King Saud University
College of Sciences
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Model Answer

Midterm Exam 2 Actuarial Mathematical Models 1

Exercise 1 ***(6 marks)***

1. *A fully continuous whole life insurance on* (30) *pays a benefit of* 17541.64386 *at the moment of death. The force of mortality is not constant for all ages and is given by:*

$$\mu_{30+t} = 0.06 \quad \textit{if} \quad 0 \leq t < 20 \quad \textit{and} \quad \mu_{30+t} = \frac{1}{60-t} \quad \textit{if} \quad 20 \leq t < 60$$

   Calculate NSP *the net single premium for this insurance when the force of interest is* 4%.

2. *On January* 1, 2020, *Sarah, aged* 40, *purchases a* 5*–payment,* 10*–year term insurance of* 1000
   (i) Death benefits are payable at the moment of death.
   (ii) Net premiums of 200 *are payable annually at the beginning of each year for* 5 *years.*
   (iii) The effective interest rate is 5%
   (iv) ${}_0L$ *is the loss random variable at time of issue.*
   Calculate the value of ${}_0L$ *if Sarah dies on June* 30, 2025.

3. *You are given:* $\ell_{40} = 500$, $\ell_{41} = 475$, $\delta = 0.04$, *and* $A_{40} = 0.520$. *Deaths are uniformly distributed over each year of age. Calculate* $\bar{A}_{41}$. *(Hint: Under UDD* $\bar{A}_x = \frac{i}{\delta} A_x$ *and* $\bar{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$*).*

<u>Solution:</u>

1. We can split the insurance into two components: a 20–year term insurance and a 20–year deferred insurance. In the initial 20–year period, the APV of the future benefits is

$$\text{NSP} = 17541.64386(\bar{A}^{1}_{30:\overline{20}|} + \bar{A}[20|]30) = 17541.64386(\bar{A}^{1}_{30:\overline{20}|} + {}_{20}E_{30}\ \bar{A}_{50})$$

   Moreover

$$\begin{aligned}\bar{A}^{1}_{30:\overline{20}|} &= \int_0^{20} e^{-\delta t}\ {}_tp_{30}\ \mu_{30+t}\ dt = \int_0^{20} 0.06 e^{-(0.04+0.06)t}\ dt \\ &= \frac{0.06}{0.10}\left(1 - e^{-(0.1)20}\right) = \frac{3}{5}\left(1 - e^{-2}\right) = 0.518798.\end{aligned}$$

   and

$${}_{20}E_{30}\ \bar{A}_{50} = e^{-(0.1)20} \int_0^{40} e^{-0.04t} \frac{1}{40} dt = \frac{e^{-2}}{40}\left(\frac{1 - e^{-(0.04)40}}{0.04}\right) = 0.0675.$$

   Thus NSP $= 17541.64386\,(0.518798 + 0.0675) =$ **10284.631**.

2. The loss at issue of this insurance ${}_0L = 1000v^{\min(T_{40},10)} - 200\ddot{a}_{\overline{\min(K_{40}+1,5)}|}$. If $T_{40} = 5.5$ then

$$
\begin{aligned}
{}_0L &= 1000v^{5.5} - 200\ddot{a}_{\overline{5}|} = 1000v^{5.5} - 200\frac{1-v^5}{d} \\
&= \frac{1000}{(1.05)^{5.5}} - 200 \times \frac{1-(1.05)^{-5}}{0.05} \times 1.05 = -\mathbf{144.55}.
\end{aligned}
$$

3. We have $\bar{A}_{41} = \frac{i}{\delta} A_{41} = \frac{e^\delta - 1}{\delta} A_{41}$ and by recursion $A_{40} = v\ q_{40} + v\ p_{40}\ A_{41}$

$$
A_{41} = \frac{A_{40} - v\ q_{40}}{v\ p_{40}} = \frac{A_{40} - v\left(1 - \frac{\ell_{41}}{\ell_{40}}\right)}{v\frac{\ell_{41}}{\ell_{40}}} = \frac{0.520 - e^{-0.04}\left(1 - \frac{475}{500}\right)}{e^{-0.04}\frac{475}{500}} = 0.5171
$$

therefore

$$
\bar{A}_{41} = \frac{e^\delta - 1}{\delta} A_{41} = \frac{e^{0.04} - 1}{0.04} 0.5171 = \mathbf{0.5276}.
$$

Exercise 2 ***(6 marks)***

1. *You are given: (i) $A_x = 0.28$ (ii) $A_{x+20} = 0.40$ (iii) ${}_{20}E_x = 0.25$ (iv) $i = 0.05$. Calculate $a_{x:\overline{20}|}$. (Hint: $a_{x:\overline{n}|} + 1 = \ddot{a}_{x:\overline{n}|} + {}_nE_x$, $\quad A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$.)*

2. *Ismail is offered the choice of two continuous annuities:*
   1. a life annuity, paying $10,000$ per year
   2. a life annuity with a 5–year certain period, i.e., an annuity paying benefits for life, but with a minimum payout period of 5 years. The benefit amount for the second annuity is chosen so that the expected present value of the two annuities is the same.
   Determine the annual benefit amount on the second annuity given $\delta = 3\%$ and $\mu = 2\%$. (Hint: $\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$).

Solution.

1. We have

$$
a_{x:\overline{20}|} = \ddot{a}_{x:\overline{20}|} - 1 + {}_{20}E_x
$$

where

$$
\begin{aligned}
\ddot{a}_{x:\overline{20}|} &= \frac{1 - A_{x:\overline{20}|}}{d} = \frac{1 - (A_x - {}_{20}E_x\ A_{x+20} + {}_{20}E_x)}{d} \\
&= \frac{1 - (0.28 - 0.25 \times 0. + 0.25)}{\frac{0.05}{1.05}} = 11.97
\end{aligned}
$$

hence $a_{x:\overline{20}|} = 11.97 - 1 + 0.25 = \mathbf{11.22}$.

2. The actuarial present value of the continuous whole life annuity of 1 is $\frac{1}{\mu+\delta} = \frac{1}{0.05} = 20$. The expected present value of the 5–year certain–and–life annuity is

$$
\bar{a}_{\overline{x:\overline{5}|}} = \bar{a}_{\overline{5}|} + {}_{5|}\bar{a}_x
$$

and

$$\bar{a}_{\overline{5}|} = \frac{1-e^{-5(0.03)}}{0.03} = 4.6431$$
$${}_{5|}\bar{a}_x = \frac{e^{-5(0.03+0.02)}}{0.05} = 15.5760$$

thus

$$\bar{a}_{\overline{x:\overline{5}|}} = 4.6431 + 15.5760 = 20.2191.$$

The APVs of the two annuities are equal thus

$$b\bar{a}_{\overline{x:\overline{5}|}} = 10000\bar{a}_{x:\overline{5}|} = 10000 \times 20 = 20000$$

So the level benefit amount of the second annuity is

$$b = \frac{200000}{20.2191} = \mathbf{9891.637115}.$$

Exercise 3 ***(6 marks)***

1. *A special temporary 3–year life annuity–due on (40) pays 1000k at the beginning of year k, k = 1, 2, 3. Find the actuarial present value of this annuity given $i = 0.06$, $q_{40} = 0.03$ and $q_{41} = 0.035$.*
2. *For a special 2–year payment whole life insurance on (50):*
   (i) Premiums of P are paid at the beginning of years 1 and 3.
   (ii) The death benefit is paid at the end of the year of death.
   (iii) There is a partial refund of premium feature: If (50) dies in either year 1 or year 3, the death benefit is $1000 + 0.7P$. Otherwise, the death benefit is 1000.
   Calculate P, using the equivalence principle given $i = 0.06$, $A_{50} = 0.250$, $q_{50} = 0.006$, ${}_2p_{50} = 0.985$ and $q_{52} = 0.007$)

Solution:

1. Denote by α the actuarial present value of this annuity, then

$$\begin{aligned}\alpha &= \sum_{k=0}^{2} v^k S_k \ {}_kp_{40} = S_0 + v\ S_1\ p_{40} + v^2\ S_2\ {}_2p_{40} \\ &= S_0\left(1 + 2v\ p_{40} + 3v^2\ p_{40}\ p_{41}\right) \\ &= 1000\left(1 + 2\frac{1-q_{40}}{1+i} + 3\frac{1-q_{40}}{(1+i)^2}(1-q_{41})\right) \\ &= 1000\left(1 + 2\frac{1-0.03}{1.06} + 3\frac{1-0.03}{(1.06)^2}(1-0.035)\right) = \mathbf{5329.4322}.\end{aligned}$$

2. The actuarial present value of the insurance is

$$\begin{aligned}\text{APV(FB)}_0 &= 1000A_{50} + 0.7P\left(v\ q_{50} + v^3\ {}_{2|}q_{50}\right) \\ & \quad 1000A_{50} + 0.7P\left(v\ q_{50} + v^3\ {}_2p_{50}\ q_{52}\right) \\ &= 250 + 0.7P\left(\frac{0.006}{1.06} + \frac{0.985 \times 0.007}{(1.06)^3}\right) \\ &= 250 + 0.008015P\end{aligned}$$

The actuarial present value of the premiums is

$$\text{APV(FP)}_0 = P\left(1 + v^2\ {}_2p_{50}\right) = P\left(1 + \frac{0.985}{(1.06)^2}\right) = 1.8766P$$

By E.P. we get by equating the two,

$$1.8766P = 250 + 0.008015P$$

Finally $P = \frac{250}{1.8766-0.008015} = \mathbf{133.7911}$.

Exercise 4 ***(6 marks)***
Consider a life aged (40) whose mortality follows De Moivre's law with a limiting age equals to 100.

1. *Find the* 20^{th}*–percentile premium of a* 15*–year payment whole life insurance of* 10000*;*
2. *Find the* 30^{th}*–percentile premium of a* 15*–year payment whole life insurance of* 10000*;*
3. *Find the* 30^{th}*–percentile premium of a* 15*–year term life insurance of* 10000
4. *Find the* 20^{th}*–percentile premium of a* 15*–year term life insurance of* 10000*;*
5. *Find the* 30^{th}*–percentile premium of a* 15*–year endowment life insurance of* 10000.
   Assume that the force of interest is $\delta = 0.05$ *and that all the policies are* ***fully continuous****.*
   Hint:

Type of plan	$t_\alpha \le n\ (\alpha = {}_{t_\alpha}q_x \le {}_nq_x)$	$t_\alpha > n\ (\alpha = {}_{t_\alpha}q_x > {}_nq_x)$
FC whole life (no n)	$\dfrac{S}{\bar{s}_{\overline{t_\alpha}\rceil}} = \dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{t_\alpha}\rceil}}$	$\dfrac{S}{\bar{s}_{\overline{t_\alpha}\rceil}} = \dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{t_\alpha}\rceil}}$
FC n–year endowment	$\dfrac{S}{\bar{s}_{\overline{t_\alpha}\rceil}} = \dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{t_\alpha}\rceil}}$	$\dfrac{S}{\bar{s}_{\overline{n}\rceil}} = \dfrac{Se^{-\delta n}}{\bar{a}_{\overline{n}\rceil}}$
FC n–year payment whole life	$\dfrac{S}{\bar{s}_{\overline{t_\alpha}\rceil}} = \dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{t_\alpha}\rceil}}$	$\dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{n}\rceil}}$
FC n–year term	$\dfrac{S}{\bar{s}_{\overline{t_\alpha}\rceil}} = \dfrac{Se^{-\delta t_\alpha}}{\bar{a}_{\overline{t_\alpha}\rceil}}$	0

Solution:

1. We first find $t_{0.20}$. This is the solution to $F_{40}(t) = \frac{t}{60} = 0.20$, that is $t_{0.20} = 06 \times 0.2 = 12 < 15$. The 20^{th}–percentile premium is

$$P_{0.20} = \frac{10000}{\bar{s}_{\overline{t_{0.20}}\rceil}} = \frac{10000 \times 0.05}{e^{0.05\times 12} - 1} = \mathbf{608.18}.$$

2. Clearly $t_{0.3} = 60 \times 0.3 = 18 > 15$, then 30^{th}–percentile premium is

$$P_{0.3} = \frac{10000e^{-0.05\times t_{0.30}}}{\bar{a}_{\overline{15}\rceil}} = \frac{10000e^{-0.05\times 18}}{\frac{1-e^{-0.05\times 15}}{0.05}} = \mathbf{385.28}.$$

3. We have $t_{0.30} > 15$ thus $P_{0.30} = \mathbf{0}$.

4. We know from Q1 that $t_{0.20} = 12 < 15$, so $P_{0.2}$ the 20^{th}–percentile premium is

$$P_{0.2} = \frac{10000}{\bar{s}_{\overline{12}|}} = \frac{10000 \times 0.05}{e^{0.05\times 12} - 1} = \mathbf{608.18}.$$

5. We have an endowment with $15 < t_{0.30}$, so the 30^{th}–percentile premium is

$$P_{0.3} = \frac{10000}{\bar{s}_{\overline{15}|}} = \frac{10000 \times 0.05}{e^{0.05\times 15} - 1} = \mathbf{447.63}.$$

Exercise 5 ***(6 marks)***
A fully discrete 4–year term insurance of 10000 *is sold to a life aged* 40. *Mortality are given by* $q_{40+k} = \frac{3+k}{1000}$ *for* $k = 0,\ 1,\ 2,\ 3$. *The level net premium is* 427.5

1. *Calculate the net premium reserve* ***at time*** 2 *if* $i = 0.04$ *in all years.*

2. *For a fully discrete* 10*–year deferred whole life insurance of* 1000 *on (*40*), you are given:* $v = 0.95$*;* $p_{48} = 0.982$*,* $p_{49} = 0.980$*,* $A_{50} = 0.352$*. The annual net premium of* 23.40 *is payable during the deferral period. Calculate* ${}_8V$ *the net premium reserve of this insurance.*

3. *For a fully discrete whole life insurance of* 3000 *on (*40*), you are given* $\mu_{40+t} = 0.02$ *for* $t < 10$ *and* $\mu_{40+t} = 0.04$ *for* $t \geq 10$ *and the annual effective interest rate* $i = 0.05$*. The net level premium is* 81.0603*. Calculate* ${}_{10}V$ *and use recursion to calculate* ${}_{10.5}V$*.*

Solution:

1. By the **prospective** method

$${}_2V = \text{APV(FB)}_2 - \text{APV(FP)}_2 = 10000\ A^{\ 1}_{42:\overline{2}|} - 427.5\ \ddot{a}_{42:\overline{2}|}$$

Moreover, we have

$$A^{\ 1}_{42:\overline{2}|} = v\ q_{42} + v^2\ p_{42}\ q_{43} = \frac{0.005}{1.04} + \frac{(0.995)(0.006)}{(1.04)^2} = 0.010327,$$

and

$$\ddot{a}_{42:\overline{2}|} = 1 + v\ p_{42} = 1 + \frac{0.995}{1.04} = 1.9567$$

hence ${}_2V = 10000 \times 0.010327 - 427.50 \times 1.9567 = \mathbf{-733.219}$.
By **recursion** we get

$$\left({}_0V + P_0\right)(1+i) = b_1\ q_{40} + \ {}_1V\ p_{40}$$

thus

$${}_1V = \frac{P_0(1+i) - b_1\ q_{40}}{1 - q_{40}} = \frac{427.50(1.04) - 10 \times 3}{1 - 0.003} = \mathbf{415.848}$$

and

$$\left({}_1V + P_1\right)(1+i) = b_2\ q_{41} + \ {}_2V\ p_{41}$$

thus

$$ {}_2V = \frac{({}_1V + P_1)(1+i) - b_2\, q_{41}}{1 - q_{41}} = \frac{(415.848 + 427.50)(1.04) - 10 \times 4}{1 - 0.004} = \mathbf{840.444}. $$

Observe that the two result are different because the premium is too high. But if take the premium as **42.750** you will get the same values for both methods.

2. In this question we will use backward recursion since ${}_{10}V = 1000\ A_{50}$ (annual net premium is payable only during the deferral period).

$$ ({}_9V + P_9)(1+i) = \frac{{}_9V + P_9}{v} = b_{10}\ q_{49} + \ {}_{10}V\ p_{49} = \ {}_{10}V\ p_{49}\ (\text{since } b_{10} = 0) $$

so

$$ {}_9V = (v\ {}_{10}V\ p_{49} - P_9) = (v\ A_{50}\ p_{49} - P) = 0.95 \times 0.980 \times 352 - 23.40 = 304.312. $$

similarly

$$ {}_8V = (v\ {}_9V\ p_{48} - P) = 0.95 \times 304.312 \times 0.982 - 23.40 = \mathbf{260.4926}. $$

3. Set $p_1 = e^{-0.02}$ and $p_2 = e^{-0.04}$, thus $q_1 = 1 - p_1 = 0.0198$ and $q_2 = 1 - p_2 = 0.03921$

By the **prospective method** we have

$$ \begin{aligned} {}_{10}V &= 3000 A_{50} - P\ddot{a}_{50} = 3000\frac{q_2}{q_2 + i} - P\frac{1+i}{q_2+i} \\ &= 3000\frac{1 - e^{-0.04}}{1.05 - e^{-0.04}} - 81.0603\frac{1.05}{1.05 - e^{-0.04}} = \mathbf{364.5125}. \end{aligned} $$

By the **retrospective method** we have

$$ \begin{aligned} {}_{10}V &= \frac{P\ddot{a}_{40:\overline{10|}} - 3000 A^{1}_{40:\overline{10|}}}{{}_{10}E_{40}} = \frac{81.0603\frac{1+i}{q_1+i}\left(1 - {}_{10}E_{40}\right) - 3000\frac{q_1}{q_1+i}\left(1 - {}_{10}E_{40}\right)}{{}_{10}E_{40}} \\ &= \left(81.0603\,(1+i) - 3000\ q_1\right)\frac{\left(1 - {}_{10}E_{40}\right)}{(q_1 + i)\ {}_{10}E_{40}} \\ &= \left(81.0603 \times 1.05 - 3000\left(1 - e^{-0.02}\right)\right)\frac{\left(1 - (1.05)^{-10} e^{-0.2}\right)}{\left(1 - e^{-0.02} + 0.05\right)(1.05)^{-10} e^{-0.2}} \\ &= \left(81.0603 \times 1.05 - 3000\left(1 - e^{-0.02}\right)\right)\frac{(1.05)^{10} e^{0.2} - 1}{1.05 - e^{-0.02}} = \mathbf{364.46760}. \end{aligned} $$

Now, by **recursion** we can write

$$ \left(\ {}_{10}V + P\right)(1+i)^{0.5} = v^{0.5}\ 3000\ {}_{0.5}q_{50} + \ {}_{10.5}V\ {}_{0.5}p_{50} $$

hence

$$ \begin{aligned} {}_{10.5}V &= \frac{\left(\ {}_{10}V + P\right)(1+i)^{0.5} - v^{0.5}\ 3000\left(1 - \ {}_{0.5}p_{50}\right)}{{}_{0.5}p_{50}} \\ &= \frac{(364.5125 + 81.0603)(1.05)^{0.5} - (1.05)^{-0.5}\, 3000\left(1 - e^{-0.04 \times 0.5}\right)}{e^{-0.04 \times 0.5}} = \mathbf{406.65624}. \end{aligned} $$