10}p_{10} \cdot {}_{15}p_{20} = {}_{10}p_{10}\,(1 - {}_{15}q_{20})$,
$$ {}_{10}p_{10} = \frac{{}_{25}p_{10}}{(1 - {}_{15}q_{20})} = \frac{0.9}{1 - 0.05} = \mathbf{0.94737}.$$
3. The required probability is given by
$$\begin{aligned} P\left(15 \leq T_{36} < 28\right) &= S_{36}(15) - S_{36}(28) = \frac{S_0(51) - S_0(64)}{S_0(36)} \\ &= \frac{\sqrt{1 - \frac{51}{100}} - \sqrt{1 - \frac{64}{100}}}{\sqrt{1 - \frac{36}{100}}} = \frac{1}{8} = \mathbf{0.125}. \end{aligned}$$

<u>**Problem 2.**</u> **(6 marks)** The force of mortality μ_x is given by

$$\mu_x = \frac{1}{100 - x} \quad \text{for} \quad 0 \leq x < 100,$$

1. **(2 marks)** Find $S_{20}(t)$ for $0 \leq t < 80$.
2. **(2 marks)** Compute ${}_{40}p_{20}$.
3. **(2 marks)** Find $f_{20}(t)$ for $0 \leq t < 80$.

Solution:

1. By definition $S_{20}(t) = e^{-\int_0^t \mu_{20+u} du} = e^{-\int_0^t \frac{1}{80-u} du} = e^{[\ln(80-u)]_0^t} = e^{\ln\left(\frac{80-t}{80}\right)} = 1 - \frac{t}{80}$.

2. We have ${}_{40}p_{20} = S_{20}(40) = 1 - \frac{40}{80} = \frac{1}{2} = \mathbf{0.5}$.

3. the p.d.f. of T_{20} is given, $f_{20}(t) = -S'_{20}(t) = \frac{1}{80}$.

Problem 3. (6 marks) You are given the following life table:

x	90	91	92	93	94	95
ℓ_x	1000	950	900	840	c_2	700
d_x	50	50	60	c_1	70	80

1. **(2 marks)** Find the values of c_1 and c_2

2. **(2 marks)** Calculate ${}_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

3. **(2 marks)** Repeat 2. by assuming constant force of mortality between integer ages.

Solution:

1. We know that $\ell_x - \ell_{x+1} = d_x \iff \ell_{x+1} = \ell_x - d_x$, thus $\ell_{95} = \ell_{94} - d_{94} = 700 = c_2 - 70$, hence $c_2 = \mathbf{770}$. And $\ell_{94} = \ell_{93} - d_{93} = 770 = 840 - c_1$, hence $c_1 = 840 - 770 = \mathbf{70}$.

2. Under UDD we can write

$$\begin{aligned} {}_{1.4}p_{90} &= p_{90}\ {}_{0.4}p_{91} = p_{90}\left(1 - {}_{0.4}q_{91}\right) = p_{90}\left(1 - 0.4q_{91}\right) \\ &= \frac{\ell_{91}}{\ell_{90}}\left(1 - 0.4\left(1 - \frac{\ell_{92}}{\ell_{91}}\right)\right) = \frac{950}{1000}\left(1 - 0.4\left(1 - \frac{900}{950}\right)\right) = \mathbf{0.93}. \end{aligned}$$

3. Under CFM we can write

$${}_{1.4}p_{90} = p_{90}\ {}_{0.4}p_{91} = p_{90} \cdot p_{91}^{0.4} = \frac{\ell_{91}}{\ell_{90}}\left(\frac{\ell_{92}}{\ell_{91}}\right)^{0.4} = \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} = \mathbf{0.92968}.$$

Problem 4. (6 marks)

1. **(2 mark)** You are given: $\ell_{[45]} = 1000$, ${}_5q_{[45]} = 0.04$, ${}_5q_{[45]+5} = 0.05$. Calculate $\ell_{[45]+10}$.

2. **(2 mark)** You are given the following extract from a select-and-ultimate mortality table with a 2–year select period:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	80625	79954	78839	62
61	79137	78402	77252	63
62	77575	76770	75578	64

Calculate under UDD ${}_{0.9}q_{[60]+0.6}$.

3. **(2 mark)** Select mortality rates for [45] are **half** of the Illustrative Life Table's mortality rates for a selection period of 3 years and $i = 6\%$. Calculate ${}_{2|2}q_{[45]}$.

Solution:

1. We have

$$\ell_{[45]+10} = \ell_{[45]}\ {}_{10}p_{[45]} = \ell_{[45]}\ {}_{5}p_{[45]}\ {}_{5}p_{[45]+5} = \ell_{[45]}\left(1-\ {}_{5}q_{[45]}\right)\left(1-\ {}_{5}q_{[45]+5}\right) = 1000\times 0.96\times 0.95 = 912.$$

2. We know that ${}_{0.9}q_{[60]+0.6} = 1-\ {}_{0.9}p_{[60]+0.6}$ and

$$\begin{aligned} {}_{0.9}p_{[60]+0.6} &= \frac{{}_{1.5}p_{[60]}}{{}_{0.6}p_{[60]}} = \frac{p_{[60]}\times\ {}_{0.5}p_{[60]+1}}{{}_{0.6}p_{[60]}} = \frac{p_{[60]}\left(1-0.5q_{[60]+1}\right)}{1-0.6q_{[60]}} \\ &= \frac{\frac{\ell_{[60]+1}}{\ell_{[60]}}\left(1-0.5\left(1-\frac{\ell_{62}}{\ell_{[60]+1}}\right)\right)}{1-0.6\left(1-\frac{\ell_{[60]+1}}{\ell_{[60]}}\right)} = \frac{\frac{79954}{80625}\left(1-0.5\left(1-\frac{78839}{79954}\right)\right)}{1-0.6\left(1-\frac{79954}{80625}\right)} = 0.9897. \end{aligned}$$

hence ${}_{0.9}q_{[60]+0.6} = 1-0.9897 = \mathbf{0.0103}$.

3. We will calculate ${}_{2}p_{[45]}$ and ${}_{2}p_{[45]+2}$. So

$${}_{2}p_{[45]} = \ p_{[45]}\ p_{[45]+1} = \left(1-\ q_{[45]}\right)\left(1-\ q_{[45]+1}\right)$$

and

$${}_{2}p_{[45]+2} = \ p_{[45]+2}\ p_{[45]+3} = \left(1-\ q_{[45]+2}\right)\left(1-\ q_{[45]+3}\right).$$

For $q_{[45]}$ and $q_{[45]+1}$ we use half of the ILT rates. Then

$${}_{2}p_{[45]} = (1-0.5(0.004))(1-0.5(0.00431) = 0.995849,$$

moreover $q_{[45]+2} = 0.5q_{47}$ but $q_{[45]+3} = q_{48}$ since the selection period ends after 3 years. Mortality for duration 3 and on is no different from standard mortality.

$${}_{2}p_{[45]+2} = (1-0.5(0.00466))(1-0.00504) = 0.992642$$

The answer is

$${}_{2|2}q_{[45]} = \ {}_{2}p_{[45]}\ {}_{2}q_{[45]+2} = \ {}_{2}p_{[45]}\left(1-\ {}_{2}p_{[45]+2}\right) = 0.995849(1-0.992642) = 0.0073275.$$

Problem 5. (6 marks)

1. **(5 marks)** Given $S_0(t) = \sqrt{1-\frac{t}{100}}$, for $0 \le t \le 100$. Evaluate a. ${}_{17}p_{19}$, b. ${}_{15}q_{36}$, c. ${}_{15|13}q_{36}$, d. μ_{36} and e. $E\left[T_{36}\right]$.
2. **(1 mark)** You are given $\mu_x = 0.02$ for all $x \ge 0$. Calculate $\mathrm{Var}(T_x)$.

Solution:

1. a. ${}_{17}p_{19} = S_{19}(17) = \frac{S_0(36)}{S_0(19)} = \frac{\sqrt{1-\frac{36}{100}}}{\sqrt{1-\frac{19}{100}}} = \frac{8}{9} = \mathbf{0.88889}$.

   b. ${}_{15}q_{36} = 1-\ {}_{15}p_{36} = 1-\frac{S_0(51)}{S_0(36)} = 1-\frac{\sqrt{1-\frac{51}{100}}}{\sqrt{1-\frac{36}{100}}} = \frac{1}{8} = \mathbf{0.125}$.

   c. ${}_{15|13}q_{36} = S_{36}(15)-S_{36}(28) = \frac{\sqrt{1-\frac{51}{100}}-\sqrt{1-\frac{64}{100}}}{\sqrt{1-\frac{36}{100}}} = \frac{1}{8} = \mathbf{0.125}$.

d. $\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)}$ now take $x = 0$ and $t = 36$ then $\mu_{36} = -\frac{S_0'(36)}{S_0(36)} = -\frac{-\frac{1}{100}\frac{1}{2}\left(1-\frac{36}{100}\right)^{-0.5}}{\sqrt{1-\frac{36}{100}}} = \mathbf{0.078125}$.

e. By definition

$$\begin{aligned} E\left[T_{36}\right] &= \int_0^{100-36} {}_tp_{36}dt = \int_0^{64} \frac{\sqrt{1-\frac{36+t}{100}}}{\sqrt{1-\frac{36}{100}}}dt = \frac{1}{\sqrt{1-\frac{36}{100}}}\int_0^{64}\sqrt{1-\frac{36+t}{100}}dt \\ &= \frac{1}{0.8}\int_0^{64}\sqrt{1-\frac{36+t}{100}}dt = \frac{34.133}{0.8} = \mathbf{42.666}. \end{aligned}$$

2. The case of CFM we have

$$E\left[T_x\right] = \int_0^{\infty} {}_tp_x dt = \int_0^{\infty} e^{-0.02t}dt = \frac{1}{0.02}\int_0^{\infty} 0.02e^{-0.02t}dt = \frac{1}{0.02} = 50,$$

and

$$E\left[T_x^2\right] = \int_0^{\infty} 2t\ {}_tp_x dt = \int_0^{\infty} 2te^{-0.02t}dt = \frac{2}{0.02}\int_0^{\infty} t0.02e^{-0.02t}dt = \frac{2}{0.02}\frac{1}{0.02} = 5000$$

then $\text{Var}(T_x) = 5000 - 50^2 = 2500$.